Constant Gaussian Curvature Hypersurfaces in Hadamard Manifolds

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Dedicated to Hennie Smith

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With affection.

Abstract: We extend recent results of Guan and Spruck, proving existence results for constant Gaussian curvature hypersurfaces in Hadamard manifolds.

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1 - Introduction.

Let M^{n+1} be a Hadamard manifold. In this paper we prove the existence of constant Gaussian curvature hypersurfaces in M satisfying prescribed boundary conditions. Explicitely, let $(\Sigma_0, \partial\Sigma_0)$ be a smooth, convex, immersed hypersurface in M with smooth boundary. Let N be the exterior unit normal vector field over Σ_0 and define $\mathcal{E} : \Sigma_0\times]-\infty, 0] \to M$ by:

$$\mathcal{E}(x,t) = \mathrm{Exp}(-t\mathsf{N}).$$

We have chosen here an unusual sign convention which we prefer for technical reasons. We say that a $C^{0,1}$ hypersurface $(\Sigma, \partial\Sigma)$ is a graph beneath Σ_0 if and only if there exists a $C^{0,1}$ function $f : \Sigma_0 \to]-\infty, 0]$ and a homeomorphism $\varphi : \Sigma_0 \to \Sigma$ such that:

(i) f vanishes along $\partial\Sigma_0$ (i.e. $\partial\Sigma = \partial\Sigma_0$); and

(ii) for all $p \in \Sigma_0$:

$$\varphi(p) = \mathrm{Exp}_p(-f(p)\mathsf{N}(p)).$$

Let $(\hat{\Sigma}, \partial\hat{\Sigma})$ be a $C^{0,1}$, convex, immersed hypersurface in M which is a graph below Σ_1. We will prove:

Theorem 1.1

Choose $k > 0$ and suppose that the Gaussian curvature of Σ_0 is less than k. Suppose, moreover, that, for some $\epsilon > 0$, the Gaussian curvature of $\hat{\Sigma}$ is no less than $k+\epsilon$ in the weak (Alexandrov) sense and that the second fundamental form of $\hat{\Sigma}$ is also no less than ϵ in the weak (Alexandrov) sense. If Σ_0 is locally and globally rigid, then there exists a smooth, convex, immersed hypersurface Σ_k such that:

(i) Σ_k is a graph beneath Σ_0;

(ii) Σ_k lies between Σ_0 and $\hat{\Sigma}$ as a graph beneath Σ_0; and

(iii) the Gaussian curvature of Σ_k is constant and equal to k.

Remark: This follows immediately from Lemma 10.1.

Remark: The weak (Alexandrov) notion of lower (and upper) bounds for curvature is defined in Section 4. Local and global rigidity are defined in Section 9.

Remark: The hypothesis that M be a Hadamard manifold is only made for simplicity of presentation. The same result, with appropriate modifications, continues to hold in more general manifolds.

The interest of this result lies in its application via the Perron method to the solution of more general boundary value problems. Indeed, let $\Gamma = (\Gamma_1, ..., \Gamma_n)$ be a disjoint collection of closed, smooth, embedded $(n-1)$-dimensional submanifolds of $\mathbb{H}^{n+1}$. Applying the machinery developed by Guan and Spruck in [10] with Lemma 11.4 (which constitutes the more precise version of 1.1 when $M = \mathbb{H}^{n+1}$) in place of Theorem 1.1 of [8], we immediately obtain:

Theorem 1.2

Choose $k > 0$. Suppose that there exists a C^2, locally convex, immersed hypersurface $\Sigma \subseteq \mathbb{H}^{n+1}$ of Gaussian curvature is no less than k such that $\partial\Sigma = \Gamma$. Suppose, moreover, that Σ is locally strictly convex along its boundary. Then there exists a smooth (up to the boundary) locally strictly convex immersed hypersurface $M \subseteq \mathbb{H}^{n+1}$ with $\partial M = \Gamma$ of constant Gaussian curvature equal to k. Moreover, M is homeomorphic to Σ.

Remark: The machinery developed in this paper is in fact also applicable in any affine flat manifold (see section 11). Theorem 1.2 also holds in any affine flat Hadamard manifold, of which a large supply is obtained by small deformations away from hyperbolic space within the family of convex $\mathbb{P}$GL manifolds, as described by Loftin in [16] and [17].

Our reasoning follows the now classical analysis of Caffarelli, Nirenberg and Spruck first laid out in [3] and first applied to constant curvature hypersurfaces by the same authors in [5]. These techniques are further developed in one direction by Guan in [8] yielding an existence result for constant curvature hypersurfaces in $\mathbb{R}^{n+1}$, of which Thereom 1.1 is a generalisation, and which forms the engine driving the existence result of [10] (also proven independantly by Trudinger and Wang in [23]). They are likewise developed in another direction by Rosenberg and Spruck in [19], yielding existence results for hypersurfaces of constant Gaussian curvature in hyperbolic space satisfying prescribed asymptotic boundary conditions. This latter result is further generalised by Guan, Spruck and Szapiel in [11] and Guan and Spruck in [12] to the case of other curvatures different from Gaussian curvature. Within this context, our results may be viewed as a refinement of [11] and [12] and a generalisation of [10]. Other related results can be found in [9] and [15].

Our work is based on two key innovations which simplify greatly the analysis required and thus permit us to obtain results in our current more general context. The first, which is merely a question of perspective, is to analyse the Gauss Curvature Equation intrinsically along the hypersurface as in Section 6, and the second is the use of Sard's Lemma in Section 8 to generate smooth families of hypersurfaces interpolating between the data and the desired solution. Such a topological approach to the deformation stage of the continuity method is already suggested by the work [8] of Guan. However, it turns out that Sard's Lemma used in conjunction with the Fredholm Theory of elliptic PDEs circumvents the necessity of studying hypersurfaces whose curvature depends, not only on position, but also on the tangent space at any point. Since this latter case probably constitutes the greatest technical difficulty in the study of constant Gaussian curvature hypersurfaces, it is a relief for it to be conveniently excluded in this manner. This case still nonetheless constitutes an interesting open problem which the techniques outlined in this paper are not yet sufficiently mature to resolve. Finally, we show in Section 12 how our techniques can be easily adapted to recover both the results [8] of Guan and [19] of Rosenberg and Spruck.

The generality of Theorem 1.1 suggests that we may further generalise Theorem 1.2 to the case of arbitrary Hadamard manifolds. The main difficulty here lies in understanding the singularities that may arise by taking limits of viscosity solutions of the Gauss Curvature

Equation. It is interesting to observe how the affine flat condition repeatedly recurs as an obstruction, and is, in fact, the only one. One is therefore led to wonder whether this condition is accidental or fundamental. At this stage, we remain confident that it is not the latter (since this would otherwise exclude manifolds as simple as $\mathbb{H}^n\times\mathbb{R}$). We therefore believe that Theorem 1.2 remains valid in any Hadamard manifold.

This paper is structured as follows:

(a) in Section 2, we show how first order bounds arise as a consequence of convexity;

(b) in Section 3, we derive the Gauss curvature equation for a graph in a general Riemannian manifold;

(c) in Section 4, we introduce the concept of weak (Alexandrov) lower and upper bounds for curvature;

(d) in Sections 5 and 6 we obtain a-priori second order bounds over the boundary and then over the whole hypersurface respectively. These bounds are then applied in Section 7 to obtain the compactness result, Lemma 7.1;

(e) in Section 8, we use Sard's Lemma to obtain smooth (albeit possibly empty) one-dimensional families of hypersurfaces interpolating between the data and the solutions. These are used in conjunction with the concept of local and global rigidity developed in Section 9 to prove in Section 10 the existence result, Lemma 10.1, which immediately yields Theorem 1.1;

(f) In Section 11, we study affine flat Riemannian manifolds and show that the singularities that may arise in such manifolds are no different from those encountered $\mathbb{R}^{n+1}$. This is used to prove Lemma 11.4, which, in conjunction with the machinery developed by Guan and Spruck in [10] immediately yields Theorem 1.2;

(g) In Section 12, we show how minor adaptations of these techniques allow us to obtain both the results [8] of Guan (Theorem 12.1) and [19] of Rosenberg and Spruck (Theorem 12.2); and

(h) In Appendix A, we prove the regularity of limiting hypersurfaces which are themselves strictly convex. This result may be found in Caffarelli [2], but, given the general public unavailability of these notes, we consider it preferable to provide our own proof here.

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2 - First Order Control.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Let $(\Sigma_0,\partial\Sigma_0)$ be a convex, immersed hypersurface with boundary. Let N_0 and A_0 denote the outward pointing unit normal and the second fundamental form respectively of Σ_0. We define $\mathcal{E}:\Sigma_0\times]-\infty,0]\rightarrow M$ by:

$$\mathcal{E}(x,t)=\mathrm{Exp}(-t\mathsf{N}_0(x)).$$

Remark: The change of sign ensures that convex hypersurfaces correspond to graphs of convex functions.

We will say that a $C^{0,1}$ hypersurface, Σ, is a graph beneath Ω if and only if there exists a $C^{0,1}$ function $f : \overline{\Omega} \to]-\infty, 0]$ and a homeomorphism $\varphi : \overline{\Omega} \to \Sigma$ such that:

(i) f vanishes along $\partial\Omega$ (i.e. $\partial\Sigma = \partial\Omega$); and

(ii) for all $p \in \Omega$:
$$\varphi(p) = \mathrm{Exp}_p(-f(p)\mathsf{N}_0(p)).$$

We refer to f as the graph function of Σ. In particular, since f is Lipschitz, its graph is never vertical, even along the boundary. Consider the family of graphs over Ω. We define the partial order "$<$" on this family such that if Σ and Σ' are two graphs over Ω and f and f' are their respective graph functions, then:

$$\Sigma < \Sigma' \Leftrightarrow f(p) < f'(p) \text{ for all } p \in \Omega.$$

Since $\partial\Omega$ is smooth, for all $p \in \partial\Omega$, the set of supporting hyperplanes in TM to $\partial\Omega$ at p is parametrised by $\mathbb{R}$. Supporting hyperplanes may be locally considered as graphs over Ω, and we obtain an analogous partial order on this set, which we also denote by $<$.

Let $\hat{\Sigma}$ be a $C^{0,1}$ convex hypersurface which is a graph over Ω. Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of convex graphs over Ω such that for all $n \in \mathbb{N}$, $\Sigma_n > \hat{\Sigma}$. For all n, let f_n be the graph function of Σ_n.

Lemma 2.1

$(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in the $C^{0,1}$ sense.

Proof: For all $n \in \mathbb{N} \cup \{\infty\}$, define U_n by:

$$U_n = \left\{\mathrm{Exp}_p(-t\mathsf{N}_0(p)) \text{ s.t. } p \in \overline{\Omega} \text{ and } 0 \leqslant t \leqslant f_n(p)\right\}.$$

By the compactness of convex sets, after extraction of a subsequence, there exists U_0 to which $(U_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense. Moreover, the supporting hyperplanes of U_0 are transverse to the normal geodesics leaving H. Indeed, suppose the contrary and let $p_0 \in \partial U_0$ be a point where the supporting hyperplane is not transverse to the normal geodesic leaving Σ_0. Taking limits $\partial U_0 \geqslant \hat{\Sigma}$. Since the tangent to $\hat{\Sigma}$ along $\partial\hat{\Sigma}$ is not vertical, it follows that p_0 lies over an interior point of Σ_0. Let $(p_n)_{n\in\mathbb{N}} \in (\partial U_n)_{n\in\mathbb{N}}$ be a sequence converging to p_0. For all $n \in \mathbb{N} \cup \{0\}$, let $q_n \in \Sigma_0$ be the orthogonal projection of p_n onto Σ_0 and let γ_n be the geodesic segment joining q_n to p_n. For all $n \in \mathbb{N}$, $\gamma_n \subseteq U_n$. Taking limits, $\gamma_0 \subseteq U_0$. It follows that γ_0 is an interior tangent to ∂U_0 at p_0. Therefore, by convexity, $\gamma_0 \subseteq \partial U_0$. In particular, U_0 has a vertical supporting tangent at q_0, which is absurd.

We assert that the supporting tangent hyperplanes of $(\partial U_n)_{n\in\mathbb{N}}$ are uniformly transverse to the foliation of normal geodesics leaving Σ_0. Indeed, suppose the contrary. For all n, let p_n be a point in ∂U_n and let P_n be a supporting tangent of U_n at p_n. Suppose that $(P_n)_{n\in\mathbb{N}}$ is not uniformly transverse to the foliation of normal geodesics leaving Σ_0. After extracting a subsequence, we may assume that there exists $p_0 \in \partial U_0$ and a supporting tangent P_0 to ∂U_0 at p_0 to which $(p_n, P_n)_{n\in\mathbb{N}}$ converges such that P_0 is tangent to the foliation of normal geodesics leaving Σ_0. This is absurd by the preceeding discussion. The assertion is thus proven, and the result follows. $\square$

3 - The Gauss Curvature Equation.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Let $(\Sigma_0, \partial\Sigma_0) \subseteq M$ be a convex immersed hypersurface with boundary. Let N_0 and A_0 denote the outward pointing unit normal and the second fundamental form respectively of Σ_0. Using the exponential map, we identify an open subset of M with $\Sigma_0\times]-\infty, 0]$.

