ZERO–HOPF BIFURCATIONS FOR A CLASS OF 3-DIMENSIONAL LOTKA-VOLTERRA SYSTEMS

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Abstract. A zero–Hopf equilibrium point p of a 3–dimensional autonomous differential system in $\mathbb{R}^3$ is an equilibrium such that the eigenvalues of the linear part of the system at p are 0 and $\pm\omega i$ with $\omega \neq 0$. A zero-Hopf bifurcation takes place when from a zero-Hopf equilibrium bifurcate some small-amplitude limit cycles moving the parameters of the system. We characterize, up to first order in the averaging theory, the eight distinct zero-Hopf bifurcations which can exhibit the following class of 3-dimensional Lotka-Volterra systems

$$\dot{x} = x(a_1(x-1) + b_1(y-1) + c_1(z-1) + d_1(x-1)^2 + e_1(y-1)^2 + f_1(z-1)^2),$$
$$\dot{y} = y(a_2(x-1) + b_2(y-1) + c_2(z-1) + d_2(x-1)^2 + e_2(y-1)^2 + f_2(z-1)^2),$$
$$\dot{z} = z(a_3(x-1) + b_3(y-1) + c_3(z-1) + d_3(x-1)^2 + e_3(y-1)^2 + f_3(z-1)^2).$$

Up to first order in the averaging theory for each one of this eight different zero-Hopf bifurcations emerges one limit cycle.

1. Introduction and statements of the main results

As usual a limit cycle of a differential system is a periodic orbit isolated in the set of all periodic orbits of the system.

While for an equilibrium point p of a 3–dimensional autonomous differential system in $\mathbb{R}^3$, such that the eigenvalues of the linear part of the system at p are ρ and $\pm\omega i$ with $\rho\omega \neq 0$, there is a general theory (see for instance pages 175–180 of [13]) for studying if moving the parameters of the system some limit cycle can bifurcate from p, for a zero-Hopf quilibrium there is no a such general theory for knowing when from this equilibrium bifurcates a small-amplitude limit cycle moving the parameters of the system. But using the averaging theory there is an algorithm for solving this problem, see for instance [14].

The Lotka-Volterra systems were initially considered independently by Lotka in 1925 [16] and by Volterra in 1926 [19], these differential systems are also known as the predator-prey systems. They are a pair of first-order, nonlinear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. Later on Kolmogorov [12] in 1936 extended these systems to arbitrary dimension and arbitrary degree, these kinds of systems are now called Kolmogorov systems.

It is known that the polynomial Lotka-Volterra differential systems in $\mathbb{R}^3$ of degree 2 cannot have isolated zero-Hopf equilibrium points, see [15]. In the article [7], we have studied the periodic orbits bifurcating from a Hopf equilibrium of 2-dimensional polynomial Kolmogorov systems of arbitrary degree.

In this paper first we study the zero-Hopf equilibrium points of a class of polynomial Lotka-Volterra differential systems in $\mathbb{R}^3$ of degree 3 via the averaging theory of first-order and we shall prove that there are eight families of such equilibrium points. We analyze when these families of zero-Hopf equilibrium points have a zero-Hopf bifurcation, i.e. when a small-amplitud limit cycle bifurcates from such equilibrium points moving the parameters of the system. We also give an example for each case, we plot their bifurcated limit cycles, and we study their stability.

For other differential systems the zero–Hopf bifurcation has been studied by many authors, for instance Guckenheimer, Han, Holmes, Kuznetsov, Marsden and Scheurle in [9, 10, 11, 13, 17]. In some cases the existence of a zero–Hopf bifurcation can imply a local birth of “chaos” see for instance the articles (cf. [1, 2, 5, 6, 17]).

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The Lotka-Voltera systems in $\mathbb{R}^3$ of degree 3 that we study are

$$\begin{aligned}\dot{x} &= x(a_1(x-1)+b_1(y-1)+c_1(z-1)+d_1(x-1)^2+e_1(y-1)^2+f_1(z-1)^2),\\ \dot{y} &= y(a_2(x-1)+b_2(y-1)+c_2(z-1)+d_2(x-1)^2+e_2(y-1)^2+f_2(z-1)^2),\\ \dot{z} &= z(a_3(x-1)+b_3(y-1)+c_3(z-1)+d_3(x-1)^2+e_3(y-1)^2+f_3(z-1)^2).\end{aligned} \tag{1}$$

where x, y and z are positive and the dot denotes the derivative with respect to the time t. Clearly systems (1) have the equilibrium point $(1,1,1)$.

In the next proposition we characterize when the equilibrium point $(1,1,1)$ of the Lotka-Volterra systems (1) is a zero–Hopf equilibrium.

Proposition 1. *There are eight one–parameter families of Lotka-Volterra systems* (1) *for which the equilibrium point* $(1,1,1)$ *is a zero–Hopf equilibrium. Namely:*

(i) $a_1 = b_1 = b_2 = c_1 = c_3 = 0,\ b_3c_2 = -\omega^2 < 0$;
(ii) $a_1 = -b_2, c_1 = c_2 = c_3 = 0,\ a_2b_1 + b_2^2 = -\omega^2 < 0$;
(iii) $a_1 = b_1 = c_1 = 0,\ c_3 = -b_2,\ b_3c_2 + b_2^2 = -\omega^2 < 0$;
(iv) $a_1 = -c_3, b_1 = b_2 = b_3 = 0,\ a_3c_1 + c_3^2 = -\omega^2 < 0$;
(v) $a_1 = -c_3,\ a_2 = -\dfrac{c_2c_3}{c_1},\ b_1 = b_2 = 0,\ a_3c_1 + b_3c_2 + c_3^2 = -\omega^2 < 0$;
(vi) $a_1 = -b_2 - c_3,\ b_3 = \dfrac{b_1c_3}{c_1},\ c_2 = \dfrac{b_2c_1}{b_1},\ a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2 = -\omega^2 < 0$;
(vii) $a_1 = -b_2 - c_3,\ a_2 = -\dfrac{b_2(b_2+c_3)}{b_1},\ c_2 = \dfrac{b_2c_1}{b_1},\ \dfrac{1}{b_1}(a_3b_1c_1 + b_2b_3c_1 + b_1c_3^2) = -\omega^2 < 0$;
(viii) $a_1 = -b_2 - c_3,\ a_3 = \dfrac{a_2(b_1c_3 - b_3c_1) + (b_2+c_3)(b_2c_3 - b_3c_2)}{b_1c_2 - b_2c_1},\ \omega^2 = \dfrac{A}{b_1c_2 - b_2c_1} > 0,$
where $A = -(-b_2^3c_1 + b_1b_2^2c_2 + b_2(-2b_3c_1c_2 + b_1c_2c_3) + c_2(b_1b_3c_2 - b_3c_1c_3 + b_1c_3^2) + a_2(b_1^2c_2 - b_3c_1^2 + b_1c_1(c_3 - b_2)))$.

Proposition 1 is proved in section 3.

Theorem 2. *Assume that one of the following eight conditions holds for the Lotka-Volterra systems* (1)

(i) $b_3c_2 < 0$ *and* $a_1 = a_{11}\varepsilon,\ b_1 = b_{11}\varepsilon,\ b_2 = b_{21}\varepsilon,\ c_1 = c_{11}\varepsilon,\ c_3 = c_{31}\varepsilon,\ b_3c_2 = -\omega^2$;

(ii) $a_2b_1 + b_2^2 < 0$ *and* $a_1 = -(b_2 + b_{21}\varepsilon),\ b_2 = b_2 + b_{21}\varepsilon,\ c_1 = c_{11}\varepsilon,\ c_2 = c_{21}\varepsilon,\ c_3 = c_{31}\varepsilon,\ a_2b_1 + b_2^2 = -\omega^2$;

(iii) $b_3c_2 + b_2^2 < 0$ *and* $a_1 = a_{11}\varepsilon,\ b_2 = b_2 + b_{21}\varepsilon,\ c_1 = c_{11}\varepsilon,\ b_1 = b_{11}\varepsilon,\ c_3 = -(b_2 + b_{21}\varepsilon),\ b_3c_2 + b_2^2 = -\omega^2$;

(iv) $a_3c_1 + c_3^2 < 0$ *and* $a_1 = -(c_3 + c_{31}\varepsilon),\ b_1 = b_{11}\varepsilon,\ b_2 = b_{21}\varepsilon,\ b_3 = b_{31}\varepsilon,\ c_3 = c_3 + c_{31}\varepsilon,\ a_3c_1 + c_3^2 = -\omega^2$;

(v) $a_3c_1 + b_3c_2 + c_3^2 < 0$ *and* $a_1 = -(c_3 + c_{31}\varepsilon),\ b_1 = b_{11}\varepsilon,\ b_2 = b_{21}\varepsilon,\ c_3 = c_3 + c_{31}\varepsilon,\ a_2 = -\dfrac{c_2c_3}{c_1} - \dfrac{c_2c_{31}}{c_1}\varepsilon,\ a_3c_1 + b_3c_2 + c_3^2 = -\omega^2$;

(vi) $a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2 < 0$ *and* $a_1 = -b_2 - c_3 - (b_{21} + c_{31})\varepsilon,\ b_2 = b_2 + b_{21}\varepsilon,\ b_3 = \dfrac{b_1c_3}{c_1} + \dfrac{b_1c_{31}}{c_1}\varepsilon,\ c_2 = \dfrac{b_2c_1}{b_1} + \dfrac{b_{21}c_1}{b_1}\varepsilon,\ c_3 = c_3 + c_{31}\varepsilon,\ a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2 = -\omega^2$;

(vii) $\dfrac{1}{b_1}(a_3b_1c_1 + b_2b_3c_1 + b_1c_3^2) < 0$ *and* $a_1 = -b_2 - c_3 - (b_{21} + c_{31})\varepsilon,\ b_2 = b_2 + b_{21}\varepsilon,\ a_2 = \dfrac{-b_2(b_2+c_3)}{b_1} - \dfrac{b_2(b_{21}+c_{31}) + b_{21}(b_2+c_3)}{b_1}\varepsilon,\ c_2 = \dfrac{b_2c_1}{b_1} + \dfrac{b_{21}c_1}{b_1}\varepsilon,\ c_3 = c_3 + c_{31}\varepsilon,\ \dfrac{1}{b_1}(a_3b_1c_1 + b_2b_3c_1 + b_1c_3^2) = -\omega^2$;

(viii) $\dfrac{A}{b_1c_2 - b_2c_1} > 0$ *and* $a_1 = -b_2 - c_3 - c_{31}\varepsilon,\ c_3 = c_3 + c_{31}\varepsilon,\ \dfrac{A}{b_1c_2 - b_2c_1} = \omega^2,\ a_3 = \dfrac{a_2(b_1c_3 - b_3c_1) + (b_2+c_3)(b_2c_3 - b_3c_2)}{b_1c_2 - b_2c_1} + \dfrac{a_2b_1c_{31} + (b_2+c_3)b_2c_{31} + c_{31}(b_2c_3 - b_3c_2)}{b_1c_2 - b_2c_1}\varepsilon$.

Then for $\varepsilon \neq 0$ *sufficiently small system* (1) *has one periodic solution* $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ *bifurcating from the zero-Hopf equilibrium* $(1,1,1)$ *when* $\varepsilon = 0$.

Theorem 2 is proved in section 3 using the averaging theory of first order.

