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# CONTINUOUS LINEAR AND QUADRATIC DIFFERENTIAL SYSTEMS ON THE 2-DIMENSIONAL TORUS

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ABSTRACT. We identify the 2-dimensional torus with $(\mathbb{R}/\mathbb{Z})^2$, and we study the dynamics of the continuous linear and quadratic polynomial differential systems on the torus $(\mathbb{R}/\mathbb{Z})^2$. The linear systems depend on two parameters, while the quadratic ones depend on six parameters. In particular we characterize all the local phase portraits of their equilibrium points and we study their limit cycles.

## 1. Introduction and main results

The objective of this paper is to study the dynamics of the linear and quadratic differential systems on the 2-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$.

Poincaré in [12] started the study of the differential systems on the 2-dimensional torus and conjectured that when such differential systems have no equilibrium points under suitable smoothness assumptions either they have a periodic solution or every solution is dense on the torus. Later on this conjecture was proved by Denjoy and Siegel in [7, 14]. Up to now there are few works on the differential systems on the torus. Later on it was shown by Saito [13] that if the flow of a differential system on the 2-dimensional torus is assumed to be measure preserving, then every orbit is either periodic or ergodic, depending on the rationality or irrationality of a certain ratio.

The *phase portrait* of a differential system provides the maximal qualitative information about its dynamics. The *phase portrait* of a differential system defined on the 2-dimensional torus consists in describing the torus as the union of all the orbits of the differential system. This is the best information which can be given for a differential system whose orbits cannot be given explicitly in function of the time. Since to draw the orbits on the curved surface of a torus is not easy, in what follows we think the torus in $(\mathbb{R}/\mathbb{Z})^2$, i.e. the plane $\mathbb{R}^2$ is reduced to the square $[0,1]\times[0,1]$ where we identify the points $(x,0)$ with $(x,1)$ for all $x\in[0,1]$, and the points $(0,y)$ with $(1,y)$ for all $y\in[0,1]$, see Figure 1. Then if a trajectory runs off an edge, it reappears on the opposite edge at the same point.

A *continuous differential system on the* 2*-dimensional torus* is a differential system of the form

$$\dot{x}=\frac{dx}{dt}=f(x,y),\qquad \dot{y}=\frac{dy}{dt}=g(x,y), \tag{1}$$

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