We will prove:

Proposition 3.1

Let $f : \Sigma_0 \to]-\infty, 0]$ be a smooth function. The Gaussian curvature of the graph of f is given by:

$$K = \psi(x, f, \nabla f)^{-1}\mathrm{Det}(\mathrm{Hess}(f) + \Psi(x, f, \nabla f))^{1/n},$$

where:

(i) $\psi = \psi(x, t, p)$ is a smooth, strictly positive function and, for all $R > 0$ there exists $\epsilon > 0$ such that, if $|t| < \epsilon$ then $\psi(x, t, p)$ is convex in p for $\|p\| \leqslant R$; and

(ii) there exists a smooth function Ψ_0 such that:

$$\Psi(x, f, \nabla f)_{ij} = A_{0,ij} + f_{;i}f_{;k}{A_0}^k{}_j + f_{;j}f_{;k}{A_0}^k{}_i + f\Psi_0(x, f, \nabla f).$$

Moreover, the graph of f is convex if and only if $\mathrm{Hess}(f) + \Psi(x, f, \nabla f)$ is positive definite.

Example: We view $\mathbb{H}^n$ as a totally geodesic, embedded hypersurface in $\mathbb{H}^{n+1}$. Let g_0 and g be the metrics of $\mathbb{H}^n$ and $\mathbb{H}^{n+1}$ respectively. We consider the foliation of $\mathbb{H}^{n+1}$ by geodesics normal to $\mathbb{H}^n$. Exceptionally, we reparametrise geodesics in a non-uniform manner in order to make this parametrisation conformal. This facilitates the calculation of the connexion 2-form. Let $\alpha :]-\pi/2, \pi/2[\to \mathbb{R}$ be such that, for all θ:

$$\cos(\theta)\cosh(\alpha(\theta)) = 1.$$

Let N be the normal vector field over $\mathbb{H}^n$ in $\mathbb{H}^{n+1}$. We define $\Phi : \mathbb{H}^n\times]-\pi/2, \pi/2[\to \mathbb{H}^{n+1}$ by:

$$\Phi(x, \theta) = \mathrm{Exp}(-\alpha(t)\mathsf{N}(x)).$$

We easily obtain:

$$\Phi^*g = \frac{1}{\cos^2(\theta)}(g_0 \oplus d\theta^2).$$

If Ω denotes the connexion 2-form of the Levi-Civita covariant derivative of Φ^*g with respect to that of the product metric, then, for all X, Y tangent to $\mathbb{H}^n$:

$$\begin{aligned}\Omega(X,Y) &= -\langle X,Y\rangle\tan(\theta)\partial_\theta,\\ \Omega(X,\partial_\theta) &= \tan(\theta)X,\\ \Omega(\partial_\theta,\partial_\theta) &= \tan(\theta)\partial_\theta.\end{aligned}$$

Thus, if $\Omega\subseteq\mathbb{H}^n$ is an open set, and if $f:\Omega\rightarrow]-\pi/2,\pi/2[$ is a smooth function, then the Gauss curvature of the graph of f is given by:

$$K = \cos(f)^3(1+\|\nabla f\|^2)^{-(n+2)/2n}\mathrm{Det}(f_{;ij}-\tan(f)(f_{;j}f_{;j}+\delta_{ij}))^{1/n}.$$

This formula will be of use in the sequel. $\square$

Let ∇^0 denote the Levi-Civita covariant derivative of the product metric on $\Sigma_0\times]-\infty,0]$. Let g denote the pull back of the metric over M through the exponential map. Let Vol denote the volume form of g and let ∇ denote the Levi Civita covariant derivative of g. Trivially, ∇ coincides with the pull back through the exponential map of the Levi-Civita covariant derivative of M.

Proposition 3.2

Let $\Omega:=\nabla-\nabla^0$ be the connection 2-form of ∇ with respect to ∇^0. There exists a smooth 2-form Ω_0 such that, if X and Y are tangent to Σ, then:

$$\begin{aligned}\Omega_{(x,t)}(X,Y) &= A_0(X,Y)\partial_t + t\Omega_{0,(x,t)}(X,Y),\\ \Omega_{(x,t)}(X,\partial_t) &= -A_0X + t\Omega_{0,(x,t)}(X,\partial_t),\\ \Omega_{(x,t)}(\partial_t,\partial_t) &= t\Omega_{0,(x,t)}(\partial_t,\partial_t).\end{aligned}$$

Proof: When $t=0$, by definition of A_0:

$$\begin{aligned}\nabla_X Y &= \nabla^0_X Y + \langle\nabla_X Y,\mathsf{N}_0\rangle\mathsf{N}_0\\ &= \nabla^0_X Y - A_0(X,Y)\mathsf{N}_0.\end{aligned}$$

Thus, since $\mathsf{N}_0=-\partial_t$, at t_0:

$$\nabla_X Y = \nabla^0_X Y + A_0(X,Y)\partial_t.$$

Likewise:

$$\nabla_X\partial_t = -\nabla_X\mathsf{N}_0 = -A_0X.$$

Finally, since the vertical lines are geodesics:

$$\nabla_{\partial_t}\partial_t = 0.$$

The result follows. $\square$

Define $\hat{f}:\Sigma_0\times]-\infty,0]\rightarrow\mathbb{R}$ by:

$$\hat{f}(x,t) = f(x) - t.$$

The graph of f is the level set $\hat{f}^{-1}(\{0\})$. Observe that $\nabla\hat{f}$ is parallel to the downwards pointing unit normal over the graph of f. Let A_f denote the second fundamental form of this graph. For all i, we define the vector field $\hat{\partial}_i = (\partial_i, f_{;i})_{(x,f(x))}$. $(\hat{\partial}_1, ..., \hat{\partial}_n)$ forms a basis of the tangent space of the graph of f.

Proof of Proposition 3.1: By definition:

$$K^n = \mathrm{Det}(A_f(\hat{\partial}_i, \hat{\partial}_j))/\mathrm{Det}(g(\hat{\partial}_i, \hat{\partial}_j)).$$

However, since the graph of f is the level set $\hat{f}^{-1}(0)$:

$$A_f = \frac{1}{\|\nabla\hat{f}\|_g}(\mathrm{Hess}(\hat{f})).$$

Moreover:

$$\mathrm{Hess}(\hat{f}) = \mathrm{Hess}^0(\hat{f}) - d\hat{f}(\Omega) = \mathrm{Hess}(f) - d\hat{f}(\Omega).$$

It follows that K has the specified form with:

$$\psi(x, f, \nabla f) = \|\nabla\hat{f}\|_g \mathrm{Det}(g(\hat{\partial}_i, \hat{\partial}_j))^{1/n},$$

and:

$$\Psi(x, f, \nabla f) = -d\hat{f}(\Omega).$$

When $t = 0$:

$$\psi(x, 0, p) = (1 + \|p\|^2)^{(n+2)/2n}.$$

Thus, since the function $p \mapsto (1 + \|p\|^2)^\alpha$ is locally uniformly strictly convex for $\alpha > 1/2$, (i) follows.

Likewise, by Proposition 3.2:

$$\begin{aligned}\Psi(x, 0, p)(\hat{\partial}_i, \hat{\partial}_j) &= d\hat{f}(\mathsf{N})A_0(\partial_i, \partial_j) + f^{;j}d\hat{f}(A_0\partial_i) + f^{;i}d\hat{f}(A_0\partial_j) \\ &= A_{0,ij} + f_{;i}f_{;k}{A_0}^k{}_j + f_{;j}f_{;k}{A_0}^k{}_i.\end{aligned}$$

(ii) follows.

Finally, the graph of f is convex if and only if A_f is positive definite, and this completes the proof. □

4 - Interlude - Maximum Principals.

Let M^{n+1} be an $(n+1)$-dimensional Riemanian manifold.

Definition 4.1

Let Σ be a $C^{0,1}$ convex, immersed hypersurface in M. Choose $k > 0$. For $P \in \Sigma$, we say that the Gaussian curvature of Σ is at least (resp. at most) k in the weak (Alexandrov) sense at P if and only if there exists a smooth, convex, immersed hypersurface Σ' such that:

(i) Σ' is an exterior (resp. interior) tangent to Σ at P; and

(ii) the Gaussian curvature of Σ' at P is equal to k.

This notion is well adapted to the weak Geometric Maximum Principal:

Lemma 4.2, Weak Geometric Maximum Principal

Let Σ_1, Σ_2 be two $C^{0,1}$, convex, immersed hypersurfaces in M. Choose $P \in \Sigma_1$. If Σ_2 is an interior tangent to Σ_1 at P, then the Gaussian curvature of Σ_2 at P is no less than the Gaussian curvature of Σ_1 at P in the weak (Alexandrov) sense.

Proof: Let Σ_1' be a smooth, convex hypersurface which is an exterior tangent to Σ_1 at P. Likewise, let Σ_2' be a smooth convex hypersurface which is an interior tangent to Σ_2 at P. Let A_1 and A_2 be the respective second fundamental forms of Σ_1' and Σ_2' respectively. Since Σ_2' is an interior tangent to Σ_1' at P:

$$A_2 \geqslant A_1.$$

The result follows. □

Remark: This result is often used in conjunction with foliations by constant curvature hypersurfaces which then act as barriers. In the case where $M = \mathbb{H}^{n+1}$, if we identify $\mathbb{H}^{n+1}$ with the upper half space in $\mathbb{R}^{n+1}$, then we obtain families of constant curvature hypersurfaces by considering intersections of spheres in $\mathbb{R}^{n+1}$ with $\mathbb{H}^{n+1}$. If the centre of such a sphere lies on $\mathbb{R}^n$, then its intersection with $\mathbb{H}^{n+1}$ has zero curvature. If the sphere is not entirely contained in $\mathbb{H}^{n+1}$, then the intersection has curvature less than 1, and if it is contained in $\mathbb{H}^{n+1}$, then the intersection has curvature greater than 1.

We also have the strong Geometric Maximum Principal:

Lemma 4.3, Strong Geometric Maximum Principal

Let Σ_1 and Σ_2 be smooth, convex, immersed hypersurfaces in M of constant Gaussian curvature equal to k. Choose $P \in \Sigma_1$. If Σ_2 is an exterior tangent to Σ_1 at P, then $\Sigma_1 = \Sigma_2$

Proof: Σ_2 is a graph beneath Σ_1 near P. Let U be a neighbourhood of P in Σ_1 over which Σ_2 is a graph. Let A be the shape operator of Σ_1 and let f be the graph function of Σ_2. By Proposition 3.1:

$$\mathrm{Det}(\mathrm{Hess}(f) + \Psi(x, f, \nabla f))^{1/n} = k\psi(x, f, \nabla f),$$

for some Ψ and ψ. However:

$$\mathrm{Det}(A) = k.$$

Thus, by concavity of $\mathrm{Det}^{1/n}$:

$$\frac{k}{n}\mathrm{Tr}(A^{-1}(\mathrm{Hess}(f) + \Psi(x, f, \nabla f) - A)) \geqslant k(\psi(x, f, \nabla f) - 1).$$

Moreover, by the proof of Proposition 3.1:

$$\psi(x, f, \nabla f) = (1 + \|\nabla f\|^2)^{n+2/2n} + f\psi_0(x, f, \nabla f),$$

For some smooth function ψ_0. Thus:

$$k(\psi(x,f,\nabla f)-1)=c_1 f+\langle b_1,\nabla f\rangle,$$

for some smooth function c_1 and vector field b_1. Likewise, by Proposition 3.1:

$$\mathrm{Tr}(A^{-1}(\Psi(x,f,\nabla f)-A))=c_2 f+\langle b_2,\nabla f\rangle,$$

for some smooth function c_2 and vector field b_2. Thus:

$$\mathrm{Tr}(A^{-1}\mathrm{Hess}(f))+\langle b,\nabla f\rangle+cf\geqslant 0,$$

for some smooth function c and vector field b. Since $f\leqslant 0$ and $f(P)=0$, it follows by the strong maximum principal that $f=0$ over a neighbourhood of P. The result follows by unique continuation of constant Gaussian curvature hypersurfaces. $\square$

5 - Second Order Bounds Along the Boundary.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Let $(\Sigma_0,\partial\Sigma_0)\subseteq M$ be a smooth, strictly convex, immersed hypersurface. Using the exponential map, we identify a subset of M with $\Sigma_0\times]-\infty,0]$. Let $\phi:M\rightarrow]0,\infty[$ be a smooth, positive function. Let $(\hat{\Sigma},\partial\hat{\Sigma})$ be a $C^{0,1}$, convex, immersed hypersurface such that:

(i) $\hat{\Sigma}$ is a graph below Σ_0;

(ii) $\partial\hat{\Sigma}=\partial\Sigma_0$; and

(iii) for all $x\in\hat{\Sigma}$, the Gaussian curvature of $\hat{\Sigma}$ is greater than $\phi(x)+\epsilon$ in the weak (Alexandrov) sense, for some $\epsilon>0$.

$\hat{\Sigma}$ serves as a lower barrier for our problem. Let $(\Sigma,\partial\Sigma)\subseteq M$ be a smooth, convex, immersed hypersurface such that:

(i) $\hat{\Sigma}\leqslant\Sigma\leqslant\Sigma_0$;

(ii) $\partial\Sigma=\partial\Sigma_0$; and

(iii) for all $x\in\Sigma$, the Gaussian curvature of Σ at x is equal to $\phi(x)$.