2. The averaging theory for periodic orbits

The averaging theory is a classical and matured tool for studying the dynamics of nonlinear smooth dynamical systems, and in particular of their periodic orbits. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of this method. The first formalization of this method is due to Fatou [8] in 1928. Important practical and theoretical contributions to this theory were made by Krylov and Bogoliubov [4] in the 1930s and Bogoliubov [3] in 1945. The averaging theory of first order for studying periodic orbits can be found in [18], see also [10].

Now we present the basic results from the averaging theory that we shall need for proving the results of this paper. The next theorem provides a first order approximation for the periodic orbits of a periodic differential system, for a see Theorems 11.5 and 11.6 of Verhulst [18].

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F(t,\mathbf{x}) + \varepsilon^2 G(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{2}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $F(t,\mathbf{x})$ and $G(t,\mathbf{x},\varepsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{3}$$

where

$$f(\mathbf{y}) = \frac{1}{T}\int_0^T F(t,\mathbf{y})dt. \tag{4}$$

Theorem 3. *Consider the two initial value problems* (2) *and* (3). *Suppose:*

- (i) *F, its Jacobian $\partial F/\partial x$, its Hessian $\partial^2 F/\partial x^2$, G and its Jacobian $\partial G/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty) \times D$ and $\varepsilon \in (0,\varepsilon_0]$.*
- (ii) *F and G are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

- (a) *If p is an equilibrium point of the averaged equation* (3) *and*

$$\det\left(\frac{\partial f}{\partial \mathbf{y}}\right)\Bigg|_{\mathbf{y}=p} \neq 0, \tag{5}$$

  then there exists a limit cycle $\mathbf{x}(t,\varepsilon)$ of period T of equation (2) *such that $\mathbf{x}(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*
- (b) *The stability or instability of the limit cycle $\mathbf{x}(t,\varepsilon)$ is given by the stability or instability of the equilibrium point p of the averaged system* (3). *In fact the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\mathbf{x}(t,\varepsilon)$.*

3. Proofs

Proof of Proposition 1. The characteristic polynomial of the linear part of the Lotka-Volterra system (1) at the equilibrium point $(1,1,1)$ is

$$\begin{aligned} p(\lambda) &= -\lambda^3 + (a_1+b_2+c_3)\lambda^2 + (a_2b_1 - a_1b_2 + a_3c_1 + b_3c_2 - a_1c_3 - b_2c_3)\lambda \\ &\quad + a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1. \end{aligned}$$

Imposing that $p(\lambda) = -\lambda(\lambda^2+\omega^2)$, we obtain the system

$$\begin{aligned} a_1+b_2+c_3 &= 0, \\ a_2b_1 - a_1b_2 + a_3c_1 + b_3c_2 - a_1c_3 - b_2c_3 &= -\omega^2, \\ a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 &= 0. \end{aligned}$$

Solving this system we get the eight families of zero–Hopf equilibria described in the statement of Proposition 1. This completes the proof. □

Proof of Theorem 2. We shall prove that a periodic orbit bifurcates from the zero–Hopf equilibrium point $(1,1,1)$ of the Lotka-Volterra system (1) when the parameters of system (1) are given by statement (i) of Proposition (1). The proof for the others zero–Hopf equilibrium are analogous and we only indicate the main steeps of their proofs.

We perturb the Lotka-Volterra system (1) with the parameters given in statement (i) of Proposition 1 satisfying the conditions of Theorem 2(i). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates doing the change of variables $x = X+1, y = Y+1, z = Z+1$. Then system (1) becomes

$$
\begin{aligned}
\dot{X} &= (1+X)\left(d_1X^2 + e_1Y^2 + f_1Z^2 + \varepsilon(a_{11}X + b_{11}Y + c_{11}Z)\right),\\
\dot{Y} &= (1+Y)\left(a_2X + d_2X^2 + e_2Y^2 + c_2Z + f_2Z^2 + \varepsilon b_{21}Y\right),\\
\dot{Z} &= (1+Z)\left(a_3X + d_3X^2 + b_3Y + e_3Y^2 + f_3Z^2 + \varepsilon c_{31}Z\right).
\end{aligned} \tag{6}
$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (6) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$
\begin{pmatrix} 0 & -\sqrt{-b_3c_2} & 0\\ \sqrt{-b_3c_2} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}.
$$

For this we do the change of variables

$$
\begin{pmatrix} u\\ v\\ w\end{pmatrix} = \begin{pmatrix} \dfrac{a_3}{\sqrt{-b_3c_2}} & -\sqrt{-\dfrac{b_3}{c_2}} & 0\\ \dfrac{a_2}{c_2} & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix} X\\ Y\\ Z\end{pmatrix}, \text{ with inverse } \begin{pmatrix} X\\ Y\\ Z\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1\\ b_3\sqrt{-\dfrac{c_2}{b_3^3}} & 0 & -\dfrac{a_3}{b_3}\\ 0 & 1 & -\dfrac{a_2}{c_2}\end{pmatrix}\begin{pmatrix} u\\ v\\ w\end{pmatrix}. \tag{7}
$$

The differential system (6) writes

$$
\begin{aligned}
\dot{u} = {} & \frac{1}{\sqrt{-b_3c_2}}\Bigg(a_3(1+w)\left(d_1w^2 + \frac{e_1(\sqrt{-b_3c_2}u - a_3w)^2}{b_3^2} + f_1\left(v - \frac{a_2w}{c_2}\right)^2\right) + \left(b_3 + \sqrt{-b_3c_2}u - a_3w\right)\left(c_2v + d_2w^2\right.\\
& \left.+\ \frac{e_2\left(\sqrt{-b_3c_2}u - a_3w\right)^2}{b_3^2} + f_2\left(v - \frac{a_2w}{c_2}\right)^2\right)\Bigg) + \varepsilon\frac{1}{\sqrt{-b_3c_2}}\Bigg(\frac{b_{21}}{b_3}\left(\sqrt{-b_3c_2}u - a_3w\right)\left(b_3 + \sqrt{-b_3c_2}u - a_3w\right) +\\
& a_3\,(1+w)\left(a_{11}w + \frac{b_{11}}{b_3}\left(\sqrt{-b_3c_2}u - a_3w\right) + c_{11}\left(v - \frac{a_2w}{c_2}\right)^2\right)\Bigg),\\
\dot{v} = {} & \frac{a_2(1+\omega)}{c_2}\left(d_1w^2 + \frac{e_1}{b_3}\left(\sqrt{-b_3c_2}u - a_3w\right)^2 + f_1\left(v - \frac{a_2w}{c_2}\right)^2\right) + \left(1 + v - \frac{a_2w}{c_2}\right)\left(\sqrt{-b_3c_2}u + d_3w^2\right.\\
& \left.+\frac{e_3}{b_3^2}\left(\sqrt{-b_3c_2}u - a_3w\right)^2 + f_3\left(v - \frac{a_2w}{c_2}\right)^2\right) + \varepsilon\frac{1}{c_2^2}\left(c_{31}\left(c_2v - a_2w\right)\left(c_2 + c_2v - a_2w\right) + a_2c_2(1+w)\left(a_{11}w\right.\right.\\
& \left.\left.+\frac{b_{11}}{b_3}\left(\sqrt{-b_3c_2}u - a_3w\right) + c_{11}\left(v - \frac{a_2w}{c_2}\right)\right)\right),\\
\dot{w} = {} & \varepsilon(1+w)\left(d_1w^2 + \frac{e_1}{b_3^2}(\sqrt{-b_3c_2}u - a_3w)^2 + f_1\left(v - \frac{a_2w}{c_2}\right)^2\right) + (1+w)\left(a_{11}w + \frac{b_{11}}{b_3}\left(\sqrt{-b_3c_2}u - a_3w\right)\right.\\
& \left.+c_{11}\left(v - \frac{a_2w}{c_2}\right)\right).
\end{aligned} \tag{8}
$$

Doing the rescaling of the variables $(u,v,w) = (\varepsilon U, \varepsilon V, \varepsilon W)$ system (8) in the new variables (U,V,W) writes

$$
\begin{aligned}
\dot{U} = {} & b_3\sqrt{-\frac{c_2}{b_3}}V + \varepsilon\frac{1}{\sqrt{-b_3c_2^5}b_3^2}\Big((a_3b_{11} + b_{21}b_3)\sqrt{-b_3c_2^5}b_3U - (a_3e_1 + b_3e_2)b_3c_2^3U^2 + a_3b_3^2c_{11}c_2^2V + \sqrt{-b_3c_2^7}b_3^2UV\\
& +(a_3f_1 + b_3f_2)b_3^2c_2^2V^2 - (a_2b_3c_{11} + a_3b_{11}c_2 - a_{11}b_3c_2 + b_{21}b_3c_2)a_3b_3c_2W - (a_3e_1 + b_3e_2)2a_3\sqrt{-b_3c_2^5}UW\\
& -(a_3c_2^2 + 2a_2a_3f_1 + 2a_2b_3f_2)b_3^2c_2VW + (a_3b_3^3c_2^2d_1 + b_3^2c_2^2d_2 + a_3^3c_2^2e_1 + a_3^2b_3c_2^2e_2 + a_2^2b_3^2f_1 + a_2^2b_3^3f_2))W^2),\\
\dot{V} = {} & \sqrt{-b_3c_2}U + \varepsilon\frac{1}{b_3^2c_2^3}\Big(a_2b_{11}\sqrt{-b_3c_2^5}b_3U - (a_2e_1 + c_2e_3)b_3c_2^3U^2 + (a_2c_{11} + c_{31})\,b_3^2c_2^2V + \sqrt{-b_3c_2^7}b_3^2UV\\
& +(a_2f_1 + c_2f_3)\,b_3^2c_2^2V^2 - (a_2b_3c_{11} + a_3b_{11}c_2 - a_{11}b_3c_2 + b_3c_2c_{31})\,a_2b_3c_2W - \left(a_2b_3^2 + 2a_2a_3e_1 + 2a_3c_2e_3\right)\\
& \sqrt{-b_3c_2^5}UW - 2(a_2f_1 + c_2f_3)a_2b_3^2c_2VW + (a_2b_3^2c_2^2d_1 + b_3^2c_2^3d_3 + a_2a_3^2c_2^2e_1 + a_3^2c_2^3e_3 + a_2^3b_3^2f_1 + a_2^2b_3^2c_2f_3\Big)\,W^2,
\end{aligned} \tag{9}
$$

$$\begin{aligned}\dot W = \;& \varepsilon\frac{1}{b_3^2c_2^3}\Big(b_{11}\sqrt{-b_3c_2^5}b_3U - b_3c_2^3e_1U^2 + b_3^2c_{11}c_2^2V + b_3^2c_2^2f_1V^2 - \big(a_2b_3^2c_{11}c_2 + a_3b_{11}b_3c_2^2 + a_{11}b_3^2c_2^2\big)W\\ & -2a_3\sqrt{-b_3c_2^5}e_1UW - 2a_2b_3^2c_2f_1VW + \big(b_3^2c_2^2d_1 + a_3^2c_2^2e_1 + a_2^2b_3^2f_1\big)W^2\Big).\end{aligned}$$

Now we pass from the differential system (3) to cylindrical coordinates (r,θ,W) defined by $U = r\cos\theta$ and $V = r\sin\theta$, and we obtain