We aim to obtain bounds for the norm of the second fundamental form of Σ along the boundary in terms of the data. To this end, we denote by $\mathcal{B}$ the family of constants which depend continuously on the data: M, Σ_0, $\hat{\Sigma}$, ϵ, ϕ and the C^1 jet of Σ (formally, $\mathcal{B}$ is the set of continuous - or even locally bounded - functions over the space of data). When supplementary data, D, (such as, for example, a vector field) is added, we denote by $\mathcal{B}(D)$ the family of constants which, in addition, also depend on D.

We will prove:

Proposition 5.1

There exists K in $\mathcal{B}$ such that, if A is the second fundamental form of Σ, then, for all $P\in\partial\Sigma$:

$$\|A(P)\|\leqslant K.$$

Remark: The strict convexity of Σ_0 is only required in the last step of the proof, where it is used to obtain uniform strict lower bounds for the restriction of the second fundamental form to the tangent space of $\partial\Sigma_0$. In the case where Σ_0 is totally geodesic, this may be shown using other geometric considerations.

Let $P \in \partial\Sigma_0$ be a point on the boundary. Let $\hat{\Sigma}_P \subseteq M$ be a smooth, convex, immersed hypersurface such that:

(i) $\hat{\Sigma}_P$ is a graph below Σ_0;

(ii) $P \in \hat{\Sigma}_P$; and

(iii) for all $x \in \hat{\Sigma}$, the Gaussian curvature of $\hat{\Sigma}_P$ at x is greater than $\phi(x) + \epsilon$.

Bearing in mind the results section 3, we will consider Σ and $\hat{\Sigma}_P$ as graphs near P over a hypersurface whose second fundamental form vanishes at P. Thus, let $\Sigma_1 \subseteq M$ be an immersed hypersurface in M which is tangent to Σ_0 at P and which is totally geodesic at P.

Let $\Omega \subseteq \Sigma_1$ be an open set with $P \in \partial\Omega$ and $f_0 : \Omega \to \mathbb{R}$ a function such that:

(i) Σ_0 is the graph of f_0 over Ω; and

(ii) $f_0(\partial\Omega) = \partial\Sigma_0$.

We observe in passing that, by convexity, after reducing Σ_1 if necessary, f_0 may be made to be positive. $\partial\Omega$ consists of two components: we denote by $\partial_b\Omega$ the subset of $\partial\Omega$ which lies above the boundary of Σ_0 and we denote by $\partial_i\Omega$ the subset of $\partial\Omega$ which lies above the interior of Σ_0.

Proposition 5.2

There exists $\delta > 0$ in $\mathcal{B}(P, \Sigma_1)$ and a neighbourhood U of P in Σ which is a graph over $B_\delta(P) \cap \Omega$.

Proof: The radius over which Σ is a graph over Σ_1 is determined by the C^1 jet of Σ, which is among the data defining $\mathcal{B}$. The result follows. $\square$

We thus replace Ω with $\Omega \cap B_\delta(P)$ and let $f, \hat{f} : \Omega \to \mathbb{R}$ be the functions of which Σ and $\hat{\Sigma}_P$ respectively are the graphs below Σ_1.

By Proposition 3.1, there exist functions ψ and Ψ and a positive number $R > 0$, which only depend on M, ϕ and Σ_1 such that:

$$\mathrm{Det}(\mathrm{Hess}(f) + \Psi(x, f, \nabla f))^{1/n} = \psi(x, f, \nabla f).$$

Moreover:

(i) $\mathrm{Hess}(f) + \Psi(x, f, \nabla f)$ is positive definite;

(ii) $\Psi(x, t, p), (\partial_{p_k}\Psi)(x, t, p) = O(d(x, P)) + O(t)$ where $d(\cdot, P)$ is the distance in M to P; and

(iii) for t sufficiently small, $p \mapsto \psi(x, t, p)$ is a convex function in p for $\|p\| \leqslant R$.

We define the matrix B by:

$$B = \frac{1}{n}\psi(x, f, \nabla f)(\mathrm{Hess}(f) + \Psi(x, f, \nabla f))^{-1}.$$

We define the operator $\mathcal{L}$ by:

$$\mathcal{L}g = B^{ij}g_{;ij} + B^{ij}(\partial_{p_k}\Psi)_{ij}g_{;k} - (\partial_{p_k}\psi)g_{;k}.$$

Proposition 5.3

There exists $\delta_1 > 0$ and $\epsilon_1 > 0$ in $\mathcal{B}(P, \Sigma_1)$ such that, for $d(x, P) < \delta_1$:

$$\mathcal{L}(f - \hat{f}) \leqslant -\epsilon_1(1 + \sum_{i=1}^{n} B^{ii}).$$

Proof: There exists $\eta_1 > 0$ in $\mathcal{B}$ such that, near p:

$$\mathrm{Det}(\mathrm{Hess}(\hat{f}) + \Psi(x, \hat{f}, \nabla\hat{f}))^{1/n} \geqslant \psi(x, \hat{f}, \nabla\hat{f}) + 2\eta_1.$$

Define $\delta : \Sigma_1 \to \mathbb{R}$ by:

$$\delta(x) = d_1(x, P)^2,$$

where d_1 denotes the intrinsic distance in Σ_1. Near P:

$$\mathrm{Hess}(\delta) \geqslant \mathrm{Id}.$$

There exists $\eta_2 > 0$ in $\mathcal{B}$ such that, if we define $\hat{g}$ by:

$$\hat{g} = \hat{f} - \eta_2\delta,$$

then, near P:

$$\mathrm{Det}(\mathrm{Hess}(\hat{g}) + \Psi(x, \hat{g}, \nabla\hat{g}))^{1/n} \geqslant \psi(x, \hat{g}, \nabla\hat{g}) + \eta_1.$$

Since $\mathrm{Det}^{1/n}$ is a concave function:

$$\begin{aligned}
&\mathrm{Det}(\mathrm{Hess}(\hat{g}) + \Psi(x, \hat{g}, \nabla\hat{g}))^{1/n} - \mathrm{Det}(\mathrm{Hess}(f) + \Psi(x, f, \nabla f))^{1/n} \\
&\quad \leqslant B^{ij}(\hat{g}_{;ij} + \Psi_{ij}(x, \hat{g}, \nabla\hat{g}) - f_{;ij} - \Psi_{ij}(x, f, \nabla f)) \\
&\quad \leqslant B^{ij}(\hat{f} - f)_{;ij} - \eta_2 \textstyle\sum_{i=1}^{n} B^{ii} + B^{ij}(\Psi_{ij}(x, \hat{g}, \nabla\hat{g}) - \Psi_{ij}(x, f, \nabla f)).
\end{aligned}$$

Since $\Psi(x, t, p) = O(d(x, P)) + O(t)$, near P:

$$B^{ij}(f - \hat{f})_{;ij} \leqslant -\eta_1 - \frac{\eta_2}{2}\sum_{i=1}^{n} B^{ii} + \psi(x, f, \nabla f) - \psi(x, \hat{g}, \nabla\hat{g}).$$

However, sufficiently close to p:

$$\psi(x, f, \nabla\hat{g}) - \psi(x, \hat{g}, \nabla\hat{g}) \leqslant \eta_1/3.$$

Moreover, by convexity of ψ:

$$\psi(x,f,\nabla f)-\psi(x,f,\nabla\hat{g})\leqslant(\partial_{p_k}\psi)(f_{;k}-\hat{f}_{;k}+\eta_2\delta_{;k}).$$

Since $\delta_{;k}$ is continuous and vanishes at P, we conclude that, near P:

$$B^{ij}(f-\hat{f})_{;ij}-(\partial_{p_k}\psi)(f_{;k}-\hat{f}_{;k})\leqslant-\eta_1/3-\eta_2/2\sum_{i=1}^{n}B^{11}.$$

Bearing in mind that, for all k, $(\partial_{p_k}\Psi)(x,t,\xi)=O(d(x,P))+O(t)$, the result follows. $\square$

Let X be a vector field over Σ_1.

Proposition 5.4

There exists K in $\mathcal{B}(P,\Sigma_1,X)$ such that, near P:

$$|\mathcal{L}(Xf)|\leqslant K(1+\sum_{i=1}^{n}B^{ii}).$$

Proof: Differentiating the Gaussian curvature equation yields, for all k:

$$B^{ij}(f_{;ijk}+(\partial_{x_k}\Psi)_{ij}+(\partial_t\Psi)_{ij}f_{;k}+(\partial_{p_l}\Psi)_{ij}f_{;lk})=(\partial_{x_k}\psi)+(\partial_t\psi)f_{;k}+(\partial_{p_l}\psi)f_{;lk}.$$

However:

$$f_{;lk}=f_{;kl}.$$

Moreover:

$$f_{;ijk}=f_{;kij}+R^{\Sigma_1}{}_{jki}{}^{p}f_{;p},$$

where R^{Σ_1} is the Riemann curvature tensor of Σ_1. There therefore exists K_1 in $\mathcal{B}(P,\Sigma_1)$ such that:

$$\left|B^{ij}(f_{;kij}+(\partial_{p_l}\Psi)_{ij}f_{;kl})-(\partial_{p_l}\psi)f_{;kl}\right|\leqslant K_1(1+\sum_{i=1}^{n}B^{ii}).$$

Moreover:

$$\begin{aligned}B^{ij}f_{;ki}&=B^{ij}((f_{;ki}+\Psi_{ki})-\Psi_{ki})\\&=n-B^{ij}\Psi_{ki}.\end{aligned}$$

However:

$$\begin{aligned}\mathcal{L}(Xf)&=X^k(B^{ij}(f_{;kij}+(\partial_{p_l}\Psi)_{ij}f_{;kl})-(\partial_{p_l}\psi)f_{;kl})\\&\quad+f_{;k}(B^{ij}(X^k{}_{;ij}+(\partial_{p_l}\Psi)_{ij}X^k{}_{;l})-(\partial_{p_l}\psi)X^k{}_{;l})\\&\quad+2B^{ij}(f_{;ki}X^k{}_{;j}).\end{aligned}$$

The result follows by combining the above relations. $\square$

Corollary 5.5

There exists K in $\mathcal{B}(P,\Sigma_1,X)$ such that, near P:

$$|\mathcal{L}X(f-f_0)|\leqslant K(1+\sum_{i=1}^{n}B^{ii}).$$

We define $\delta : \Sigma_1 \rightarrow]0,\infty[$ by:

$$\delta(x) = d_1(x,P)^2,$$

where $d_1(\cdot,P)$ denotes the distance in Σ_1 to P.

Proposition 5.6

There exists K in $\mathcal{B}(P,\Sigma_1,X)$ such that, near p:

$$|\mathcal{L}\delta| \leqslant K(1+\sum_{i=1}^{n} B^{ii}).$$

Proof: Trivial. □

Proof of Proposition 5.1: Let P, Σ_1 and Ω be as before. Let X be a vector field over Ω which is tangent to $\partial_b\Omega$. By Propositions 5.3 and 5.6 and Corollary 5.5, there exists $\eta, K > 0$ in $\mathcal{B}(P,\Sigma_1,X)$ such that:

$$\begin{array}{ll} |\mathcal{L}X(f-f_0)| & \leqslant K(1+\sum_{i=1}^n B^{ii}),\\ |\mathcal{L}\delta| & \leqslant K(1+\sum_{i=1}^n B^{ii}),\\ \mathcal{L}(f-\hat{f}) & \leqslant -\eta(1+\sum_{i=1}^n B^{ii}). \end{array}$$

Moreover, we may assume that, throughout Ω:

$$|X(f-f_0)| \leqslant K.$$

By definition of X, $X(f-f_0)$ vanishes along $\partial_b\Omega$. Since $\partial_i\Omega$ is bounded away from P, there therefore exists $A_+ > 0$ in $\mathcal{B}(P,\Sigma_1,X)$ such that, over $\partial\Omega$:

$$X(f-f_0) - A_+\delta \leqslant 0.$$

There exists $B_+ > 0$ in $\mathcal{B}(P,\Sigma_1,X)$ such that, throughout Ω:

$$\mathcal{L}(X(f-f_0) - A_+\delta - B_+(f-\hat{f})) \geqslant 0.$$

Moreover, since $f-\hat{f} \geqslant 0$, this function is also negative along $\partial\Omega$. Thus, by the Maximum Principal, throughout Ω:

$$X(f-f_0) \leqslant A_+\delta + B_+(f-\hat{f}).$$

Likewise, there exists $A_-, B_- > 0$ in $\mathcal{B}(P,\Sigma_1,X)$ such that:

$$X(f-f_0) \geqslant -A_-\delta - B_-(f-\hat{f}).$$

There therefore exists $K_1 > 0$ in $\mathcal{B}(P,\Sigma_1,X)$ such that:

$$|d(X(f-f_0))(P)| \leqslant K_1.$$

Thus, increasing K_1 if necessary:

$$|d(Xf)(P)| \leqslant K_1.$$

Let N be the unit normal vector field along $\partial\Sigma$ pointing into Σ. There exists $K_2 > 0$ in $\mathcal{B}$ such that, for any vector field, X, tangent to $\partial\Sigma$:

$$\|A(X, \mathsf{N})\| \leqslant K_2\|X\|.$$

The restriction of A to $\partial\Sigma$ is determined by the norm of the second fundamental form of $\partial\Sigma = \partial\Sigma_0$. There therefore exists $K_3 > 0$ in $\mathcal{B}$ such that, if X and Y are vector fields tangent to $\partial\Sigma$, then:

$$\|A(X, Y)\| \leqslant K_3\|X\|\|Y\|.$$

Finally, since Σ lies between Σ_0 and $\hat{\Sigma}$, both of which are strictly convex, there exists $\epsilon_1 > 0$ in $\mathcal{B}$ such that, throughout $\partial\Sigma$:

$$A|_{T\partial\Sigma} \geqslant \epsilon_1\mathrm{Id}.$$

Since $\mathrm{Det}(A) = \phi$, $A(\mathsf{N}, \mathsf{N})$ may be estimated in terms of the other components of A, and there therefore exists $K_4 > 0$ in $\mathcal{B}$ such that, throughout $\partial\Sigma$:

$$\|A(\mathsf{N}, \mathsf{N})\| \leqslant K_4.$$

The result now follows. $\square$

6 - Second Order Bounds Over the Interior.