(10)

$$\begin{aligned}\dot r =\;& \frac{\varepsilon}{b_3^2c_2^2}\Big(\big(\sqrt{-b_3c_2^3}(a_3e_1+b_3e_2)\cos^3\theta + b_3c_2(b_3c_2-e_3c_2-a_2e_1)\sin\theta\cos^2\theta + \sqrt{-b_3c_2}b_3(b_3c_2-a_3f_1-b_3f_2)\\ & \sin^2\theta\cos\theta + b_3^2(a_2f_1+c_2f_3)\sin^3\theta\big)r^2 + b_3c_2\big((a_3b_{11}+b_{21}b_3)c_2\cos^2\theta + \sqrt{-b_3c_2}(a_2b_{11}-a_3c_{11})\sin\theta\cos\theta\\ & +b_3(a_2c_{11}+c_2c_{31})\sin^2\theta\big)r - W\big(2a_3c_2^2(a_3e_1+b_3e_2)\cos^2\theta + \sqrt{-b_3c_2}\big(a_2c_2b_3^2 - 2a_2f_2b_3^2 - a_3c_2^2b_3 - 2a_2a_3f_1b_3\\ & +2a_2a_3c_2e_1 + 2a_3c_2^2e_3\big)\sin\theta\cos\theta + 2a_2b_3^2(a_2f_1+c_2f_3)\sin^2\theta\big)r + W\big(a_3\sqrt{-b_3c_2}(a_2b_3c_{11}+a_3b_{11}c_2-a_{11}b_3c_2\\ & +b_{21}b_3c_2)\cos\theta - a_2b_3(a_2b_3c_{11}+a_3b_{11}c_2-a_{11}b_3c_2+b_3c_2c_{31})\sin\theta\big) - W^2\big(\sqrt{-b_3c_2}\big(c_2^2e_1a_3^3 + b_3c_2^2e_2a_3^2 + b_3^2c_2^2d_1a_3\\ & +a_2^2b_3^2f_1a_3 + b_3^3c_2^2d_2 + a_2^2b_3^3f_2\big)\cos\theta - b_3\big(b_3^2f_1a_2^3 + b_3^2c_2f_3a_2^2 + b_3^2c_2^2d_1a_2 + a_3^2c_2^2e_1a_2 + b_3^2c_2^3d_3 + a_3^2c_2^3e_3\big)\sin\theta\big)\Big),\\
\dot\theta =\;& \sqrt{-b_3c_2} + \frac{\varepsilon}{b_3^2c_2^2}\Big(\frac{1}{b_3c_2r}\big(b_3\big(b_3^2f_1a_2^3 + b_3^2c_2f_3a_2^2 + b_3^2c_2^2d_1a_2 + a_3^2c_2^2e_1a_2 + b_3^2c_2^3d_3 + a_3^2c_2^3e_3\big)\cos\theta + \sqrt{-b_3c_2}\big(c_2^2e_1a_3^3\\ & +b_3c_2^2e_2a_3^2 + b_3^2c_2^2d_1a_3 + a_2^2b_3^2f_1a_3 + b_3^3c_2^2d_2 + a_2^2b_3^3f_2\big)\sin\theta\big)W^2 - \cos\theta\big(\sqrt{-b_3c_2^3}\big(a_2b_3^2 + 2a_2a_3e_1 + 2a_3c_2e_3\big)\cos\theta\\ & -2\big(-a_2^2f_1b_3^2 - a_2c_2f_3b_3^2 + a_3c_2^2e_2b_3 + a_3^2c_2^2e_1\big)\sin\theta\big)W + \big(-\sqrt{-b_3c_2}b_3\big(a_3c_2^2 + 2a_2a_3f_1 + 2a_2b_3f_2\big)\sin^2\theta\\ & -\frac{\sqrt{-b_3c_2}}{r}\big(b_{11}c_2a_3^2 + a_2b_3c_{11}a_3 - a_{11}b_3c_2a_3 + b_{21}b_3c_2a_3\big)\sin\theta - \frac{1}{r}\big(a_2^2c_{11}b_3^2 - a_{11}a_2c_2b_3^2 + a_2c_2c_{31}b_3^2\\ & +a_2a_3b_{11}c_2b_3\big)\cos\theta\big)W + b_3c_2\big(a_2b_{11}\sqrt{-b_3c_2}\cos^2\theta + (a_2b_3c_{11} - a_3b_{11}c_2 - b_{21}b_3c_2 + b_3c_2c_{31})\sin\theta\cos\theta +\\ & a_3\sqrt{-b_3}c_{11}\sqrt{c_2}\sin^2\theta\big) - \tfrac12 c_2r\big(-b_3\big(b_3c_2^2 - 2e_3c_2^2 - 2a_2e_1c_2 - b_3f_3c_2 - a_2b_3f_1\big)\cos^3\theta - \sqrt{-b_3c_2}\big(2c_2b_3^2 - f_2b_3^2\\ & -2c_2e_2b_3 - a_3f_1b_3 - 2a_3c_2e_1\big)\sin\theta\cos^2\theta + b_3^2\big(c_2^2 - f_3c_2 - a_2f_1\big)\sin^2\theta\cos\theta + b_3^2\big(c_2^2 - f_3c_2 - a_2f_1\big)\cos\theta\\ & -\sqrt{-b_3c_2}b_3(a_3f_1+b_3f_2)\sin^3\theta - \sqrt{-b_3}b_3\sqrt{c_2}(a_3f_1+b_3f_2)\sin\theta\big)\Big),\\
\dot W =\;& \frac{\varepsilon}{b_3^2c_2^2}\Big(b_3r\big(b_{11}\sqrt{-b_3c_2}\cos\theta + b_3c_{11}\sin\theta\big)c_2^2 + b_3r^2\big(b_3f_1\sin^2\theta - c_2e_1\cos^2\theta\big)c_2^2 - b_3c_2(a_2b_3c_{11} + a_3b_{11}c_2\\ & -a_{11}b_3c_2)W - 2rW\big(a_2f_1\sin\theta b_3^2 + a_3\sqrt{-b_3}c_2^{3/2}e_1\cos\theta\big)c_2 + \big(c_2^2d_1b_3^2 + a_2^2f_1b_3^2 + a_3^2c_2^2e_1\big)W^2\Big).\end{aligned}$$

We change the independent variable from t to θ, and denoting the derivative with respect to θ by a dot, then the differential system (10) becomes

(11)

$$\begin{aligned}\dot r =\;& \frac{\varepsilon}{(-b_3c_2)^{5/2}}\Big(\big(\sqrt{-b_3c_2^3}(a_3e_1+b_3e_2)\cos^3\theta + b_3c_2(b_3c_2-e_3c_2-a_2e_1)\sin\theta\cos^2\theta + \sqrt{-b_3c_2}b_3(b_3c_2-a_3f_1-b_3f_2)\\ & \sin^2\theta\cos\theta + b_3^2(a_2f_1+c_2f_3)\sin^3\theta\big)r^2 + b_3c_2\big((a_3b_{11}+b_{21}b_3)c_2\cos^2\theta + \sqrt{-b_3c_2}(a_2b_{11}-a_3c_{11})\sin\theta\cos\theta\\ & +b_3(a_2c_{11}+c_2c_{31})\sin^2\theta\big)r - W\big(2a_3c_2^2(a_3e_1+b_3e_2)\cos^2\theta + \sqrt{-b_3c_2}\big(a_2c_2b_3^2 - 2a_2f_2b_3^2 - a_3c_2^2b_3 - 2a_2a_3f_1b_3\\ & +2a_2a_3c_2e_1 + 2a_3c_2^2e_3\big)\sin\theta\cos\theta + 2a_2b_3^2(a_2f_1+c_2f_3)\sin^2\theta\big)r + W\big(a_3\sqrt{-b_3c_2}(a_2b_3c_{11}+a_3b_{11}c_2-a_{11}b_3c_2\\ & +b_{21}b_3c_2)\cos\theta - a_2b_3(a_2b_3c_{11}+a_3b_{11}c_2-a_{11}b_3c_2+b_3c_2c_{31})\sin\theta\big) - W^2\big(\sqrt{-b_3c_2}\big(c_2^2e_1a_3^3 + b_3c_2^2e_2a_3^2 + b_3^2c_2^2d_1a_3\\ & +a_2^2b_3^2f_1a_3 + b_3^3c_2^2d_2 + a_2^2b_3^3f_2\big)\cos\theta - b_3\big(b_3^2f_1a_2^3 + b_3^2c_2f_3a_2^2 + b_3^2c_2^2d_1a_2 + a_3^2c_2^2e_1a_2 + b_3^2c_2^3d_3 + a_3^2c_2^3e_3\big)\sin\theta\big)\Big)\\
=\;& F_1(\theta,r,W),\\
\dot W =\;& \frac{\varepsilon}{(-b_3c_2)^{5/2}}\Big(b_3r\big(b_{11}\sqrt{-b_3c_2}\cos\theta + b_3c_{11}\sin\theta\big)c_2^2 + b_3r^2\big(b_3f_1\sin^2\theta - c_2e_1\cos^2\theta\big)c_2^2 - b_3c_2(a_2b_3c_{11} + a_3b_{11}c_2\\ & -a_{11}b_3c_2)W - 2rW\big(a_2f_1\sin\theta b_3^2 + a_3\sqrt{-b_3}c_2^{3/2}e_1\cos\theta\big)c_2 + \big(c_2^2d_1b_3^2 + a_2^2f_1b_3^2 + a_3^2c_2^2e_1\big)W^2\Big) = F_2(\theta,r,W).\end{aligned}$$

We apply the averaging theory described in Theorem 3 to the differential system (11). Using the notation of section 2 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,W)^T$ and

$$F(\theta,r,W) = \begin{pmatrix} F_1(\theta,r,W),\\ F_2(\theta,r,W)\end{pmatrix}, \quad \text{and} \quad f(r,W) = \begin{pmatrix} f_1(r,W),\\ f_2(r,W)\end{pmatrix}.$$

It is immediate to check that system (11) satisfies all the assumptions of Theorem 3. Now we compute the integrals (4), i.e.

$$\begin{aligned} f_1(r,W) = &\ \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r,W)d\theta, \\ = &\ \frac{r\left(-2a_2^2b_3^2f_1W + a_2b_3^2c_2(c_{11}-2f_3W) + c_2^2(b_3^2(b_{21}+c_{31}) - 2a_3^2e_1W + a_3b_3(b_{11}-2e_2W))\right)}{2(-b_3c_2)^{5/2}}, \\ f_2(r,W) = &\ \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r,W)d\theta, \\ = &\ \frac{\left(b_3c_2^2(-c_2e_1+b_3f_1)r^2 + 2b_3c_2(-a_2b_3c_{11} - a_3b_{11}c_2 + a_{11}b_3c_2)W + 2(a_3^2c_2^2e_1 + b_3^2(c_2^2d_1 + a_2^2f_1))W^2\right)}{2(-b_3c_2)^{5/2}}. \end{aligned}$$

The system $f_1(r,W) = f_2(r,W) = 0$ has a unique solution (r^*,W^*) with $r^* > 0$, namely

$$(r^*,W^*) = \left(\frac{1}{T_1}\sqrt{\frac{C_1N_1T_1 - D_1N_1^2}{R_1}}, -\frac{N_1}{T_1}\right),$$

if $T_1 > 0$, $R_1 \neq 0$ and $R_1(C_1N_1T_1 - D_1N_1^2) > 0$. If the Jacobian (5) at (r^*,W^*) takes the value $N_1(C_1T_1 - D_1N_1)/(2T_1b_3^5c_2^5) \neq 0$, where

$$\begin{aligned} C_1 = &\ 2b_3c_2(-a_2b_3c_{11} - a_3b_{11}c_2 + a_{11}b_3c_2),\ D_1 = 2\big(a_3^2c_2^2e_1 + b_3^2(a_2^2f_1 + c_2^2d_1)\big),\ R_1 = b_3c_2^2(b_3f_1 - c_2e_1), \\ N_1 = &\ a_2b_3^2c_2c_{11} + c_2^2\big(b_3^2(b_{21}+c_{31}) + a_3b_3b_{11}\big),\ T_1 = -2a_2^2b_3^2f_1 - 2a_2b_3^2c_2f_3 + c_2^2\big(-2a_3^2e_1 - 2a_3b_3e_2\big), \end{aligned}$$

then Theorem 3 guarantees for $\varepsilon > 0$ sufficiently small the existence of a periodic solution $(r(\theta,\varepsilon), W(\theta,\varepsilon))$ of system (11) such that $(r(0,\varepsilon), W(0,\varepsilon)) \to (r^*,W^*)$ when $\varepsilon \to 0$. So for $\varepsilon > 0$ sufficiently small system (10) has the periodic solution

$$(r(t,\varepsilon), \theta(t,\varepsilon), W(t,\varepsilon)) \quad \text{where } \theta(t,\varepsilon) = \cos(\sqrt{-b_3c_2}\,t) + O(\varepsilon).$$

Consequently system (3) has the periodic solution

$$(U(t,\varepsilon), V(t,\varepsilon), W(t,\varepsilon)) = \big(r(t,\varepsilon)\cos(\sqrt{-b_3c_2}\,t), r(t,\varepsilon)\sin(\sqrt{-b_3c_2}\,t), W(t,\varepsilon)\big) + O(\varepsilon),$$

for $\varepsilon > 0$ sufficiently small. Therefore system (8) for $\varepsilon > 0$ sufficiently small has the periodic solution

$$(u(t,\varepsilon), v(t,\varepsilon), w(t,\varepsilon)) = \varepsilon\big(r(t,\varepsilon)\cos(\sqrt{-b_3c_2}\,t), r(t,\varepsilon)\sin(\sqrt{-b_3c_2}\,t), W(t,\varepsilon)\big) + O(\varepsilon^2). \tag{12}$$

Finally for $\varepsilon > 0$ sufficiently small system (6) has the periodic solution $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon))$ obtained from (12) through the change of variables (7). This periodic solution tends to the origin of coordinates when $\varepsilon \to 0$. Therefore it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon = 0$. This completes the proof of statement (i) of Theorem 2.

Example 1. Consider the Lotka-Volterra system

$$\dot{x} = x(2(x-1)^2 + (z-1)^2),\ \dot{y} = y(-1 + z + 2(y-1)^2 + (z-1)^2),\ \dot{z} = z(x - y + (x-1)^2 + (y-1)^2). \tag{13}$$

This system in the new variables (X,Y,Z) writes

$$\dot{X} = (X+1)(2X^2 + 2X\varepsilon + Z^2),\ \dot{Y} = (Y+1)(2Y^2 + Y\varepsilon + Z^2 + Z),\ \dot{Z} = (Z+1)(X^2 + Y^2 + Z\varepsilon + X - Y).$$

The corresponding system associated to system (11) satifies

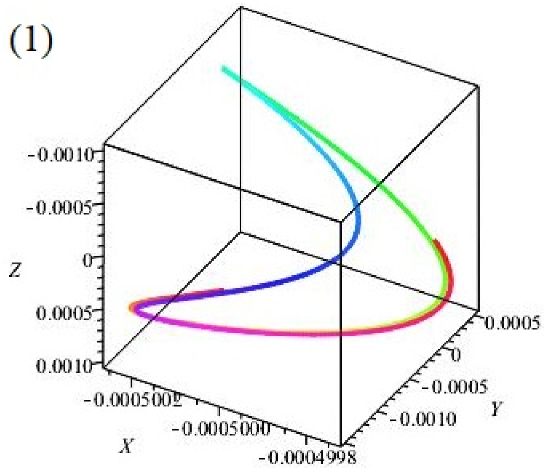
$$\begin{aligned} &F_1(\theta,r,W) = (-3\cos\theta^3 + 2\cos\theta^2\sin\theta + \cos\theta)r^2 + (4\cos\theta^2 W - 3\cos\theta\sin\theta W + 1)r + 2\sin\theta W^2 + \cos\theta W, \\ &F_2(\theta,r,W) = \sin\theta^2 r^2 + 2W^2 + 2W. \end{aligned}$$

To look for the limit cycles we must solve the system

$$f_1(r,W) = r(2W+1) = 0, \quad f_2(r,W) = 2W^2 + \frac{1}{2}r^2 + 2W = 0.$$

This system has the four solutions (r,W) given by $(0,0)$, $(0,-1)$, $(1,-1/2)$, $(-1,-1/2)$. Since r must be positive and W real, the unique admissible solution is $(1,-1/2)$.

Since the determinant (5) at the previous solution is -2, system (1) under assumptions (i) has exactly one limit cycle bifurcating from the origin. We plot this limit cycle for $\varepsilon = 10^{-3}$ in the next figure.



Since the eigenvalues of the Jacobian matrix of (f_1, f_2) at singular point $(r, W) = (1, 1/2)$ are $\pm\sqrt{2}$, by Theorem 3 corresponding limit cycle is unstable, and locally it has a stable manifold formed by two cylinders and an unstable manifold also formed by two cylinders. From the proof of (i) we obtain that the limit cycle bifurcating from the equilibrium point $(1, 1, 1)$ of system (13) is

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (1 - \varepsilon/2, 1 - \varepsilon\cos t, 1 + \varepsilon\sin t) + O(\varepsilon^2).$$

This completes the proof of Example 1.

Now we perturb the Lotka-Volterra system (1) with the parameters given in statement (ii) of Proposition 1 satisfying the conditions of Theorem 2(ii). We translate the equilibrium point $(1, 1, 1)$ to the origin of coordinates doing the change of variables $x = X + 1, y = Y + 1, z = Z + 1$. Then system (1) becomes

$$\begin{aligned}
\dot{X} &= (X+1)\big((-2Xb_{21} + Zc_{11})\varepsilon + d_1X^2 + e_1Y^2 + f_1Z^2 - b_2X + b_1Y\big),\\
\dot{Y} &= (Y+1)\big((Yb_{21} + Zc_{21})\varepsilon + d_2X^2 + e_2Y^2 + f_2Z^2 + a_2X + b_2Y\big),\\
\dot{Z} &= (Z+1)\big(X^2d_3 + Y^2e_3 + Z^2f_3 + Z\varepsilon c_{31} + Xa_3 + Yb_3\big).
\end{aligned} \tag{14}$$

We write the linear part of system (14) with $\varepsilon = 0$ at the equilibrium point $(0, 0, 0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{-a_2b_1 - b_2^2} & 0 \\ \sqrt{-a_2b_1 - b_2^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this we do the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\dfrac{b_2}{b_1\sqrt{-a_2b_1 - b_2^2}} & \dfrac{1}{\sqrt{-a_2b_1 - b_2^2}} & 0 \\ \dfrac{1}{b_1} & 0 & 0 \\ -\dfrac{a_2b_3 - a_3b_2}{a_2b_1 + b_2^2} & -\dfrac{a_3b_1 + b_2b_3}{a_2b_1 + b_2^2} & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

and following the same steps as the case of Theorem 2(i), we get the system

(15)

$$\begin{aligned}
\dot{r} = {} & -\frac{\varepsilon}{b_1\big(-a_2b_1 - b_2^2\big)^2}\Big(\sqrt{-a_2b_1 - b_2^2}\Big(\Big((2(a_3b_1 + b_2b_3))b_3(b_1f_2 - b_2f_1) + (a_2b_1 + b_2^2)((a_2 - b_2)b_1^2 + 2b_2^2(b_1 - e_1)\\
& + 2b_1b_2e_2) - (a_3b_1 + b_2b_3)^2f_1 - (a_2b_1 + b_2^2)^2e_1\Big)\cos^2\theta\sin\theta r^2 + r\big(\cos^2\theta\big((2(a_3b_1 + b_2b_3))W(b_1f_2 - b_2f_1)\\
& + (a_3b_1 + b_2b_3)(b_1c_{21} - b_2c_{11}) + (a_2b_1 + b_2^2)b_1b_{21}\big) + (a_2b_1 + b_2^2)(2Wb_3f_1 - 2b_1b_{21} + b_3c_{11})\sin^2\theta\big)\Big) + (-a_2b_1 - b_2^2)\\
& \big(\big(b_3f_1(2a_3b_1 + 3b_2b_3) - b_2^2(b_1e_2 - b_2e_1) - b_1b_3^2f_2 + (a_2b_1 + b_2^2)(b_1^2 - b_1b_2 + 2b_2e_1) - b_1^2(b_1d_2 - b_2d_1)\big)\cos\theta\sin^2\theta r^2\\
& + (2W((a_3b_1 + b_2b_3)f_1 - b_3(b_1f_2 - b_2f_1)) - b_1(3b_2b_{21} + b_3c_{21}) + c_{11}(a_3b_1 + 2b_2b_3))\cos\theta\sin\theta r - \cos\theta W\\
& (W(b_1f_2 - b_2f_1) + b_1c_{21} - b_2c_{11})) - (-a_2b_1 - b_2^2)^{3/2}(\sin\theta W(Wf_1 + c_{11}) + \sin^3\theta r^2(b_1^2d_1 + b_2^2e_1 + b_3^2f_1))\\
& - \cos^3\theta r^2((a_3b_1 + b_2b_3)^2(b_1f_2 - b_2f_1) + (a_2b_1 + b_2^2)^2(b_1b_2 + b_1e_2 - b_2e_1))\Big) = F_1(\theta, r, W),
\end{aligned}$$