Let M^{n+1} be a Hadamard manifold. Let $(\Sigma, \partial\Sigma) \subseteq M^{n+1}$ be a smooth convex hypersurface. Let N and A be the unit exterior normal vector and the shape operator of Σ respectively. Let $\phi : M \to]0, \infty[$ be a strictly positive smooth function. We prove second order estimates given second order estimates along the boundary for the problem:

$$\mathrm{Log}(\mathrm{Det}(A)) = \phi(x),$$

We denote by $\|A|_{\partial\Sigma_0}\|$ the supremum over $\partial\Sigma_0$ of the norm of A. We will prove:

Proposition 6.1

There exists $K > 0$ in $\mathcal{B}(\|A|_{\partial\Sigma_0}\|)$ such that:

$$\|A\| \leqslant K.$$

In the sequel, we raise and lower indices with respect to A. Thus:

$$A^{ij}A_{jk} = \delta^i{}_k,$$

where δ is the Krönecker delta function.

Proposition 6.2

(i) For all p:

$$A^{ij}A_{ij;p} = \phi_{;p}.$$

(ii) For all p, q:

$$A^{ij}A_{ij;pq} = A^{im}A^{jn}A_{ij;p}A_{mn;q} + \phi_{;pq}.$$

Proof: This follows by differentiating the equation $\mathrm{Log}(\mathrm{Det}(A)) = \phi$. $\square$

We recall the commutation rules of covariant differentiation in a Riemannian manifold:

Lemma 6.3

Let R^Σ and R^M be the Riemann curvature tensors of Σ and M respectively. Then:

(i) For all i, j, k:

$$A_{ij;k} = A_{kj;i} + R^M_{ki\nu j},$$

where ν represents the direction normal to Σ; and

(ii) For all i, j, k, l:

$$A_{ij;kl} = A_{ij;lk} + R^{\Sigma}_{kli}{}^{p}A_{pj} + R^{\Sigma}_{klj}{}^{p}A_{pi}.$$

Corollary 6.4

For all i, j, k and l:

$$A_{ij;kl} = A_{kl;ij} + R^M_{kj\nu i;l} + R^M_{li\nu k;j} + R^{\Sigma}_{jlk}{}^{p}A_{pi} + R^{\Sigma}_{jli}{}^{p}A_{pk}.$$

Proof:

$$\begin{aligned} A_{ji;kl} &= A_{ki;jl} + R^M_{kj\nu i;l} \\ &= A_{ik;lj} + R^M_{kj\nu i;l} + R^{\Sigma}_{jlk}{}^{p}A_{pi} + R^{\Sigma}_{jli}{}^{p}A_{pk} \\ &= A_{lk;ij} + R^M_{kj\nu i;l} + R^M_{li\nu k;j} + R^{\Sigma}_{jlk}{}^{p}A_{pi} + R^{\Sigma}_{jli}{}^{p}A_{pk} \end{aligned}$$

The result follows. $\square$

Choose $P \in \Sigma$. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A at P. Choose an orthonormal basis, $(e_1, ..., e_n)$ of $T_P\Sigma$ with respect to which A is diagonal such that $a := \lambda_1 = A_{11}$ is the highest eigenvalue of A at P. We extend this to a frame in a neighbourhood of P by parallel transport along geodesics. We likewise extend a to a function defined in a neighbourhood of P by:

$$a = A(e_1, e_1).$$

Viewing λ_1 also as a function defined near P, $\lambda_1 \geqslant a$ and $\lambda_1 = a$ at P.

Proposition 6.5

For all i, at P:

$$a_{;ii} = A_{11;ii}.$$

Proof: Bearing in mind that $\nabla_{e_i}e_i = 0$ at P:

$$\begin{aligned} a_{;ii} &= D_{e_i}D_{e_i}a \\ &= D_{e_i}D_{e_i}A(e_1,e_1) \\ &= D_{e_i}(\nabla A)(e_1,e_1;e_i) - 2D_{e_i}A(\nabla_{e_i}e_1,e_1) \\ &= (\nabla^2 A)(e_1,e_1;e_i,e_i) - 2A(\nabla_{e_i}\nabla_{e_i}e_1,e_1). \end{aligned}$$

The result now follows since $\nabla_{e_i}\nabla_{e_i}e_1 = 0$ at P for all i. $\square$

We define the Laplacian Δ such that, for all functions f:

$$\Delta f = A^{ij}f_{;ij}.$$

Proposition 6.6

There exists $K > 0$, which only depends on M and ϕ such that, if $a > 1$, then:

$$\Delta\mathrm{Log}(a)(P) \geqslant -K(1 + \sum_{i=1}^n \frac{1}{\lambda_i}).$$

Proof: By Corollary 6.4:

$$\begin{aligned} a_{;ii} &= A_{11;ii} \\ &= A_{ii;11} + R^M_{i1\nu 1;i} + R^M_{i1\nu i;1} + R^{\Sigma}_{1ii}{}^{p}A_{p1} + R^{\Sigma}_{1i1}{}^{p}A_{pi}. \end{aligned}$$

However, at P:

$$\sum_{i=1}^n \frac{1}{\lambda_1\lambda_i}A_{ii;11} = \sum_{i,j=1}^n \frac{1}{\lambda_i\lambda_j\lambda_1}A_{ij;1}A_{ij;1} + \frac{1}{\lambda_1}\phi_{;11}.$$

Thus, at P:

$$\begin{aligned} \Delta\mathrm{Log}(a) \geqslant\ & \tfrac{1}{\lambda_1}\phi_{;11} + \textstyle\sum_{i,j=1}^n \frac{1}{\lambda_i\lambda_j\lambda_1}A_{ij;1}A_{ij;1} - \sum_{i=1}^n \frac{1}{\lambda_1^n\lambda_i}A_{11;i}A_{11;i} \\ & + \textstyle\sum_{i=1}^n \frac{1}{\lambda_1\lambda_i}(R^M_{i1\nu 1;i} + R^M_{i1\nu i;1}) \\ & + \textstyle\sum_{i,j=1}^n \frac{1}{\lambda_1\lambda_i}(R^{\Sigma}_{1ii}{}^{p}A_{p1} + R^{\Sigma}_{1i1}{}^{p}A_{pi}). \end{aligned}$$

We consider each contribution seperately. Since, for all $a,b \in \mathbb{R}$, $(a+b)^2 \leqslant 2a^2 + 2b^2$, by Lemma 6.3, for all $i \geqslant 2$:

$$A_{11;i}^2 = (A_{i1;1} + R^M_{i1\nu 1})^2 \leqslant 2A_{i1;1}^2 + 2(R^M_{i1\nu 1})^2$$

Thus, bearing in mind that $\lambda_1 \geqslant 1$, there exists K_1, which only depends on M such that:

$$\sum_{i,j=1}^n \frac{1}{\lambda_i\lambda_j\lambda_1}A_{ij;1}A_{ij;1} - \sum_{i=1}^n \frac{1}{\lambda_1^n\lambda_i}A_{11;i}A_{11;i} \geqslant -K_1\sum_{i=1}^n \frac{1}{\lambda_i}.$$

For all ξ, X and Y:

$$\begin{aligned} \nabla^\Sigma\xi(Y;X) &= \nabla^M\xi(Y;X) - A(X,Y)\xi(N);\ \text{and}\\ X\xi(\mathsf{N}) &= \nabla^M\xi(\mathsf{N};X) + \xi(AX). \end{aligned}$$

Thus:

$$\begin{aligned} R^M_{i1\nu1;i} &= (\nabla^M R^M)_{i1\nu1;i} + \lambda_i(1-\delta_{i1})R^M_{1\nu\nu1} + \lambda_i R^M_{i1i1},\\ R^M_{i1\nu i;1} &= (\nabla^M R^M)_{i1\nu i;1} - \lambda_1(1-\delta_{i1})R^M_{i\nu\nu i} + \lambda_1 R^M_{i1i1}. \end{aligned}$$

Bearing in mind that $\lambda_1 \geqslant 1$, there exists K_3, which only depends on M such that:

$$\sum_{i=1}^n \frac{1}{\lambda_1\lambda_i}(R^M_{i1\nu1;i} + R^M_{i1\nu i;1}) \geqslant -K_3(1+\sum_{i=1}^n\frac{1}{\lambda_i}).$$

Moreover:

$$R^{\Sigma\ \ p}_{1ii}A_{p1} + R^{\Sigma\ \ p}_{1i1}A_{pi} = R^M_{1ii1}(\lambda_1-\lambda_i) + \lambda_1\lambda_i(\lambda_1-\lambda_i).$$

Bearing in mind that $\lambda_1 \geqslant 1$ and that $\lambda_1 \geqslant \lambda_i$ for all i, there exists K_2, which only depends on M such that:

$$\sum_{i,j=1}^n \frac{1}{\lambda_1\lambda_i}(R^{\Sigma\ \ p}_{1ii}A_{p1} + R^{\Sigma\ \ p}_{1i1}A_{pi}) \geqslant -K_2(1+\sum_{i=1}^n\frac{1}{\lambda_i}).$$

Since $\nabla^\Sigma_{e_1}e_1 = 0$:

$$\begin{aligned} \nabla^M_{e_1}e_1 &= \nabla^\Sigma_{e_1}e_1 + \langle\nabla^M_{e_1}e_1,\mathsf{N}\rangle\mathsf{N}\\ &= -A(e_1,e_1)\mathsf{N}\\ &= -\lambda_1\mathsf{N} \end{aligned}$$

Thus:

$$\begin{aligned} \phi_{;11} &= \partial_1\partial_1\phi\\ &= \mathrm{Hess}^M(\phi)(e_1,e_1) - d\phi(\nabla^M_{e_1}e_1)\\ &= \mathrm{Hess}^M(\phi)(e_1,e_1) - \lambda_1 d\phi(\mathsf{N}) \end{aligned}$$

Bearing in mind that $\lambda_1 \geqslant 1$, there thus exists K_3, which only depends on M and ϕ such that:

$$\frac{1}{\lambda_1}\phi_{;11} \geqslant -K_3.$$

The result now follows by combining the above relations. □

We recall that a function f is said to satisfy $\Delta f \geqslant g$ in the weak sense if and only if, for all $P \in \Sigma$, there exists a smooth function φ, defined near P such that:

(i) $f \geqslant \varphi$ near P;

(ii) $f = \varphi$ at P; and

(iii) $\Delta\varphi \geqslant g$ at P.

Corollary 6.7

With the same K as in Proposition 6.6, if $\lambda_1 \geqslant 1$, then:

$$\Delta\mathrm{Log}(\lambda_1) \geqslant -K(1+\sum_{i=1}^n\frac{1}{\lambda_i}),$$

in the weak sense.

Proof: Near $P \in \Sigma$, $\lambda_1 \geqslant a$ and $\lambda_1 = a$ at P. Since $P \in \Sigma_0$ is arbitrary, and since a is smooth at P, the result follows. □

Choose $x_0 \in M$. Define δ by:

$$\delta = \frac{1}{2}d(x, x_0)^2.$$

Proposition 6.8

There exists c, which only depends on M, Ω, ϕ and x_0 such that:

$$\lambda_1 \geqslant c \ \Rightarrow\ \Delta^\Sigma\delta \geqslant \frac{1}{2}(1 + \sum_{i=1}^n \frac{1}{\lambda_i}).$$

Proof: Since M has non-positive curvature:

$$\begin{array}{lll} & \mathrm{Hess}^M(\delta) & \geqslant \mathrm{Id} \\ \Rightarrow & \mathrm{Hess}^\Sigma(\delta) & \geqslant \mathrm{Id} - d(x, x_0)\langle \mathsf{N}, \nabla d\rangle A \\ \Rightarrow & \Delta\delta & \geqslant \sum_{i=1}^n \frac{1}{\lambda_i} - nd(x, x_0). \end{array}$$

By compactness of Ω, there exists $K_1 > 0$ such that, throughout Ω:

$$e^\phi \leqslant K_1.$$

Thus:

$$\begin{array}{lll} & \lambda_1\lambda_n^{n-1} & \leqslant K_1 \\ \Rightarrow & \lambda_n & \leqslant (K_1\lambda_1^{-1})^{1/(n-1)} \\ \Rightarrow & \frac{1}{\lambda_n} & \geqslant (\lambda_1/K_1)^{1/(n-1)} \\ \Rightarrow & \sum_{i=1}^n \frac{1}{\lambda_i} & \geqslant (\lambda_1/K_1)^{1/(n-1)}. \end{array}$$

There thus exists $c_1 > 0$ such that, for $\lambda_1 \geqslant c_1$, and for $x \in \Omega$:

$$\begin{array}{lll} & \sum_{i=1}^n \frac{1}{\lambda_i} & \geqslant 2nd(x, x_0) + 1 \\ \Rightarrow & \Delta^\Sigma\delta & \geqslant \frac{1}{2}(1 + \sum_{i=1}^n \frac{1}{\lambda_i}). \end{array}$$

The result now follows. □

Corollary 6.9

There exists $\lambda > 0$ and $c > 0$, which only depend on M, Ω, ϕ and x_0 such that:

$$\lambda_1 \geqslant c \ \Rightarrow\ \Delta(\mathrm{Log}(a) + \lambda\delta) > 0,$$

in the weak sense.