$$\begin{aligned}
\dot W = \; & -\frac{\varepsilon}{(-a_2b_1-b_2^2)^{5/2}}\Big(\sqrt{-a_2b_1-b_2^2}\Big(\Big(W\Big(2(a_3b_1+b_2b_3)((a_3b_1+b_2b_3)f_2+(a_2b_3-a_3b_2)f_1-(a_2b_1+b_2^2)f_3)\\
& -(a_2b_1+b_2^2)^2b_3\Big)+(a_3b_1+b_2b_3)((a_3b_1+b_2b_3)c_{21}+(a_2b_3-a_3b_2)c11)+(a_3b_1+b_2b_3)(a_2b_1+b_2^2)(b_{21}-c_{31})\Big)\\
& \cos\theta r+\Big(2(a_3b_1+b_2b_3)^2b_3f_2+(a_3b_1+b_2b_3)(2(a_2b_3-a_3b_2)b_3f_1+(a_2b_1+b_2^2)(b_1(a_2-a_3)+b_2(2b_2-b_3+2e_2)\\
& -2b_3f_3)\Big)+(a_2b_3-a_3b_2)(a_2b_1+b_2^2)(b_1^2+2b_2e_1)-(a_2b_1+b_2^2)^2(2b_2e_3+b_3^2)\Big)\sin\theta\cos\theta r^2\Big)-(-a_2b_1-b_2^2)\\
& (\sin\theta r(W(2b_3((a_3b_1+b_2b_3)f_2+(a_2b_3-a_3b_2)f_1)-(a_2b_1+b_2^2)(a_3b_1+b_2b_3+2b_3f_3)\Big)+(a_3b_1+b_2b_3)(b_2b_{21}\\
& +b_3c_{21})-(a_2b_3-a_3b_2)(2b_1b_{21}-b_3c_{11})-(a_2b_1+b_2^2)b_3c_{31})+W\Big((a_3b_1+b_2b_3)(Wf_2+c_{21})+(a_2b_3-a_3b_2)\\
& (Wf_1+c_{11})-(a_2b_1+b_2^2)(Wf_3+c_{31})\Big)\Big)-\Big(r^2\Big((a_3b_1+b_2b_3)^3f_2-(a_2b_1+b_2^2)^3e_3\Big)+(a_2b_1+b_2^2)^2r^2((b_2-b_3\\
& +e_2)(a_3b_1+b_2b_3)+(a_2b_3-a_3b_2)e_1)+(a_3b_1+b_2b_3)^2r^2\Big((a_2b_3-a_3b_2)f_1-(a_2b_1+b_2^2)f_3)\Big)\cos\theta\\
& -\Big(-(a_2b_1+b_2^2)r^2((a_2b_1b_2+b_1^2d_2+b_2^3+b_2^2e_2+b_3^2f_2)(a_3b_1+b_2b_3)+(a_2b_3-a_3b_2)(b_1^2d_1+b_2^2e_1+b_3^2f_1))\\
& +(a_2b_1+b_2^2)^2r^2(a_3b_1b_3+b_1^2d_3+b_2^2e_3+(b_2+f_3)b_3^2)\Big)\sin^2\theta\Big)=F_2(\theta,r,W).
\end{aligned}$$

Now the integrals (4) are

$$\begin{aligned}
f_1(r,W) = \; & \frac{1}{2\pi}\int_0^{2\pi}F_1(\theta,r,W)d\theta = -\frac{r(T_2W+N_2)}{2\big(-a_2b_1-b_2^2\big)^{3/2}},\\
f_2(r,W) = \; & \frac{1}{2\pi}\int_0^{2\pi}F_2(\theta,r,W)d\theta = \frac{D_2W^2+R_2r^2+C_2W}{2\big(-a_2b_1-b_2^2\big)^{5/2}}.
\end{aligned}$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r,W) with $r>0$, namely

$$(r^*,W^*)=\left(\frac{1}{T_2}\sqrt{\frac{C_2N_2T_2-D_2N_2^2}{R_2}},-\frac{N_2}{T_2}\right),$$

if $T_2>0$, $R_2\neq 0$ and $R_2(C_2N_2T_2-D_2N_2^2)>0$. The Jacobian (5) at (r^*,W^*) is $N_2(C_2T_2-D_2N_2)/(2T_2\big(a_2b_1+b_2^2\big)^4)$, where

$$\begin{aligned}
C_2 = \; & -2\big(a_2b_1+b_2^2\big)((a_3b_1+b_2b_3)c_{21}+(a_2b_3-a_3b_2)c_{11}-(a_2b_1+b_2^2)c_{31}\big),\\
D_2 = \; & -2(a_2b_1+b_2^2)\big((a_3b_1+b_2b_3)f_2+(a_2b_3-a_3b_2)f_1-(a_2b_1+b_2^2)f_3\big),\\
N_2 = \; & -b_{21}(a_2b_1+b_2^2)+(a_3b_1+b_2b_3)c_{21}+(a_2b_3-a_3b_2)c_{11},\\
R_2 = \; & (a_2b_1+b_2^2)^2\big(a_3b_1b_3+b_1^2d_3+b_2^2e_3+(a_2b_3-a_3b_2)e_1+(a_3b_1+b_2b_3)(b_2-b_3+e_2)+(b_2+f_3)b_3^2\big)+(a_3b_1+b_2b_3)^2\\
& ((a_2b_3-a_3b_2)f_1-(a_2b_1+b_2^2)f_3)+(a_3b_1+b_2b_3)^3f_2-(a_2b_1+b_2^2)^3e_3-(a_2b_1+b_2^2)((a_2b_1b_2+b_1^2d_2+b_2^3+b_2^2e_2\\
& +b_3^2f_2)(a_3b_1+b_2b_3)+(a_2b_3-a_3b_2)(b_1^2d_1+b_2^2e_1+b_3^2f_1)\big),\\
T_2 = \; & 2\big((b_3(a_2f_1+b_2f_2)+a_3(b_1f_2-b_2f_1)\big)\big).
\end{aligned}$$

If this Jacobian is non-zero by Theorem 3 for $\varepsilon>0$ sufficiently small system (14) has a periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ tending to $(1,1,1)$ when $\varepsilon\to 0$. Therefore it is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon=0$. This completes the proof of statement (ii) of Theorem 2.

Example 2. Consider the Lotka-Volterra system

$$\dot x = x\Big(-x+y-\frac{161}{162}(x-1)^2+(z-1)^2\Big),\;\; \dot y = y(-2x+1+y+(z-1)^2),\;\; \dot z = z(-y+1). \tag{16}$$

This system in the variables (X,Y,Z) writes

$$\dot X=(X+1)\Big(\frac{1}{9}Z\varepsilon-\frac{161}{162}X^2+Z^2-X+Y\Big),\;\; \dot Y=(Y+1)(Z^2-2X+Y),\;\; \dot Z=-(Z+1)Y.$$

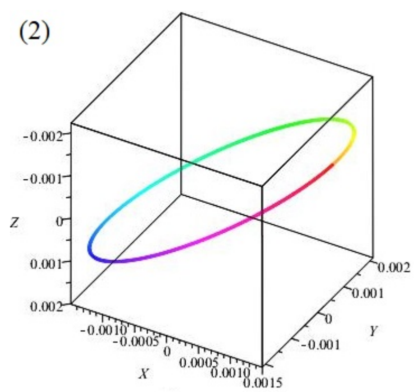
The corresponding system (3) is

$$\begin{aligned} F_1(\theta,r,W) = \; & -r\Big(\frac{1}{9}\cos^2\theta - \Big(-2W-\frac{1}{9}\Big)\sin^2\theta\Big) - \frac{163}{162}\cos\theta\sin^2\theta r^2 - \Big(-2W-\frac{2}{9}\Big)\cos\theta\sin\theta r \\ & -\frac{1}{9}\cos\theta W + \sin\theta W\Big(W+\frac{1}{9}\Big) + \frac{1}{162}\sin^3\theta r^2 + \cos^3\theta r^2 \\ F_2(\theta,r,W) = \; & -\Big(-W-\frac{2}{9}\Big)\cos\theta r + \sin\theta r\Big(-3W-\frac{2}{9}\Big) + W\Big(W+\frac{2}{9}\Big) - r^2\cos^2\theta + \frac{82}{81}r^2\sin^2\theta. \end{aligned}$$

To look for the limit cycles we must solve the system

$$f_1(r,W) = -\frac{1}{2}r\Big(2W+\frac{2}{9}\Big) = 0, \quad f_2(r,W) = W^2 + \frac{1}{162}r^2 + \frac{2}{9}W = 0.$$

This system possesses the solutions $/r,W)$ given by $(0,0)$, $(0,-2/9)$, $(\sqrt{2},-1/9)$, $(-\sqrt{2}.-1/9)$. Since r must be positive the unique admissible solution is $(\sqrt{2},-1/9)$. The determinant (5) at this solution is $2/81$. Hence system (16) has exactly one limit cycle bifurcating from $(1,1,1)$. We plot this bifurcated limit cycle for $\varepsilon = 10^{-3}$ in the next figure.



The eigenvalues of the Jacobian matrix of (f_1,f_2) at $(\sqrt{2},-1/9)$ are $\pm\sqrt{2/9}$. Therefore this limit cycle has the same kind of stability than the limit cycle of Example 1. This completes the proof of Example 2.

Now we perturb the Lotka-Volterra system (1) with the parameters given in statement (iii) of Proposition 1 satisfying the conditions of Theorem 2(iii). We translate the equilibrium point $(1,1,1)$ to the origin and system (1) becomes

$$\begin{aligned} \dot X &= (X+1)\big((Xa_{11}+Yb_{11}+Zc_{11})\varepsilon + X^2d_1 + Y^2e_1 + Z^2f_1\big), \\ \dot Y &= (Y+1)\big(X^2d_2 + Y^2e_2 + Y\varepsilon b_{21} + Z^2f_2 + Xa_2 + Yb_2 + Zc_2\big), \\ \dot Z &= (Z+1)(X^2d_3 + Y^2e_3 + Z^2f_3 - Z\varepsilon b_{21} + Xa_3 + Yb_3 - Zb_2). \end{aligned} \tag{17}$$

We write the linear part of system (17) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{-b_2^2-b_3c_2} & 0 \\ \sqrt{-b_2^2-b_3c_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this we do the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\dfrac{a_2b_2+a_3c_2}{b_2^2+b_3c_2} & -1 & 0 \\ \dfrac{a_2}{\sqrt{-b_2^2-b_3c_2}} & \dfrac{b_2}{\sqrt{-b_2^2-b_3c_2}} & \dfrac{c_2}{\sqrt{-b_2^2-b_3c_2}} \\ \dfrac{1}{b_2^2+b_3c_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

and by following the same steps as the case of Theorem 2(i), we get the system

(18)

$$
\begin{aligned}
\dot r = & \frac{\varepsilon}{c_2^2(-b_2^2-b_3c_2)^{3/2}}\Big((-b_2^2-b_3c_2)\Big(\cos\theta\sin^2\theta r^2(f_2(3b_2^2+b_3c_2)+f_1(3a_2b_2+a_3c_2)-2b_2c_2(b_2-f_3)-c_2^2(b_2\\
& +b_3))+(a_2f_1-b_2c_2+b_2f_2+c_2f_3)\sqrt{-b_2^2-b_3c_2}\sin^3\theta r^2\Big)+sqrt(-b_2^2-b_3c_2)\Big(\cos^2\theta\sin\theta r^2\Big((b_2^2+b_3c_2)\\
& (2b_2f_2-c_2^2)-b_2^3(c_2-f_2)-b_2c_2^2(b_3-e_2)+b_2f_1(3a_2b_2+2a_3c_2)+c_2^2(a_2e_1+c_2e_3)+b_2^2c_2f_3\Big)-c_2\cos\theta\sin\theta r\\
& \Big(\Big((b_2^2+b_3c_2)b_2c_2(a_2-a_3)-(-a_2b_3+a_3b_2)(2b_2(a_2f_1+c_2f_3)-2b_2^2(c_2-f_2)-c_2^2(b_2+b_3))+(a_2b_2+a_3c_2)\\
& ((b_2^2+b_3c_2)c_2-2(-a_2b_3+a_3b_2)f_1)-2(b_2^2+b_3c_2)(-a_2b_3+a_3b_2)f_2-(a_2b_2+a_3c_2)c_2(b_2(b_2-b_3+2e_2)+2a_2e_1\\
& +2c_2e_3)\Big)W-2b_2(a_2c_{11}-b_{21}c_2)+c_2(a_2b_{11}-a_3c_{11})\Big)+c_2^2\sin\theta W\Big(\Big((a_2b_2+a_3c_2)^2(a_2e_1+b_2^2+b_2e_2+c_2e_3)\\
& -(a_2b_2+a_3c_2)(b_2^2(a_2b_2+a_3c_2)-b_3c_2(a_2b_3-a_3b_2))+(b_2^2+b_3c_2)^2(a_2d_1+b_2d_2+c_2d_3)+(-a_2b_3+a_3b_2)^2(a_2f_1\\
& -b_2c_2+b_2f_2+c_2f_3)+(b_2^2+b_3c_2)(-a_2b_3+a_3b_2)a_3c_2)W+(b_2^2+b_3c_2)a_2a_{11}-(a_2b_2+a_3c_2)(a_2b_{11}+b_2b_{21})+\\
& (-a_2b_3+a_3b_2)(a_2c_{11}-b_{21}c_2)\Big)\Big)+c_2r\Big(\Big((b_2^2+b_3c_2)((a_2b_2+a_3c_2)c_2(b_2+b_3)-(b_2^2+b_3c_2)a_3c_2-2(-a_2b_3+\\
& a_3b_2)(a_2f_1-b_2c_2+b_2f_2+c_2f_3)\Big)\sin^2\theta+\Big((a_2b_2+a_3c_2)((b_2^2+b_3c_2)(b_2+2e_2)c_2+2(-a_2b_3+a_3b_2)b_2f_1)\\
& +(b_2^2+b_3c_2)(-a_2b_3+a_3b_2)(2b_2f_2-c_2^2)+2(a_2b_2+a_3c_2)^2c_2e_1-(b_2^2+b_3c_2)^2a_2c_2\Big)\cos^2\theta\Big)W+\cos^2\theta\Big((a_2b_2\\
& +a_3c_2)(b_2c_{11}-b_{11}c_2)-(b_2^2+b_3c_2)b_{21}c_2\Big)-(b_2^2+b_3c_2)\sin^2\theta(a_2c_{11}-b_{21}c_2)\Big)-c_2^2\Big(\Big((a_2b_2+a_3c_2)\Big((a_2-d_1)\\
& (b_2^2+b_3c_2)^2+(-a_2b_3+a_3b_2)((b_2^2+b_3c_2)c_2-(-a_2b_3+a_3b_2)f_1)\Big)-(a_2b_2+a_3c_2)^2(b_2^2+b_3c_2)(e_2+b_2)-(b_2^2+\\
w\quad & b_3c_2)((b_2^2+b_3c_2)^2d_2+(-a_2b_3+a_3b_2)^2f_2\Big)-(a_2b_2+a_3c_2)^3e_1\Big)W+(a_2b_2+a_3c_2)^2b_{11}-(a_2b_2+a_3c_2)\big((a_{11}-b_{21})\\
& (b_2^2+b_3c_2)+(-a_2b_3+a_3b_2)c_{11}\big)\Big)\cos\theta W+((a_2b_2+a_3c_2)(b_2^2f_1+c_2^2e_1)+(b_2^2+b_3c_2)(b_2^2f_2+c_2^2e_2))\cos^3\theta r^2\Big),\\
= & F_1(\theta,rW),\\
\dot W = & -\frac{\varepsilon}{c_2^2(-b_2^2-b_3c_2)^{3/2}}\Big(\sqrt{-b_2^2-b_3c_2}(c_2\sin\theta r(2(-a_2b_3+a_3b_2)Wf_1+c_{11})+2\cos\theta\sin\theta r^2b_2f_1\Big)+c_2\cos\theta r\\
& \big(2W((a_2b_2+a_3c_2)c_2e_1+(-a_2b_3+a_3b_2)b_2f_1)+b_2c_{11}-b_{11}c_2\big)+c_2^2W\Big(W\big((a_2b_2+a_3c_2)^2e_1+(b_2^2+b_3c_2)^2d_1+\\
& (-a_2b_3+a_3b_2)^2f_1\big)-(a_2b_2+a_3c_2)b_{11}+(b_2^2+b_3c_2)a_{11}+(-a_2b_3+a_3b_2)c_{11}\Big)+r^2\Big(\cos^2\theta(b_2^2f_1+c_2^2e_1+(b_2^2\\
& +b_3c_2)f_1)-(b_2^2+b_3c_2)f_1\Big)\Big)=F_2(\theta,rW).
\end{aligned}
$$

We compute the integrals (4) and we obtain

$$
f_1(r,W)=\frac{r(T_3W+N_3)}{2c_2\big(-b_2^2-b_3c_2\big)^{3/2}},\quad f_2(r,W)=-\frac{D_3W^2-R_3r^2+C_3W}{2c_2^2\big(-b_2^2-b_3c_2\big)^{3/2}}.
$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$
(r^*,W^*)=\left(\frac{1}{T_3}\sqrt{\frac{D_3N_3^2-C_3N_3T_3}{R_3}},-\frac{N_3}{T_3}\right),
$$

if $T_3>0$, $R_3\neq 0$ and $R_3(D_3N_3^2-C_3N_3T_3)>0$. The Jacobian (5) at (r^*,W^*) is $-N_3\big(C_3T_3-D_3N_3\big)/(2c_2^3\big(b_2^2+b_3c_2\big)^3T_3)$, where

$$
\begin{aligned}
C_3 = & -2c_2^2\big((a_2b_2+a_3c_2)b_{11}-(b_2^2+b_3c_2)a_{11}-(-a_2b_3+a_3b_2)c_{11}\big),\ R_3=-b_2^2f_1-c_2^2e_1+(b_2^2+b_3c_2)f_1,\\
N_3 = & (a_2b_2+a_3c_2)(b_2c_{11}-b_{11}c_2)-(b_2^2+b_3c_2)a_2c_{11},\ D_3=2c_2^2\Big((a_2b_2+a_3c_2)^2e_1+\big(b_2^2+b_3c_2\big)^2d_1+(-a_2b_3+a_3b_2)^2f_1\Big),\\
T_3 = & \Big(2\big(a_2b_2+a_3c_2\big)^2c_2e_1+(a_2b_2+a_3c_2)\big((b_2^2+b_3c_2)c_2(2b_2+b_3+2e_2)+2(-a_2b_3+a_3b_2)b_2f_1\big)-\big(b_2^2+b_3c_2\big)^2c_2(a_2\\
& +a_3)-\big(b_2^2+b_3c_2\big)(-a_2b_3+a_3b_2)(2a_2f_1-c_2(2b_2-c_2-2f_3))\Big).
\end{aligned}
$$

If this Jacobian is non-zero then by Theorem 3 for $\varepsilon > 0$ sufficiently small system (17) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ which tends to the equilibrium $(1,1,1)$ when $\varepsilon \to 0$. Therefore it is a periodic orbit starting at the zero-Hopf equilibrium point located at $(1,1,1)$ when $\varepsilon = 0$. This completes the proof of statement (iii) of Theorem 2.

Example 3. Consider the Lotka-Volterra system

$$\text{(19)}\quad \dot{x} = x\Big((z-1)^2 - \frac{1}{16}(x-1)^2\Big),\ \dot{y} = y\big(x-2-y+2z+(x-1)^2+(y-1)^2+(z-1)^2\big),\ \dot{z} = z\big(z-y+(x-1)^2+(z-1)^2\big).$$

This system in the new variables (X, Y, Z) writes

$$\dot{X} = (X+1)\Big(\frac{1}{9}Z\varepsilon - \frac{1}{16}X^2 + Z^2\Big),\ \dot{Y} = (Y+1)\big(X^2+Y^2+Z^2+X-Y+2Z\big),\ \dot{Z} = (Z+1)\big(X^2+Z^2-Y+Z\big).$$

The corresponding system (18) is

$$\begin{aligned} F_1(\theta, r, W) = &\ -\frac{1}{9}\cos\theta\sin\theta r(18W+1) + \Big(\frac{1}{144}(279W^2+144r^2+16W)\Big)\sin\theta - \frac{1}{144}\cos\theta\big(567W^2+72r^2 \\ &+16W\big) + 2rW(\cos^2\theta+1) - \cos^3\theta r^2 + \frac{1}{18}r, \\ F_2(\theta, r, W) = &\ \frac{1}{2}\cos\theta\sin\theta r^2 + \frac{1}{18}r(18W+1)(-\sin\theta+\cos\theta) - \frac{15}{16}W^2 - \frac{1}{4}r^2 - \frac{1}{9}W. \end{aligned}$$

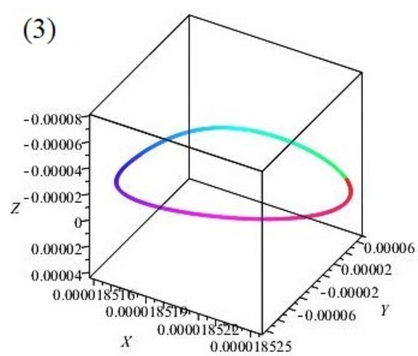
To look for the limit cycles we must solve the system

$$f_1(r, W) = \frac{1}{4}r\Big(12W + \frac{2}{9}\Big) = 0, \quad f_2(r, W) = -\frac{15}{16}W^2 - \frac{1}{4}r^2 - \frac{1}{9}W = 0.$$

This system has the solutions (r, W) given by

$$(0,0), \quad (0, -16/135), \quad (1/12, -1/54), \quad (-1/12, -1/54).$$

Since r must be positive the unique admissible solution is $(1/12, -1/54)$. The determinant of the Jacobian matrix of (f_1, f_2) at this solution $1/96$. Hence system (17) has exactly one limit cycle bifurcating from the origin. We plot this bifurcated limit cycle for $\varepsilon = 10^{-3}$ in the next figure.



The eigenvalues of the Jacobian matrix of (f_1, f_2) at $(1/12, -1/54)$ are $(-11 \pm \sqrt{743}\,i)/288$. So this limit cycle for the differential system (19) is stable. This completes the proof of Example 3.