Interior bounds now follow by the maximum principal:

Proof of Proposition 6.1: Consider the function $\|A\|e^{\lambda\delta} = \lambda_1 e^{\lambda\delta}$. If this function achieves its maximum along $\partial\Sigma$, then the result follows since $e^{\lambda\delta}$ is uniformly bounded above and below. Otherwise, it acheives its maximum in the interior of Σ, in which case, by Corollary 6.9 and the Maximum Principal, at this point:

$$\|A\| = \lambda_1 \leqslant c.$$

The result follows. □

7 - Compactness.

Let M^{n+1} be an $(n+1)$-dimensional Hadamard manifold. Let $(\Sigma_0, \partial\Sigma_0) \subseteq M^{n+1}$ be a smooth, strictly convex hypersurface. Let N_0 and A_0 be the unit exterior normal vector field and the shape operator of Σ_0 respectively. Using the exponential map, we identify $\Sigma\times]-\infty,0]$ with a subset of M.

Let $\mathrm{Conv} \subseteq C^\infty(\Sigma_0,]-\infty,0])$ be the family of smooth, negative valued functions over Σ_0 which vanish along $\partial\Sigma_0$ and whose graphs are strictly convex. We define the Gauss Curvature operator $K : \mathrm{Conv} \to C^\infty(\Sigma_0)$ such that, for all f, $(Kf)(x)$ is the Gauss curvature of the graph of f at the point $(x, f(x))$. The formula for K is given by Proposition 3.1.

Let $f_0, \hat{f} \in \mathrm{Conv}$ be such that:

$$\hat{f} \leqslant f_0 < 0, \qquad K\hat{f} - \epsilon > Kf_0 > 0,$$

for some $\epsilon > 0$. Denote $\phi_0 = Kf_0$ and $\hat{\phi} = K\hat{f}$. Denote by $\mathrm{Conv}(f_0, \hat{f})$ the set of all $f \in \mathrm{Conv}$ such that:

$$\hat{f} \leqslant f \leqslant f_0, \text{ and } \hat{\phi} - \epsilon \geqslant Kf \geqslant \phi_0.$$

We prove a slightly stronger version of the assertion that the restriction of K to $\mathrm{Conv}(f_0, \hat{f})$ is a proper mapping:

Lemma 7.1

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Conv}(f_0, \hat{f})$. Suppose there exists $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(M)$ such that, for all n, and for all $x \in \Sigma_0$:

$$(Kf_n)(x) = \phi_n(x, f_n(x)).$$

If there exists $\phi_0 \in C^\infty(M)$ to which $(\phi_n)_{n\in\mathbb{N}}$ converges, then there exists $f_0 \in \mathrm{Conv}(f_0, \hat{f})$ to which $(f_n)_{n\in\mathbb{N}}$ subconverges.

Corollary 7.2

The restriction of K to $\mathrm{Conv}(f_0, \hat{f})$ is a proper mapping.

Proof of Lemma 7.1: By Lemma 2.1 and Propositions 5.1 and 6.1, there exists $C_1 > 0$ in $\mathcal{B}$ such that, for all n:

$$\|f_n\|_{C^2} \leqslant C_1.$$

By Proposition 3.1:

$$Kf = F(\mathrm{Hess}(f), \nabla f, f, x),$$

where $F(M, p, t, x)$ is elliptic in the sense of [4] and is concave in M. It follows by Theorem 1 of [4] that there exists $\alpha > 0$ and $C_2 > 0$ in $\mathcal{B}$ such that, for all n:

$$\|f_n\|_{C^{2,\alpha}} \leqslant C_2.$$

Thus, by the Schauder Esimates (see [7]), for all $k \in \mathbb{N}$, there exists $B_k > 0$ such that, for all n:

$$\|f_n\|_{C^k} \leqslant B_k.$$

The result now follows by the Arzela-Ascoli Theorem. $\square$

8 - One Dimensional Families of Solutions.

Let M^{n+1} be an $(n+1)$-dimensional Hadamard manifold. Let $(\Sigma_0, \partial\Sigma_0) \subseteq M^{n+1}$ be a smooth, convex hypersurface. Let N_0 and A_0 be the unit exterior normal vector field and the shape operator of Σ_0 respectively. Using the Exponential Map, we identify $\Sigma\times]-\infty, 0]$ with a subset of M.

Let f_0,$\hat{f}$,ϕ_0 and $\hat{\phi}$ be as in the previous section. Let $\gamma : [0,1] \to C^\infty(\Sigma_0)$ be a smooth family of smooth functions such that, for all τ:

$$\phi_0 + \epsilon \leqslant \gamma(\tau) \leqslant \hat{\phi} - \epsilon,$$

for some $\epsilon > 0$. As before, let $K : \mathrm{Conv} \to C^\infty(\Sigma_0)$ be the Gauss Curvature Operator. For all $\phi \in C^\infty(\Sigma_0)$, define $\Gamma_\phi \subseteq I \times \mathrm{Conv}(f_0, \hat{f})$ by:

$$\Gamma_\phi = \left\{(t, f) \text{ s.t. } Kf = \gamma(t) + \phi\right\}.$$

Viewing Conv as a Banach manifold (strictly speaking, the intersection of an infinite nested family of Banach manifolds), we will prove:

Proposition 8.1

There exists $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma_0)$ which converges to 0 such that, for all n:

(i) $\Gamma_n := \Gamma_{\phi_n}$ is a (possibly empty) smooth, embedded 1-dimensional submanifold of $I \times \mathrm{Conv}(f_0, \hat{f})$; and

(ii) $\partial\Gamma_n$ lies inside $\{0,1\} \times \mathrm{Conv}(f_0, \hat{f})$.

Proposition 8.2

(i) For all ϕ, Γ_ϕ is compact; and

(ii) For any neighbourhood Ω of Γ_0 in $I \times \mathrm{Conv}(f_0, \hat{f})$, there exists a neighbourhood U of 0 in $C^\infty(\Sigma_0)$ such that if $\phi \in U$, then $\Gamma_\phi \subseteq \Omega$.

Proof: (i). This assertion follows from Corollary 7.2.

(ii). Suppose the contrary. Let $(\tau_n)_{n\in\mathbb{N}} \in [0,1]$, $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma_0)$ and $(f_n)_{n\in\mathbb{N}} \in \mathrm{Conv}(f_0, \hat{f})$ be such that $(\tau_n)_{n\in\mathbb{N}}$ converges to $\tau_0 \in [0,1]$, $(\phi_n)_{n\in\mathbb{N}}$ converges to 0 and, for all n:

$$(\tau_n, f_n) \notin \Omega.$$

Suppose moreover that, for all n:

$$Kf_n = \gamma(\tau_n) + \phi_n.$$

By Lemma 7.1, $(f_n)_{n\in\mathbb{N}}$ subconverges to $f_0 \in \mathrm{Conv}(f_0, \hat{f})$ such that:

$$\begin{array}{rll} & Kf_0 & = \gamma(\tau_0) \\ \Rightarrow & (\tau_0, f_0) & \in \Gamma_0. \end{array}$$

Thus, for sufficiently large n, $(\tau_n, f_n) \in \Omega$, which is absurd. The result follows. □

We denote by $C^\infty_0(\Sigma_0)$, the set of smooth functions on Σ_0 which vanish along $\partial\Sigma_0$, and we identify this with the tangent space of Conv in the natural manner. We consider the derivative of K:

Proposition 8.3

At every point of Conv, DK defines a uniformly elliptic operator from $C^\infty_0(\Sigma_0)$ to $C^\infty(\Sigma_0)$.

Proof: This follows by differentiating the formula for the Gauss Curvature Operator given by Proposition 3.1. □

DK is therefore Fredholm. Since it is defined on the space of smooth functions over a compact manifold with boundary, which themselves vanish over the boundary, it is of index zero.

Proof of Proposition 8.1: Define $\hat{K} : [0,1] \times \mathrm{Conv}(f_0, \hat{f}) \times C^\infty(\Sigma_0) \to C^\infty(\Sigma_0)$ by:

$$\hat{K}(\tau, f, \phi) = \gamma(\tau) - Kf + \phi.$$

By compactness, there exists a neighbourhood, Ω, of Γ in $[0,1]\times\mathrm{Conv}(f_0, \hat{f})$ and a subspace $E \subseteq C^\infty(\Sigma_0)$ of dimension $m < \infty$ such that the restriction of $D\hat{K}$ to $\Omega \times E$ is always surjective. This restriction is Fredholm of index $(m+1)$. Define $\hat{\Gamma}$ by:

$$\hat{\Gamma} = \hat{K}^{-1}(\{0\}).$$

By the Implicit Function Theorem for Banach manifolds, $\hat{\Gamma}$ is an $(m+1)$-dimensional smooth manifold. Let $\pi_3 : [0,1] \times \mathrm{Conv}(f_0, \hat{f}) \times E \to E$ denote projection onto the third factor. By Sard's Lemma, there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \in E$ which tends to 0 such that, for all n, ϕ_n is a regular value of the restriction of π_3 to $\hat{\Gamma}$. However, for all n:

$$\Gamma_n := \Gamma_{\phi_n} = \hat{\Gamma} \cap \pi_3^{-1}(\phi_n).$$

Moreover, since ϕ_n is a regular value of π_3, Γ_n is a (possibly empty) smooth 1-dimensional embedded manifold. By Proposition 8.2, for all n, Γ_n is compact, and for sufficiently large n, Γ_n lies entirely inside $[0,1] \times \Omega$. Therefore:

$$\partial\Gamma_n \subseteq \partial(I \times \Omega) \subseteq (\{0,1\} \times \Omega) \cup ([0,1] \times \partial\mathrm{Conv}(f_0, \hat{f})).$$

However, if $(\tau, f) \in \Gamma_n$, then:

$$f_0 \leqslant f \leqslant \hat{f}, \qquad Kf_0 + \epsilon/2 \leqslant Kf \leqslant K\hat{f} - \epsilon/2.$$

Thus, by the geometric maximum principal (away from $\partial\Sigma_0$):

$$f_0 < f < \hat{f}.$$

Likewise, a similar relation holds for the derivative of f in the internal normal direction along $\partial\Sigma_0$. It follows that Γ_n lies in the interior of $\mathrm{Conv}(f_0, \hat{f})$ and so:

$$\partial\Gamma_n \subseteq \{0,1\} \times \Omega.$$

The result follows. □

9 - Local and Global Rigidity.

Let M^{n+1} be an $(n+1)$-dimensional Hadamard manifold. Let $(\Sigma_0, \partial\Sigma_0) \subseteq M^{n+1}$ be a smooth, convex hypersurface. Choose $f_0 < \hat{f} \in \text{Conv}$.

Definition 9.1

(i) We say that $\phi \in C^\infty(\Sigma_0)$ is locally rigid over $\text{Conv}(f_0, \hat{f})$ if and only if for all $f \in \text{Conv}(f_0, \hat{f})$ such that $Kf = \phi$, DK is invertible at ϕ (in other words, ϕ is a regular value of K).

(ii) We say that $\phi \in C^\infty(\Sigma_0)$ is globally rigid over $\text{Conv}(f_0, \hat{f})$ if and only if there exists at most one $f \in \text{Conv}(f_0, \hat{f})$ such that $Kf = \phi$.

Example: Let $\mathbb{H}^{n+1}$ be $(n+1)$-dimensional hyperbolic space. Let H be a totally geodesic hypersurface. For $D > 0$, let $H(D)$ be the equidistant hypersurface at a distance D from H. $H(D)$ has constant Gaussian curvature equal to $\tanh(D)$. Let $\Omega \subseteq H(D)$ be any bounded open subset with smooth boundary and consider the hypersurface $(\Sigma_0, \partial\Sigma_0) = (\Omega, \partial\Omega)$. Define $f_0 = 0$ and $\phi_0 = Kf_0 = \tanh(D)$. By the strong Geometric Maximum Principal and the homogeneity of $\mathbb{H}^{n+1}$, ϕ_0 is both globally and locally rigid. $\square$

Example: The above example is a special case of a more general construction. Let M be a Riemannian manifold. Let $P \in M$ be a point, let $\mathsf{N} \in UM$ be a unit vector at P, let A be a positive-definite symmetric 2-form over $\mathsf{N}^\perp$ and let $k = \text{Det}(A)$. There is no algebraic obstruction to the construction of a hypersurface Σ such that:

(i) $P \in \Sigma$;

(ii) N is normal to Σ at P;

(iii) the second fundamental form of Σ at P is equal to A; and

(iv) if $\psi = \text{Det}(A)$ is the Gaussian curvature of Σ, then $\psi = k$ up to infinite order at P.

Since $\psi = k$ up to infinite order at P, for $\epsilon > 0$ small, there exists a smooth family $(\psi_t)_{t<\epsilon}$ of smooth functions such that:

(i) $\psi_0 = \psi$; and

(ii) for all t, $\psi_t = k$ over the geodesic ball of radius t about P.

Suppose moreover that M has negative sectional curvature bounded above by -1 and that $A = k\text{Id}$ for $k < 1$. In this case, the derivative of the Gauss Curvature Operator is invertible over a geodesic ball of small radius about P (see [15] for details in the 2-dimensional case). We may therefore assume by the Inverse Function Theorem for Banach Manifolds that $\psi = k$ over a geodesic ball of small radius about P. Moreover, Σ may be extended to a foliation, $(\Sigma_t)_{t\in]-\epsilon,\epsilon[}$, of a neighbourhood of P in M by hypersurfaces of constant curvature equal to k. Now let $B \subseteq M$ be a geodesic ball in M centred on P which is covered by this foliation. Let $\Omega \subseteq \Sigma$ be an open set with smooth boundary contained in $B \cap \Sigma$. If Σ' is any other hypersurface of constant Gaussian curvature equal to k such that $\partial\Sigma' = \partial\Omega$, then, by the Geometric Maximum Principal, Σ' is contained inside B, and, by the strong Geometric Maximum Principal, Σ' coincides with a leaf of the foliation. It is therefore equal to Ω, and we have thus shown that $\psi = k$ is both locally and globally rigid over Ω. $\square$

Proposition 9.2

(i) If ϕ_0 is locally rigid, then ϕ' is also locally rigid for all ϕ' sufficiently close to ϕ_0.

(ii) If ϕ_0 is locally and globally rigid, then ϕ' is globally rigid for all ϕ' sufficiently close to ϕ_0.

Proof: (i). Suppose the contrary. Let $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma_0)$ be a sequence of non-locally rigid functions converging to ϕ_0. Since ϕ_0 is locally rigid, DK is invertible at f for all $f \in K^{-1}(\{\phi\})$. There therefore exists a neighbourhood Ω of $K^{-1}(\{\phi\})$ in $\mathrm{Conv}(f_0, \hat{f})$ such that DK is invertible at f for all $f \in \Omega$. However, by Corollary 7.2, for all sufficiently large n:

$$K^{-1}(\{\phi_n\}) \subseteq \Omega.$$

ϕ_n is therefore locally rigid for sufficiently large n, which is absurd, and the assertion follows.

(ii). Suppose the contrary. There exists a sequence $(\phi'_n)_{n\in\mathbb{N}}$ which converges to ϕ such that ϕ'_n is not globally rigid. Thus, for all n, there exists $f_{1,n} \neq f_{2,n} \in \mathrm{Conv}(f_0, \hat{f})$ such that:

$$Kf_{1,n} = Kf_{2,n} = \phi'_n.$$

By Corollary 7.2, there exist $f_{1,0}, f_{2,0} \in \mathrm{Conv}(f_0, \hat{f})$ to which $(f_{1,n})_{n\in\mathbb{N}}$ and $(f_{2,n})_{n\in\mathbb{N}}$ respectively converge. In particular:

$$Kf_{1,0} = Kf_{2,0} = \phi_0.$$

Since ϕ_0 is globally rigid, it follows that:

$$f_{1,0} = f_{2,0} = f_0.$$

Since ϕ_0 is locally rigid at f_0, DK is invertible at f_0 and thus K is locally invertible over a neighbourhood of f_0. In particular, for sufficiently large n:

$$f_{1,n} = f_{2,n}.$$

This is absurd, and the result follows. $\square$

10 - Existence.

Let M^{n+1} be an $(n+1)$-dimensional Hadamard manifold. Let $(\Sigma_0, \partial\Sigma_0) \subseteq M^{n+1}$ be a smooth, convex hypersurface. Let N_0 and A_0 be the unit exterior normal vector field and the shape operator of Σ_0 respectively. Using the Exponential Map, we identify $\Sigma\times]-\infty, 0]$ with a subset of M. Let f_0,$\hat{f}$,ϕ_0 and $\hat{\phi}$ be as in section 7.

Lemma 10.1

If Σ_0 is locally and globally rigid, then, for all ϕ such that:

$$\phi_0 \leqslant \phi \leqslant \hat{\phi} - \epsilon,$$

for some $\epsilon > 0$, there exists a smooth, convex, immersed hypersurface Σ_ϕ such that:

(i) $\hat{\Sigma} < \Sigma_\phi \leqslant \Sigma_0$, and

(ii) the Gaussian curvature of Σ_ϕ at the point p is equal to $\phi(p)$.

Proof: Assume first that:

$$\phi_0 + \epsilon \leqslant \phi \leqslant \hat{\phi} - \epsilon.$$

By local rigidity, reducing ϵ is necessary, there exists $f_0' \in \mathrm{Conv}(f_0, \hat{f})$ such that:

$$f_0' < 0; \qquad \phi_0' := K f_0' \geqslant \phi_0 + \epsilon.$$

By Proposition 9.2, we may assume moroever that ϕ_0' is both locally and globally rigid. Let $\gamma : [0,1] \rightarrow C^\infty(\Sigma_0)$ be a smooth family of smooth functions such that:

(i) $\gamma(0) = \phi_0'$, $\gamma(1) = \phi$, and

(ii) for all $t \in [0,1]$:

$$\phi_0 + \epsilon \leqslant \gamma(t) \leqslant \hat{\phi} - \epsilon.$$

By Proposition 8.1, there exists $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma_0)$ which converges to 0 such that, for all n, $\Gamma_n := \Gamma_{\phi_n}$ is a (possibly empty) smooth, 1-dimensional embedded submanifold of $[0,1] \times \mathrm{Conv}(f_0, \hat{f})$. Moreover, for all n, Γ_n is compact, and:

$$\partial\Gamma_n \subseteq \{0,1\} \times \mathrm{Conv}(f_0, \hat{f}).$$

By Proposition 9.2, we may assume that, for all n, $\phi_0' + \phi_n$ is both locally and globally rigid. Likewise, since ϕ_0' is locally rigid, we may assume that, for all n, there exists $f_n \in \mathrm{Conv}(f_0, \hat{f})$ such that:

$$(0, f_n) \in \Gamma_n.$$

Γ_n is therefore non-empty for all n. Since it is compact, it is either an embedded, compact interval or an embedded closed loop. We claim that Γ_n is not a closed loop. Indeed, by local rigidity, DK is invertible at $(0, f_n)$. Consequently, if $\pi_1 : [0,1] \times \mathrm{Conv}(f_0, \hat{f}) \rightarrow [0,1]$ is the projection onto the first factor, the restriction of $D\pi_1$ to $T\Gamma_n$ is invertible at f_n.

Since $0 = \pi_1(f_n)$ is an end point of $[0, 1]$, we deduce that Γ_n is not a closed loop and the assertion follows.

For all n, let g_n by the other end of Γ_n. Since $(\phi_0' + \phi_n)$ is globally rigid:

$$g_n \in \{1\} \times \mathrm{Conv}(f_0, \hat{f}).$$

In other words:

$$Kg_n = \phi + \phi_n.$$

By Corollary 7.2, there exists $g_0 \in \mathrm{Conv}(f_0, \hat{f})$ to which $(g_n)_{n\in\mathbb{N}}$ subconverges. In particular:

$$Kg_0 = \phi.$$

This proves existence in the case where $\phi_0 + \epsilon \leqslant \phi \leqslant \hat{\phi} - \epsilon$. The general case follows by taking limits. □

11 - Affine Flat Manifolds.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Let g be the metric on M and let ∇ be the Levi-Civita covariant derivative of g. Let g' be another metric on M. We say that g and g' are affine equivalent if and only if their geodesics coincide up to reparametrisation. In particular, we say that g is affine flat if and only if g is everywhere locally affine equivalent to a flat metric (see [16] and [17]).

We will prove:

Proposition 11.1

Suppose that M is affine flat. Choose $k > 0$, and let $(K_n)_{n\in\mathbb{N}} \subseteq M$ be a sequence of convex subsets of M with smooth boundary such that, for all n, the Gaussian curvature of ∂K_n is equal to k. Suppose that $(K_n)_{n\in\mathbb{N}}$ converges to $K_0 \subseteq M$ and that K_0 has non-empty interior. Then the set of points on ∂K_0 where K_0 is not strictly convex is closed

Example: Consider a convex polygon, P, in $\mathbb{R}^n$. The set of boundary points where P is not strictly convex is the complement of the set of vertices of P, which is not closed. □

Remark: If K_0 is not strictly convex at $P \in \partial K_0$, then there exists a geodesic segment passing through P which is contained in ∂K_0.

Trivially, if M is affine flat, then, for every $P \in M$ and for every subspace $E \subseteq T_P M$, there exists a (unique) totally geodesic submanifold whose tangent space at P is E. We recall the following characterisation of affine flat metrics:

Lemma 11.2

Let ∇' be the Levi-Civita covariant derivative of g' and let $\Omega' = \nabla' - \nabla$ be the connection 2-form of ∇' with respect to ∇. g' and g are affine equivalent if and only if there exists a 1-form, α such that, for all $X, Y \in TM$:

$$\Omega(X, Y) = \alpha(X)Y + \alpha(Y)X.$$

Proof: ($\Rightarrow$). Choose $P \in M$ and $X \in T_PM$. Let $\gamma :]-\epsilon,\epsilon[\rightarrow M$ be a ∇-geodesic in M passing through P such that $\partial_t\gamma(0) = X$. We extend X to a vector field defined near P such that, for all t:

$$X(\gamma(t)) = \partial_t\gamma.$$

Since, up to reparametrisation, γ is a ∇'-geodesic, there exists $\lambda \in \mathbb{R}$ such that:

$$\Omega(X,X) = \lambda X.$$

Since Ω is bilinear, λ depends linearly on X, and the result follows by polarisation.

($\Leftarrow$). Let $\gamma :]-\epsilon,\epsilon[\rightarrow M$ be a ∇-geodesic. Then:

$$\nabla'_{\partial_t\gamma}\partial_t\gamma = \nabla_{\partial_t\gamma}\partial_t\gamma + \Omega(\partial_t\gamma,\partial_t\gamma) = 2\alpha(\partial_t\gamma)\partial_t\gamma.$$

It follows that, up to reparametrisation, γ is also a ∇'-geodesic, and g' and g are therefore affine equivalent. This completes the proof. □

When M is a locally affine manifold, we recover a simple form for the Gauss Curvature Equation:

Proposition 11.3

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set containing the origin. Let g be a metric over U which is affine equialent to the Euclidean metric. Let $B_\epsilon(0)$ be the ball of radius ϵ in $\mathbb{R}^n$ and let $f : B_\epsilon(0) \rightarrow]-\epsilon,\epsilon[$ be a smooth function. There exists a smooth function ψ which depends only on g (and n) such that, if Kf the Gaussian curvature with respect to g of the graph of f, then:

$$Kf = \psi(x,f,\nabla f)^{-1}\mathrm{Det}(\mathrm{Hess}(f))^{1/n}.$$

Proof: Let Ω be the connection 2-form of g with respect to the Euclidean metric. By Lemma 11.2, using the notation of Proposition 3.1, for all i and j:

$$d\hat{f}(\Omega(\hat{\partial}_i,\hat{\partial}_j)) = 0.$$

The result follows. □

Proof of Proposition 11.1: We identify M with a subset of $\mathbb{R}^{n+1}$ furnished with an affine flat metric, g_0. Choose $P \in \partial K_0$ and let H be a supporting totally geodesic hyperspace to K_0 at P. We identify H with $\mathbb{R}^n$. Since K_0 has non-empty interior, there exists a neighbourhood $U \subset \mathbb{R}^n$ of P over which K_0 is the graph of some convex function f_0, say. Moreover, there exists a sequence of affine flat smooth metrics $(g_n)_{n\in\mathbb{N}}$ and a sequence of convex functions $(f_n)_{n\in\mathbb{N}}$ such that:

(i) $(g_n)_{n\in\mathbb{N}}$ converges to g_0 in the C^∞ sense;

(ii) $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in the $C^{0,1}$ sense and converges to f in the $C^{0,\alpha}$ sense for all α; and

(iii) for all n, the Gaussian curvature with respect to g_n of the graph of f_n is equal to k.

Thus, by Proposition 11.3, there exists $0 < \lambda < \Lambda$ such that, for all n:

$$\lambda \leqslant \mathrm{Det}(\mathrm{Hess}(f_n)) \leqslant \Lambda.$$

Taking limits yields:

$$\lambda \leqslant \mathrm{Det}(\mathrm{Hess}(f_0)) \leqslant \Lambda,$$

in the Alexandrov sense (see, for example, [13]), and the result now follows by [1]. □

In particular, using the Kleinian representation of $\mathbb{H}^{n+1}$, we see that $\mathbb{H}^{n+1}$ is locally affine.

Lemma 11.4

Let $\mathbb{H}^{n+1}$ be $(n+1)$-dimensional hyperbolic space, and let $H \subseteq \mathbb{H}^{n+1}$ be a totally geodesic hypersurface. Choose $k > 0$, and let $\Omega \subseteq H$ be a bounded open set such that there exists a convex hypersurface, $\hat{\Sigma}$ such that:

(i) $\hat{\Sigma}$ is a graph below Ω;

(ii) the second fundamental form of $\hat{\Sigma}$ is at least ϵ in the Alexandrov sense, for some $\epsilon > 0$; and

(iii) the Gaussian curvature of Σ is at least k in the Alexandrov sense.