Now we perturb the Lotka-Volterra system (1) with the parameters given in statement (iv) of Proposition 1 satisfying the conditions of Theorem 2(iv). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates and system (1) becomes

$$\text{(20)}\quad \begin{aligned} \dot{X} &= (X+1)\Big((-2Xc_{31} + Yb_{11})\varepsilon + X^2d_1 + Y^2e_1 + Z^2f_1 - Xc_3 + Zc_1\Big), \\ \dot{Y} &= (Y+1)\Big(X^2d_2 + Y^2e_2 + Y\varepsilon b_{21} + Z^2f_2 + Xa_2 + Zc_2\Big), \\ \dot{Z} &= (Z+1)\Big((Yb_{31} + Zc_{31})\varepsilon + X^2d_3 + Y^2e_3 + Z^2f_3 + Xa_3 + Zc_3\Big). \end{aligned}$$

We write the linear part of system (20) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{-a_3c_1 - c_3^2} & 0 \\ \sqrt{-a_3c_1 - c_3^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this we do the change of variables

$$\begin{pmatrix} u\\ v\\ w\end{pmatrix} = \begin{pmatrix} -\dfrac{c_3}{c_1\sqrt{-a_3c_1-c_3^2}} & 0 & \dfrac{1}{\sqrt{-a_3c_1-c_3^2}}\\ \dfrac{1}{c_1} & 0 & 0\\ a_2c_3-a_3c_2 & a_3c_1+c_3^2 & -a_2c_1-c_2c_3\end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix},$$

and by following the same steps as the case of Theorem 2(i) we get the system

(21)

$$\begin{aligned}
\dot r = {}& \frac{\varepsilon}{c_1(-a_3c_1-c_3^2)^{7/2}}\Big((-a_3c_1-c_3^2)^{3/2}(((a_3c_1+c_3^2)(a_2b_{11}c_1+2b_{11}c_2c_3-b_{31}c_1c_2-3c_1c_3c_{31})+2W(a_2c_1e_1-\\
&c_1c_2e_3+2c_2c_3e_1))\cos\theta\sin\theta r+(c_1(a_2(c_1e_3-c_3e_1)(a_2c_1+2c_2c_3)+a_3(c_1c_3+c_1f_3-c_3f_1)(a_3c_1+2c_3^2)+\\
&c_3^2(c_2^2e_3+c_3^3+c_3^2f_3))-c_2^2c_3^3e_1-c_3^5f_1)\cos^3\theta r^2)-(-a_3c_1-c_3^2)^{5/2}(3c_2^2c_3e_1+3c_3^3f_1+c_1(2a_2c_2e_1+(c_1^2-\\
&c_1c_3+2c_3f_1)a_3-(c_1d_3-c_3^2-c_3d_1)c_1-c_2^2e_3-c_3^3-c_3^2f_3))r^2\cos\theta\sin^2\theta-(-c_1(a_3(a_2c_1(a_3c_1+2c_3^2)(a_2c_1e_1\\
&-2c_1c_2e_3+4c_2c_3e_1)-a_3c_1(a_3c_1(a_3c_1^2-a_3c_1f_1-c_1^2c_3+5c_1c_3^2+2c_1c_3f_3-6c_3^2f_1)-c_1c_3(5c_1c_3^2-2c_2^2e_3-9c_3^3\\
&-6c_3^2f_3)-3c_2^2c_3^2e_1-12c_3^4f_1)+c_1c_3^3(5c_1c_3^2-4c_2^2e_3-7c_3^3-6c_3^2f_3)+6c_2^2c_3^4e_1+10c_3^6f_1)+c_3^4(c_1(a_2^2e_1-2a_2c_2e_3\\
&+c_3^3)+2c_3(2a_2c_2e_1-c_2^2e_3-c_3^3-c_3^2f_3)))-6e_1c_2^2c_3^6-3c_3^8f_1)\cos^2\theta\sin\theta r^2+(-a_3c_1-c_3^2)\Big(r((2W(a_2c_1^2e_3-\\
&a_2c_1c_3e_1-a_3c_1c_2e_1+c_1c_2c_3e_3-2c_2c_3^2e_1)-(a_3c_1+c_3^2)(a_2b_{11}c_1c_3-a_2b_{31}c_1^2+a_3b_{11}c_1c_2-3a_3c_1^2c_{31}+2b_{11}c_2c_3^2\\
&-b_{31}c_1c_2c_3-3c_1c_3^2c_{31}))\cos^2\theta+2Wc_2e_1(a_3c_1+c_3^2)+(a_3c_1+c_3^2)^2(b_{11}c_2-2c_1c_{31}))+\sin\theta W(b_{11}(a_3c_1+c_3^2)\\
&+We_1)\Big)+(-a_3c_1-c_3^2)^3(c_1^2d_1+c_2^2e_1+c_3^2f_1)r^2\sin^3\theta+\sqrt{-a_3c_1-c_3^2}((-b_{11}c_3+b_{31}c_1)(a_3c_1+c_3^2)+W(c_1e_3\\
&-c_3e_1))\cos\theta W\Big)=F_1(\theta,r,W),\\
\dot W = {}& \frac{\varepsilon}{(-a_3c_1-c_3^2)^{5/2}}\Big(W(a_3c_1+c_3^2)(a_2(b_{11}c_3-b_{31}c_1)-a_3(b_{11}c_2-b_{21}c_1)+b_{21}c_3^2-b_{31}c_2c_3)-W^2(a_2(c_1e_3-\\
&c_3e_1)-a_3(c_1e_2-c_2e_1)+c_2c_3e_3-c_3^2e_2)-(-a_3c_1-c_3^2)^{3/2}\sin\theta\cos\theta r^2(2c_3(c_3(c_2^2(a_2e_1-c_2e_3)+c_3(c_3\\
&(a_2f_1+c_2(c_2-c_3-f_3)+c_3f_2)-c_2(a_3f_1-c_2e_2)))-a_3c_2^3e_1)+c_1(c_3(2c_2(a_2^2e_1-2a_2c_2e_3-a_3^2f_1+a_3c_2e_2)\\
&+c_3(2a_2(a_3f_1+c_2c_3+c_2e_2-c_3^2-c_3f_3)-a_3(3c_2c_3+2c_2f_3-4c_3f_2)))-a_3c_2^2(2a_2e_1-3c_3^2)+c_1(a_3c_1(a_2(a_2\\
&-a_3+c_3)-a_3c_2)-a_2(a_2(2c_2e_3-c_3^2)-a_3(2c_2c_3+2c_2e_2-3c_3^2-2c_3f_3)-c_3^3)+a_3((c_2^2-c_2c_3+2c_3f_2)a_3\\
&-c_2c_3^2))))-(-a_3c_1-c_3^2)\Big(r^2((a_3c_1^4(a_2d_3-a_3d_2)+c_1^3(a_2^2(a_2e_3-a_3e_2)-a_3^2(a_3f_2-c_2d_1)+a_3c_3(c_2d_3-\\
&2c_3d_2)-a_2(a_3^2(2c_2-2c_3-f_3)+c_3(a_3d_1-c_3d_3)))-c_1^2(-a_3^3c_2f_1+a_3^2(c_2^2(2c_3+e_2)-c_2c_3(2c_3+f_3)+c_3\\
&(a_2f_1+4c_3f_2))-a_3(a_2(c_2^2e_3-2c_2c_3(2c_3+e_2)+c_3^2(4c_3+3f_3))+c_2(a_2^2e_1+c_3^2d_1))+c_3(a_2^3e_1-a_2^2(3c_2e_3-\\
&c_3e_2)+c_3^2(a_2d_1-c_2d_3+c_3d_2)))-c_1(-a_3^2c_2(c_2^2e_1+3c_3^2f_1)-a_3c_3(e_1c_2^2a_2-3c_3^2f_1a_2+e_3c_2^3-5c_3^3f_2-c_2c_3\\
&(c_2(4c_3+3e_2)-4c_3^2-3c_3f_3))-2a_2c_3^2(2c_2^2e_3+c_3^3+f_3c_3^2-c_2c_3(c_3+e_2))+2a_2^2c_2c_3^2e_1)-2c_3^2(c_2c_3(c_2(a_2e_1\\
&+c_3^2+c_3e_2)-a_3c_3f_1-c_2^2e_3-c_3^3-f_3c_3^2)-a_3c_2^3e_1+c_3^3f_1a_2+c_3^4f_2))\cos^2\theta+(a_3c_1+c_3^2)(a_3a_2c_1^2c_2+c_3^4f_2\\
&+c_2^2c_3^3-c_2c_3^4-c_2c_3^3f_3-e_3c_2^3c_3+e_2c_2^2c_3^2+c_3^3f_1a_2-d_3a_2c_1^3+c_1^2c_3^2d_2-a_2c_1c_3^3+a_3c_1^3d_2-a_3c_2^3e_1-a_2a_3c_1^2c_3\\
&-f_1a_3c_2c_3^2+f_2a_3c_1c_3^2-a_3c_1^2c_2d_1+a_3c_1c_2^2c_3-a_3c_1c_2c_3^2+e_1c_2^2a_2c_3+a_2c_1^2c_3d_1-c_1^2c_2c_3d_3-a_2e_3c_1c_2^2-\\
&a_2f_3c_1c_3^2+a_2c_1c_2c_3^2+a_3c_1e_2c_2^2))+\sin\theta r(W(c_1((a_3c_1-2c_2e_3+c_3^2)a_2+a_3c_2(c_3+2e_2))+_{c}2(c_3(2a_2e_1-\\
&2c_2e_3+c_3^2+2c_3e_2)-2a_3c_2e_1))+(a_3c_1+c_3^2)(a_2(b_{11}c_2c_3-b_{31}c_1c_2-3c_1c_3c_{31})-a_3c_2(b_{11}c_2-(b_{21}+2c_{31})\\
&c_1)+c_2c_3(c_3(b_{21}-c_{31})-b_{31}c_2)))\Big)-\sqrt{-a_3c_1-c_3^2}((c_1^2(2a_2^2e_3-2a_2a_3e_2-a_3^2c_2)-2c_1(a_2(c_3(a_2e_1-2c_2e_3\\
&+c_3e_2)-a_3c_2e_1)+a_3c_2c_3(c_3+e_2))-c_2c_3(c_3(2a_2e_1-2c_2e_3+c_3^2+2c_3e_2)-2a_3c_2e_1))W-(a_3c_1+c_3^2)\\
&(a_2c_1+c_2c_3)(a_2(b_{11}c_3-b_{31}c_1)-a_3(b_{11}c_2-b_{21}c_1+c_1c_{31})+c_3^2(b_{21}-c_{31})-b_{31}c_2c_3))\cos\theta r\Big)=F_2(\theta,r,W),
\end{aligned}$$

Then

$$f_1(r,W)=\frac{(T_4W-N_4)r}{2\big(-a_3c_1-c_3^2\big)^{5/2}},\quad f_2(r,W)=\frac{2D_4W^2-R_4r^2-2C_4W}{2\big(-a_3c_1-c_3^2\big)^{3/2}}.$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$(r^*,W^*)=\left(\frac{1}{T_4}\sqrt{\frac{2D_4N_4^2-2C_4N_4T_4}{R_4}},\frac{N_4}{T_4}\right),$$

if $T_4>0$, $R_4\neq 0$ and $R_4(2D_4N_4^2-2C_4N_4T_4)>0$. The Jacobian (5) at (r^*,W^*) is $-N_4\big(C_4T_4-D_4N_4\big)/(T_4\big(a_3c_1+c_3^2\big)^4)$, where

$$\begin{aligned}
N_4 =&\ \big(a_3c_1+c_3^2\big)\big(a_2b_{11}c_3-a_2b_{31}c_1-a_3b_{11}c_2+a_3c_1c_{31}-b_{31}c_2c_3+c_3^2c_{31}\big),\\
T_4 =&\ 2(a_2c_1e_3-a_2c_3e_1+a_3c_2e_1+c_2c_3e_3),\\
D_4 =&\ \frac{a_2c_1e_3-a_2c_3e_1-a_3c_1e_2+a_3c_2e_1+c_2c_3e_3-c_3^2e_2}{a_3c_1+c_3^2},\\
C_4 =&\ a_2b_{11}c_3-a_2b_{31}c_1-a_3b_{11}c_2+a_3b_{21}c_1+b_{21}c_3^2-b_{31}c_2c_3,\\
R_4 =&\ c_1\Big(a_3^3c_1(c_2f_1-c_1f_2)-a_3^2\big((a_2c_3f_1-c_2^2e_2-c_2c_3f_3+2c_3^2f_2-2a_2f_3c_1)c_1+c_1^2(a_2f_3-c_1d_2+c_2d_1)+c_2(c_2^2e_1\\
&-c_3^2f_1)\big)-a_3\big(c_1^2(a_2^2e_2-a_2c_3d_1+c_2c_3d_3-2c_3^2d_2)-c_1(a_2^2c_2e_1-a_2(c_2^2e_3+2c_2c_3e_2-c_3^2f_3)-c_2c_3^2d_1)+a_2\\
&(c_1^3d_3-3c_2^2c_3e_1+c_3^3f_1)+c_3(c_2^3e_3-c_2^2c_3e_2-c_2c_3^2f_3+c_3^3f_2)\big)+a_2^2\big(a_2c_1^2e_3-2c_2c_3^2e_1\big)-c_1c_3\big(a_2^3e_1-3a_2^2c_2e_3\\
&+a_2^2c_3e_2-a_2c_3^2d_1+c_2c_3^2d_3-c_3^3d_2\big)-a_2c_3^2\big(c_1^2d_3-2c_2^2e_3+2c_2c_3e_2\big)\Big).
\end{aligned}$$

By applying Theorem 3 as in the case (i) for $\varepsilon>0$ sufficiently small system (20) has one periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ which tends to the equilibrium $(1,1,1)$ when $\varepsilon\to 0$. Hence it is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon=0$. This completes the proof of statement (iv) of Theorem 2.