There exists a unique convex, immersed hypersurface $(\Sigma, \partial\Sigma)$ such that:

(i) Σ is a graph below Ω and $\partial\Sigma = \partial\Omega$;

(ii) Σ lies above $\hat{\Sigma}$;

(iii) Σ has C^∞ interior and is $C^{0,1}$ up to the boundary; and

(iv) the Gaussian curvature of Σ is equal to k.

Moreover if $\partial\Omega$ is smooth, then Σ is smooth up to the boundary.

Proof: We begin by smoothing the upper barrier. Choose $k' < k$. As in Lemma 2.13 of [20], there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}} \in]0, k-k'[$ of positive numbers and a sequence of smooth, convex immersed hypersurfaces $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ such that:

(i) $(\epsilon_n)_{n\in\mathbb{N}}$ converges to 0 and $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ converges to $\hat{\Sigma}$ in the $C^{0,\alpha}$ sense for all α;

(ii) for all n, $\hat{\Sigma}_n$ is a graph over a bounded open subset of H; and

(iii) for all n, the Gaussian curvature of $\hat{\Sigma}_n$ is greater than $k - \epsilon_n$.

Let $(\delta_n)_{n\in\mathbb{N}} > 0$ be a sequence of positive numbers converging to 0. For all n, let H_n be the equidistant hypersurface at distance δ_n from H. We may assume that, for all n, a portion of $\hat{\Sigma}_n$ is a graph over H_n. Let Ω_n be the subset of H_n over which it as a graph.

For all n, since $(\Omega_n, \partial\Omega_n)$ is locally and globally rigid, it follows by Theorem 1.1 that there exists a smooth, convex hypersurface, Σ_n, which is a graph below Ω_n such that $\Sigma_n > \hat{\Sigma}_n$ and whose Gaussian curvature is equal to k'.

Suppose now that $\partial\Omega$ is smooth. There exists $\epsilon > 0$ such that, for all n and for all $P \in \partial\Omega_n$, there exists a geodesic ball $B \subseteq \Omega_n$ such that $P \in \partial B$. For all such B, we consider the foliation of constant Gaussian curvature hypersurfaces which are graphs below B and whose boundary is ∂B (in the upper half space model of $\mathbb{H}^{n+1}$, these are merely intersections of spheres in $\mathbb{R}^{n+1}$ with $\mathbb{H}^{n+1}$). Using these foliations and the Geometric Maximum Principal, we find that there exists $\theta > 0$ such that, for all n, $T\Sigma_n$ makes an angle of at least θ with H_n along $\partial\Sigma_n$. Bearing in mind the remark following Proposition 5.1, this yields uniform lower bounds for the restriction to $\partial\Omega_n$ of the the second fundamental form of Σ_n. Taking limits now yields the desired hypersurface, Σ.

Consider now the general case. By Lemma 2.1, we may nonetheless assume that $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ converges to a $C^{0,1}$, convex hypersurface Σ which is a graph below Ω such that $\Sigma \geqslant \hat{\Sigma}$. Let B be a geodesic ball such that $\overline{B} \subseteq \Omega$. Using the geometric maximum principal, by considering the foliation of constant Gaussian curvature hypersurfaces which are graphs below B and whose boundary is B, we may show that Σ lies strictly below Ω over its interior. We now assert that Σ is everywhere strictly convex. Indeed, suppose the contrary. By Proposition 11.1 the set of points where $\partial\Sigma$ is not strictly convex is closed. Thus, if $P \in \partial\Sigma$ is a point where $\partial\Sigma$ is not strictly convex, and if H' is a supporting totally geodesic hyperplane of $\partial\Sigma$ at P, then P lies in the convex hull of the intersection of H' with $\partial\Omega$ (see, for example, [21]). In particular, P lies in H and thus, by convexity, $\Sigma = \Omega$, which is absurd. The assertion follows and it now follows by [2] that Σ is smooth over its interior, and this proves existence (see also Appendix A).

Let Σ be a graph over Ω of constant Gaussian curvature equal to k. Let f be the graph function of Σ in conformal coordinates about H (see the example following Proposition 3.1). f satisfies the following equation:

$$\mathrm{Det}(f_{;ij} - \tan(f)(f_{;j}f_{;j} + \delta_{ij}))^{1/n} = k\frac{1}{\cos(f)^3}(1 + \|\nabla f\|^2)^{(n+2)/2n}.$$

Let Σ' be another such hypersurface and suppose that $\Sigma' \neq \Sigma$. Let f' be the graph function of Σ' in conformal coordinates about H. Without loss of generality, there exists $P \in H$ such that $f'(P) > f(P)$ and $f' - f$ is maximised at P. Define the field of matrices, A, by:

$$A = (\mathrm{Hess}(f) - \tan(f)(\nabla f \otimes \nabla f + \mathrm{Id}))^{-1}.$$

(This matrix is invertible by convexity of Σ). A is positive definite. Thus, near P, by concavity of $\mathrm{Det}^{1/n}$, and since $f' > f$, for some $\epsilon, \hat{k} > 0$ that we need not calculate:

$$\begin{aligned}&\hat{k}\mathrm{Tr}(A^{-1}(f'_{;ij} - f_{;ij})) - \hat{k}\tan(f)\mathrm{Tr}(A^{-1}(f'_{;i}f'_{;j} - f_{;i}f_{;j}))\\ &\qquad \geqslant \epsilon + \tfrac{k}{\cos(f)^3}((1 + \|\nabla f'\|^2)^{(n+2)/2} - (1 + \|\nabla f\|^2)^{(n+2)/2}).\end{aligned}$$

At P, since $(f' - f)$ is maximised, $\nabla f' = \nabla f$. Thus, near P;

$$\mathrm{Tr}(A^{-1}(f'_{;ij} - f_{;ij})) > 0$$

This yields a contradiction by the Maximum Principal. Uniqueness follows and this completes the proof. $\square$

12 - Relations to Existing Results.

With small modifications, these techniques may be adapted to yield existing results. First, considering $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ in the natural manner, we recover the following theorem of Guan (see [8]), which is the analogue in Euclidean space of Lemma 11.4:

Theorem 12.1, [Guan, 1998]

Choose $k > 0$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Suppose that there exists a convex hypersurface, $\hat{\Sigma}$ such that:

(i) $\hat{\Sigma}$ is a graph below Ω;

(ii) the second fundamental form of $\hat{\Sigma}$ is at least ϵ in the Alexandrov sense, for some $\epsilon > 0$; and

(iii) the Gaussian curvature of Σ is at least k in the Alexandrov sense.

There exists a unique convex, immersed hypersurface $(\Sigma, \partial\Sigma)$ such that:

(i) Σ is a graph below Ω and $\partial\Sigma = \partial\Omega$;

(ii) Σ lies above $\hat{\Sigma}$;

(iii) Σ has C^∞ interior and is $C^{0,1}$ up to the boundary; and

(iv) the Gaussian curvature of Σ is equal to k.

Moreover, if $\partial\Omega$ is smooth, then Σ is smooth up to the boundary.

Remark: Although, as in Lemma 11.4, if we identify $(\Sigma_0, \partial\Sigma_0) = (\Omega, \partial\Omega)$, then the Gauss Curvature Equation is not elliptic at $f_0 = 0$, this, in itself, does not present a serious difficulty since there exist functions arbitrarily close to f_0 where the Gauss Curvature Equation is elliptic. The particular difficulty in Euclidean space lies in obtaining functions near f_0 for which the Gauss Curvature Equation is also locally rigid. We circumvent this by approximating $\mathbb{R}^n$ by spaces of constant negative sectional curvature.

Proof: Using polar coordinates for $\mathbb{R}^n$, we identify $\mathbb{R}^{n+1}$ with $\Sigma^{n-1}\times]0,\infty[\times\mathbb{R}$, where Σ^{n+1} is the unit sphere. We thus denote a point in $\mathbb{R}^{n+1}$ by the coordinates $(\theta, r, t) \in \Sigma^{n-1}\times]0,\infty[\times\mathbb{R}$. Let g^Σ denote the standard metric over Σ^{n-1}. For $\epsilon > 0$, we define the metric g_ϵ over $\mathbb{R}^{n+1}$ such that, at (θ, r, t):

$$g = \cosh^2(\epsilon t)(\sinh^2(\epsilon r)g^\Sigma \oplus dr^2) \oplus dt^2.$$

This metric is smooth and has constant curvature equal to $-\epsilon$. Indeed, this formula is obtained by using polar coordinates for $\mathbb{H}^n$ about a point and subsequently by identifying $\mathbb{H}^{n+1}$ with $\mathbb{H}^n\times\mathbb{R}$ using the foliation by geodesics normal to a totally geodesic hypersurface.

With respect to this metric, $\mathbb{R}^n$ is identified with a totally geodesic hypersurface, and, for all $k' < k$, there exists $\epsilon > 0$ such that $\hat{\Sigma}$ satisfies the hypotheses of Lemma 11.4, with k' instead of k. There therefore exists $\Sigma_\epsilon \subseteq \mathbb{R}^{n+1}$ possessing the desired properties and of constant Gaussian curvature equal to k' with respect to the metric g_ϵ. Existence follows by taking limits as in Lemma 11.4.

To prove uniqueness, let Σ_1 and Σ_2 be two solutions. Suppose that $\Sigma_1 \neq \Sigma_2$. Without loss of generality, there is a point of Σ_1 lying below Σ_2. There therefore exists a translate Σ_1' of Σ_1 in the vertical direction which lies strictly above Σ_1 and which is an exterior tangent to Σ_2 at some point P'. Since $\partial\Sigma_1'$ lies strictly above $\partial\Sigma_2$, P' is an interior point of Σ_1' and Σ_2. It follows by the strong Geometric Maximum Principal that Σ_1' and Σ_2 coincide, which is absurd. Uniqueness follows and this completes the proof. $\square$

If M is a Riemannian manifold, we say that a bounded open subset $\Omega \subseteq M$ satisfies a uniform exterior ball condition if and only if there exists $\epsilon > 0$ such that for every $P \in \partial\Omega$, there exists an open geodesic ball $B \subseteq \Omega^c$ of radius ϵ such that:

$$P \in \partial B \cap \partial\Omega.$$

By compactness, Ω satisfies a uniform exterior ball condition for a given metric over M if and only if it satisfies this condition for any metric over M, and we thus extend this condition to subsets of arbitrary C^∞ manifolds.

Example: Any compact, open subset with smooth boundary satisfies a uniform exterior ball condition. $\square$

Example: Any convex, open subset satisfies a uniform exterior ball condition. $\square$

We now recover the following theorem of Rosenberg and Spruck (see [19]), which has also recently been proven in a more general setting by Guan, Spruck and Szapiel (see [11]):

Theorem 12.2, [Rosenberg, Spruck, (1994)]

Let $\Omega \subseteq \partial_\infty\mathbb{H}^{n+1}$ be a non-trivial open subset whose boundary satisfies the uniform exterior ball condition. Then, for all $k \in]0,1[$, there exists a convex, immersed hypersurface $\Sigma_k \subseteq \mathbb{H}^{n+1}$ such that:

(i) identifying $\partial_\infty\mathbb{H}^{n+1}$ with $\mathbb{R}^n \cup \{\infty\}$ and viewing Ω as a subset of $\mathbb{R}^n$, Σ_k is a graph over Ω;

(ii) Σ_k is smooth and $C^{0,1}$ up to the boundary;

(iii) $\partial\Sigma_k = \partial\Omega$; and

(iv) Σ_k has constant Gaussian curvature equal to k.

Moreover, if Ω is star-shaped, then Σ_k is unique.

Proof: We identify $\mathbb{H}^{n+1}$ with the upper half space $\mathbb{R}^n\times]0,\infty[$ in the standard manner. We thus identify $\partial_\infty\mathbb{H}^{n+1}$ with $\mathbb{R}^n \cup \{\infty\}$ and view Ω as a subset of $\mathbb{R}^n$. For $\epsilon > 0$, let $H_\epsilon = \mathbb{R}^n \times \{\epsilon\}$ be the horosphere at height ϵ above $\mathbb{R}^n$. We define $\Omega_\epsilon \subseteq H_\epsilon$ by

$$\Omega_\epsilon = \{(x,\epsilon) \text{ s.t. } x \in \Omega\}.$$

By the uniform exterior ball condition, for ϵ sufficiently small, $\partial\Omega_\epsilon$ is uniformly strictly convex as a subset of $\mathbb{H}^{n+1}$ with respect to the outward pointing unit normal in H_ϵ.

Let K_ϵ be the complement of Ω_ϵ in H_ϵ. Let $\hat{K}_\epsilon$ be the convex hull of K_ϵ in $\mathbb{H}^{n+1}$. We denote by $\Sigma_{0,\epsilon}$ the portion of $\partial\hat{K}_\epsilon$ lying above H_ϵ. In other words:

$$\partial\hat{K}_\epsilon = (\partial\hat{K}_\epsilon \cap H_\epsilon) \cup \Sigma_{0,\epsilon}.$$

Since it is locally ruled, $\Sigma_{0,\epsilon}$ serves as a lower barrier for the problem (see [21]). We define $(\hat{\Sigma}_\epsilon, \partial\hat{\Sigma}_\epsilon) = (\Omega, \partial\Omega)$. The only difference between our current framework and that of Theorem 1.1 is that it is the upper barrier, $\hat{\Sigma}_\epsilon$ that is smooth and the lower barrier, $\Sigma_{0,\epsilon}$ that is not. The only change required to adapt the proof to our framework is therefore to replace $(f-f_0)$ in Corollary 5.5 with $(f-\hat{f})$. The uniform strict convexity of Ω_ϵ as a subset of $\mathbb{H}^{n+1}$ with respect to the normal in H_ϵ ensures uniform lower bounds of the restriction to $\partial\Omega$ of the second fundamental form of any surface Σ which is a graph above Ω such that $\partial\Sigma = \partial\Omega$. Thus proceeding as in Theorem 1.1, we show that there exists a graph Σ_ϵ over Ω_ϵ which is smooth up to the boundary and has constant Gaussian curvature equal to k.

Taking limits yields a $C^{0,1}$ graph, Σ, over Ω such that $\partial\Sigma = \partial\Omega$. By Proposition 11.1, the set where Σ is not strictly convex is closed. It is therefore the convex hull of the intersection of some totally geodesic hyperplane H with $\partial\Omega$ (see [20] for details). In particular, if Σ is not strictly convex at some point, then, viewed as a graph, it is vertical at some point on the boundary. However, consider a point $P \in \partial\Omega_\epsilon$ and geodesic ball $B \subseteq H_\epsilon$ such that $B \subseteq \Omega^c$ and $P \in \partial B$. Using the foliation of constant Gaussian curvature hypersurfaces in $\mathbb{H}^n$ whose boundary coincides with ∂B, we deduce by the Geometric Maximum Principal that there exists $\theta > 0$ such that, for ϵ sufficiently small, Σ_ϵ makes an angle at P of at least θ with the foliation of vertical geodesics along $\partial\Omega$. Moreover, θ may be chosen independant of P. Taking limits, it follows that Σ is everywhere strictly convex and is therefore smooth over the interior by [2]. This proves existence.

Suppose now that Ω is star-shaped, and let Σ_1 and Σ_2 be two solutions. Suppose that $\Sigma_1 \neq \Sigma_2$. Without loss of generality, there exists a point $P \in \Sigma_1$ lying below Σ_2. As before, we identify $\mathbb{H}^{n+1}$ with $\mathbb{R}^n \times]0,\infty[$. Without loss of generality, we may suppose that Ω is star-shaped about $(0,0)$. Consider the family $(M_\lambda)_{\lambda>1}$ of isometries of $\mathbb{H}^{n+1}$ given by:

$$M_\lambda(x,t) = (\lambda x, \lambda t).$$

There exists $\lambda > 1$ such that $M_\lambda\Sigma_1$ is an exterior tangent to Σ_2 at some point P'. Since $M_\lambda\partial\Sigma_1 \cap \partial\Sigma_2 = \emptyset$, P' is an interior point of Σ_1 and Σ_2. It follows by the strong Geometric Maximum Principal that $M_\lambda\Sigma_1 = \Sigma_2$, which is absurd. Uniqueness follows and this completes the proof. $\square$

A - Regularity of Limit Hypersurfaces.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Choose $k > 0$ let $(\Sigma_m)_{m\in\mathbb{N}}$ be a sequence of smooth, convex, immersed hypersurfaces in M of constant Gaussian curvature equal to k. Suppose that there exists a $C^{0,1}$ locally convex, immersed hypersurface, Σ_0 to which $(\Sigma_m)_{m\in\mathbb{N}}$ converges in the $C^{0,\alpha}$ sense for all α. For all $m \in \mathbb{N}$, let N_m and A_m be the unit normal vector field and the second fundamental form respectively of Σ_m. Choose

$p_0 \in \Sigma_0$ and let $(p_m)_{m\in\mathbb{N}} \in (\Sigma_m)_{m\in\mathbb{N}}$ be a sequence converging to p_0. For all $r > 0$, and for all $m \in \mathbb{N} \cup \{0\}$, let $B_{m,r}$ be the ball of radius r (with respect to the intrinsic metric) about p_m in Σ_m.

We will say that Σ_0 is functionally strictly convex at p_0 if there exists a smooth function, φ, defined on M near p_0 such that:

(i) φ is strictly convex;

(ii) $\varphi(p_0) > 0$; and

(iii) the connected component of $\varphi^{-1}([0,\infty[) \cap \Sigma_0$ containing p_0 is compact.

Observe that if M is affine flat (in particular, if $M = \mathbb{H}^{n+1}$), then Σ_0 is functionally strictly convex whenever it is strictly convex. We will prove:

Lemma A.1

If Σ_0 is funtionally strictly convex at p_0, then there exists $r > 0$ such that $(B_{m,r}, p_m)_{m\in\mathbb{N}}$ converges to $(B_{0,r}, p_0)$ in the pointed C^∞-Cheeger Gromov sense. In particular, $B_{0,r}$ is a smooth, convex immersion of constant Gaussian curvature equal to k.

As in section 5, we denote by $\mathcal{B}$ the family of constants which depend continuously on the data: M, k, (Σ_0, p_0) and the C^1 jets of $(\Sigma_m, p_m)_{m\in\mathbb{N}}$. In this section, for any positive quantity, X, we denote by $O(X)$ any term which is bounded in magnitude by $K\,|X|$ for some K in $\mathcal{B}$.

The following elementary lemma will be of use in the proof:

Lemma A.2

For $\lambda > 0$ and for all $a, b \in \mathbb{R}$:

$$(a+b)^2 \leqslant (1+\lambda)a^2 + (1+\lambda^{-1})b^2.$$

Proof of Lemma A.1: Since $(\Sigma_m)_{n\in\mathbb{N}}$ converges to Σ_0 and since Σ_0 is functionally strictly convex at p_0, there exists $\epsilon, h > 0$, open sets $\Omega_0, (\Omega_m)_{m\in\mathbb{N}} \subseteq M$ and, for every m, a smooth function $\varphi_m : \Omega_m \to [0,\infty[$ such that:

(i) for all m, Ω_m is a neighbourhood of p_m and $(\Omega_m)_{m\in\mathbb{N}}$ converges to Ω_0 in the Hausdorff sense;

(ii) $(\varphi_m)_{m\in\mathbb{N}}$ converges to φ_0 in the C^∞ sense;

(iii) for all m, $\mathrm{Hess}(\varphi_m) \geqslant \epsilon\mathrm{Id}$;

(iii) for all m, $\varphi_m(p_0) = 2h$; and

(iv) for all m, the connected component of p_m in $\Sigma_m \cap \Omega_m$ is compact with smooth boundary and φ_m equals zero along the boundary: we denote this connected component by $\Sigma_{m,0}$.

We may assume that, for all m, $\varphi_m \leqslant 1$ over $\Sigma_{m,0}$. Finally, after reducing ϵ if necessary, there exists a smooth, unit length vector field X defined over a neighbourhood of p_0 such that, for all m, throughout $\Sigma_{m,0}$, $\langle X, \mathsf{N}_m\rangle \geqslant \epsilon$. We now follow an adaptation of reasoning presented by Pogorelov in [18].

Choose $\alpha \geqslant 1$. For all m, we define the function Φ_m by:

$$\Phi_m = \alpha\mathrm{Log}(\varphi_m) - \langle X, \mathsf{N}_m\rangle + \mathrm{Log}(\|A_m\|),$$

where $\|A_m\|$ is the operator norm of A_m, which is equal to the highest eigenvalue of A_m. We aim to obtain a priori upper bounds for Φ_m for some α. We trivially obtain a-priori bounds whenever $\|A_m\| \leqslant 1$. We thus consider the region where $\|A_m\| \geqslant 1$. Choose $m \in \mathbb{N}$ and $P \in \Sigma_{m,0}$. Let $\lambda_1 \geqslant ... \geqslant \lambda_n$ be the eigenvalues of A_m at P. In particular, $\lambda_1 = \|A_m\|$. Let $e_1, ..., e_n$ be the corresponding orthonormal basis of eigenvectors. In the sequel, we will suppress m.

Let the subscript ";" denote covariant differentiation with respect to the Levi-Civita covariant derivative of Σ. Thus, for example:

$$A_{ij;k} = (\nabla^{\Sigma}_{e_k} A)(e_i, e_j).$$

We consider the Laplacian, Δ, defined on functions by:

$$\Delta f = \sum_{i=1}^{n} \frac{1}{\lambda_i} f_{;ii}.$$

We aim to use the Maximum Principal in conjunction with Δ. Thus, in the sequel, we will only be interested in the orders of magnitude of potentially negative terms.

In analogy to Corollary 6.7, at P:

$$\Delta\mathrm{Log}(\lambda_1) \geqslant \sum_{i,j=1}^{n} \frac{1}{\lambda_1\lambda_i\lambda_j} A_{ij;1}A_{ij;1} - \sum_{i=1}^{n} \frac{1}{\lambda_1\lambda_1\lambda_i} A_{11;i}A_{11;i} - O(1) - O(\|A^{-1}\|),$$

in the weak sense. However, by Lemma 6.3, for all i:

$$A_{11;i} = A_{i1;1} + R^M_{i1\nu 1},$$

where ν represents the exterior normal direction to Σ. Thus, bearing in mind Lemma A.2, and that $\lambda_1 \geqslant 1$, we obtain:

$$\Delta\mathrm{Log}(\lambda_1) \geqslant \sum_{i=2}^{n} \frac{1}{2\lambda_1\lambda_1\lambda_i} A_{i1;1}A_{i1;1} - O(1) - O(\|A^{-1}\|),$$

in the weak sense. Differentiating the Gauss Curvature Equation yields, for all j:

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} A_{ii;j} = 0,$$

Thus:

$$\begin{aligned}-\Delta\langle X,\mathsf{N}\rangle &\geqslant \langle X,\mathsf{N}\rangle\mathrm{Tr}(A) - O(1) - O(\|A^{-1}\|)\\ &\geqslant \epsilon\lambda_1 - O(1) - O(\|A^{-1}\|).\end{aligned}$$

Finally:

$$\Delta(\alpha\mathrm{Log}(\varphi)) \geqslant \frac{\alpha}{\varphi}\epsilon\mathrm{Tr}(A^{-1}) - \frac{\alpha}{\varphi^2}\sum_{i=1}^n\frac{1}{\lambda_i}\varphi_{;i}\varphi_{;i} - O(\alpha).$$

However, bearing in mind Lemma 6.3:

$$\Phi_{;i} = \frac{\alpha}{\varphi}\varphi_{;i} - X^\nu{}_{;i} - X^i\lambda_i + \frac{1}{\lambda_1}A_{i1;1} + \frac{1}{\lambda_1}R^M_{i1\nu1},$$

where ν is the exterior normal direction over Σ. Thus, by induction on Lemma A.2, modulo $\nabla\Phi$:

$$\left|\frac{\alpha}{\varphi}\varphi_{;i}\right|^2 \leqslant \frac{4}{\lambda_1^2}A_{i1;1}A_{i1;1} + \frac{4}{\lambda_1^2}(R^M_{i1\nu1})^2 + 4(X^i\lambda_i)^2 + 4(X^\nu_{;i})^2.$$

Thus, bearing in mind that $\lambda_1 \geqslant \lambda_i$ for all i and that $\lambda_1 \geqslant 1$, we obtain, modulo $\nabla\Phi$:

$$\frac{\alpha}{\varphi^2}\sum_{i=2}^n\frac{1}{\lambda_i}\varphi_{;i}\varphi_{;i} = O(\alpha^{-1}\|A^{-1}\|) + O(\alpha^{-1}\lambda_1) + \sum_{i=2}^n\frac{4}{\alpha\lambda_1^2\lambda_i}A_{i1;1}A_{i1;1}.$$

Since φ is bounded above (and thus φ^{-1} is bounded below), for sufficently large α we obtain, modulo $\nabla\Phi$:

$$\begin{aligned}\Delta\Phi &\geqslant \tfrac{\epsilon}{2}\lambda_1 - O(\lambda_1^{-1}\varphi^{-2}) - O(1)\\ &= (\varphi^{2\alpha}\|A\|)^{-1}(\tfrac{\epsilon}{2}(\varphi^\alpha\|A\|)^2 - O(\varphi^\alpha\|A\|) - O(1)).\end{aligned}$$

There therefore exists $K_1 > 0$ in $\mathcal{B}$ such that if $(\varphi^\alpha\|A\|) \geqslant K$, then the right hand side is positive. However, for all $m \in \mathbb{N}$, $\Phi_m = -\infty$ along $\partial\Sigma_{m,0}$. There thus exists a point $P \in \Sigma_{m,0}$ where Φ_m is maximised. By the Maximum Principal, at this point, either $\|A\| \leqslant 1$ or $\varphi^\alpha\|A\| \leqslant K_1$. Taking exponentials, there therefore exists $K_2 > 0$ in $\mathcal{B}$ such that, for all $m \in \mathbb{N}$, throughout $\Sigma_{m,0}$:

$$\varphi^\alpha\langle X,\mathsf{N}_m\rangle^{-1}\|A_m\| \leqslant K_2.$$

Since $\langle X,\mathsf{N}_m\rangle \leqslant 1$, this yields a-priori bounds for $\|A_m\|$ over the intersection of $\Sigma_{m,0}$ with $\varphi_m \geqslant h$. Using, for example, an adaptation of the proof of Theorem 1.2 of [22] in conjunction with the Bernstein Theorem [6], [14] & [18] of Calabi, Jörgens, Pogorelov, we obtain a-priori C^k bounds for $\Sigma_{m,0}$ over the region $\varphi_m \geqslant 3h$ for all k. The result now follows by the Arzela-Ascoli Theorem. □

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