Example 4. Consider the Lotka-Volterra system

(22) $\dot x=x\big(-x+z+(y-1)^2+(z-1)^2\big),\ \dot y=y\big(1-z+(x-1)^2+(y-1)^2\big),\ \dot z=z\big(-2x+1+z+(x-1)^2+(z-1)^2\big).$

This system in the variables (X,Y,Z) writes

$$\dot X=\big(X+1\big)\Big(\frac{1}{9}Y\varepsilon+Y^2+Z^2-X+Z\Big),\ \dot Y=\big(Y+1\big)\big(X^2+Y^2-Z\big),\ \dot Z=\big(Z+1\big)\big(X^2+Z^2-2X+Z\big).$$

The corresponding system associated to system (21) satifies

$$\begin{aligned}
F_1(\theta,r,W)=&\ 2r^2\big(4\cos\theta^2+1\big)\sin\theta+\frac{1}{9}r(18W-1)\big(\sin\theta-\cos\theta\big)^2+\frac{1}{9}W(9W-1)\big(\sin\theta-\cos\theta\big),\\
F_2(\theta,r,W)=&\ \frac{2}{9}W-3W^2+2\sin\theta\cos\theta r^2-r^2\big(-4\cos\theta^2+6\big)-\sin\theta r\Big(7W-\frac{2}{9}\Big)-\Big(-5W+\frac{2}{9}\Big)\cos\theta r.
\end{aligned}$$

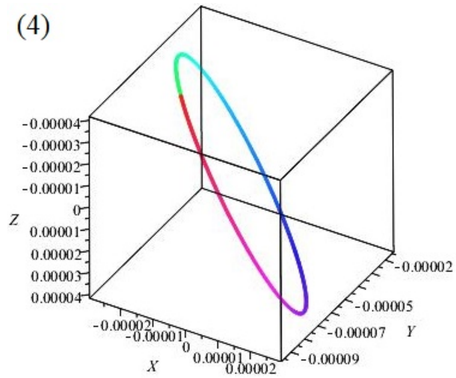
To look for the limit cycles we must solve the system

$$f_1(r,W)=\frac{1}{2}\Big(4W-\frac{2}{9}\Big)r,\quad f_2(r,W)=-3W^2-4r^2+\frac{2}{9}W.$$

This system has the solutions

$$(0,0),\quad (0,2/27),\quad (1/36,1/18),\quad (-1/36,1/18),$$

The unique admissible root is $(1/36,1/18)$. The determinant of the Jacobian matrix (f_1,f_2) is $1/81$. Hence system (22) has exactly one limit cycle bifurcating from the origin. We plot this limit cycle for $\varepsilon=10^{-3}$ in the following figure.



The eigenvalues of the Jacobian matrix of $f_1, f_2)$ at $(1/36, 1/18)$ are $() - 1 \pm \sqrt{3}\, i)/18$. So this limit cycle is stable. This completes the proof of Example 4.

Now we perturb the Lotka-Volterra system (1) with the parameters given in statement (v) of Proposition 1 satisfying the conditions of Theorem 2(v). We translate the equilibrium point $(1, 1, 1)$ to the origin of coordinates, then system (1) becomes

$$
\begin{aligned}
\dot X &= (X+1)\Big(\big(-2Xc_{31}+Yb_{11}\big)\varepsilon + X^2d_1 + Y^2e_1 + Z^2f_1 - Xc_3 + Zc_1\Big),\\
\dot Y &= (Y+1)\Big(\Big(\frac{-c_2c_{31}X}{c_1} + b_{21}Y\Big)\varepsilon - \frac{c_2c_3X}{c_1} + c_2Z + d_2X^2 + e_2Y^2 + f_2Z^2\Big),\\
\dot Z &= (Z+1)\big(X^2d_3 + Y^2e_3 + Z^2f_3 + c_{31}Z\varepsilon + Xa_3 + Yb_3 + Zc_3\big).
\end{aligned} \tag{23}
$$

As usual we write the linear part of system (23) with $\varepsilon = 0$ at $(0, 0, 0)$ in its real Jordan normal form

$$
\begin{pmatrix}
0 & -\sqrt{-a_3c_1 - b_3c_2 - c_3^2} & 0\\
\sqrt{-a_3c_1 - b_3c_2 - c_3^2} & 0 & 0\\
0 & 0 & 0
\end{pmatrix}.
$$

For this we do the change of variables

$$
\begin{pmatrix} u\\ v\\ w \end{pmatrix} =
\begin{pmatrix}
\dfrac{a_3c_1 + c_3^2}{c_1\big(a_3c_1 + b_3c_2 + c_3^2\big)b_3} & \dfrac{1}{a_3c_1 + b_3c_2 + c_3^2} & 0\\
\dfrac{c_3}{c_1\sqrt{-a_3c_1 - b_3c_2 - c_3^2}\, b_3} & 0 & -\dfrac{1}{b_3\sqrt{-a_3c_1 - b_3c_2 - c_3^2}}\\
-\dfrac{c_2}{c_1\big(a_3c_1 + b_3c_2 + c_3^2\big)} & \dfrac{1}{a_3c_1 + b_3c_2 + c_3^2} & 0
\end{pmatrix}
\begin{pmatrix} X\\ Y\\ Z \end{pmatrix},
$$

and by following the same steps as the case (i) we obtain the system

(24)

$$
\begin{aligned}
\dot W = {}& \frac{\varepsilon}{c_1\big(-a_3c_1 - b_3c_2 - c_3^2\big)^{3/2}}\Big(\sqrt{-a_3c_1 - b_3c_2 - c_3^2}\big(\sin\theta W r b_3\big(2b_3c_2c_3f_1 - c_1c_3(2b_3f_2 - c_2c_3) + c_1^2c_2(a_3 + b_3)\big)\\
& - \cos\theta\sin\theta r^2 b_3^2\big(c_2(c_1^2 + 2c_3f_1) - c_1(c_2^2 + 2c_3f_2)\big)\big) - \big(c_1^3(a_3^2e_2 + b_3^2d_2) - c_1c_3^2(2a_3c_2e_1 - b_3^2f_2 - c_3^2e_2) - c_1^2\\
& (a_3^2c_2e_1 - 2a_3c_3^2e_2 + b_3^2c_2d_1) - c_2c_3^2(b_3^2f_1 + c_3^2e_1)\big)W^2 - \big((a_3c_1 + c_3^2)(-b_{11}c_2 + b_{21}c_1) - b_3c_1c_2c_{31}\big)W - r^2b_3^2\big(\big((c_1f_2 -\\
& c_2f_1)(a_3c_1 + b_3c_2) + c_1(c_2^2e_2 + 2c_3^2f_2) - c_2(c_2^2e_1 + 2c_3^2f_1) + c_1^2(c_1d_2 - c_2d_1)\big)\cos\theta^2 - (c_1f_2 -_c 2f_1)(a_3c_1 + b_3c_2 + c_3^2)\big)\\
& - \cos\theta r b_3\big(\big(2c_2(c_1e_2 - c_2e_1)(a_3c_1 + c_3^2) - 2b_3\big(c_1^2(c_1d_2 - c_2d_1) + c_3^2(c_1f_2 - c_2f_1)\big)\big)W - c_2(b_{11}c_2 - b_{21}c_1 - c_1c_{31})\big)\Big)\\
= {}& F_2(\theta, r, W),
\end{aligned}
$$

$$
\begin{aligned}
\dot r = \; & \frac{\varepsilon}{c_1\big(-a_3c_1-b_3c_2-c_3^2\big)^{5/2}}\Big(\big(-a_3c_1-b_3c_2-c_3^2\big)^{3/2}\big(-\big(W\big(a_3c_1\big(b_3(c_1^2-c_1c_2-c_1c_3+2c_3f_1)+2c_2(c_1e_3-\\
& c_3e_1)\big)-b_3c_1\big(c_3(b_3c_2-2b_3f_2-2c_1d_1)-c_3^2(c_1-c_2-c_3-2f_3)+2c_1^2d_3\big)+2c_3^2(2b_3c_3f_1+c_1c_2e_3-c_2c_3e_1)\big)\\
& -c_3(b_{11}c_2-3c_1c_{31})\big)r\cos\theta\sin\theta+(a_3c_1(c_1^2-c_1c_3+2c_3f_1)+b_3c_1(c_2^2-c_2c_3+2c_3f_2)-c_1(c_2^2e_3+c_3^3+c_3^2f_3)\\
& +c_3(c_3+d_1)c_1^2-d_3c_1^3+c_3c_2^2e_1+3c_3^3f_1)r^2\cos^2\theta\sin\theta b_3+(c_1c_3+c_1f_3-c_3f_1)(a_3c_1+b_3c_2+c_3^2)r^2\sin^3\theta b_3\big)\\
& -\big(-a_3c_1-b_3c_2-c_3^2\big)\Big(r(((a_3c_1+c_3^2)(2c_2e_1(a_3c_1+c_3^2)-b_3(c_1^2c_3+2c_1^2d_1-2c_1c_2e_2-c_1c_3^2-2c_1c_3f_3+4c_3^2f_1))\\
& -b_3^2(2c_1^3d_2+c_1^2c_2c_3+2c_2c_3^2f_1-c_1c_3(c_2c_3+2c_2f_3-2c_3f_2)))W+b_3c_1c_2(b_{21}-2c_{31})+(b_{11}c_2-3c_1c_{31})(a_3c_1+\\
& c_3^2))\cos^2\theta+r((c_1^2-c_1c_3-2c_1f_3+2c_3f_1)(b_3c_3(a_3c_1+c_3^2)+b_3^2c_2c_3)W+c_1c_{31}(a_3c_1+b_3c_2+c_3^2))+(\cos\theta W\\
& \big(\big((a_3c_1+c_3^2)\big(b_3^2(c_1^2d_1+c_3^2f_1)+b_3c_1e_2(a_3c_1+c_3^2)+e_1(a_3c_1+c_3^2)^2\big)+c_1b_3^3(c_1^2d_2+c_3^2f_2)\big)W+(a_3c_1+c_3^2)(b_3c_1\\
& (b_{21}+2c_{31})+b_{11}(a_3c_1+c_3^2))+b_3^2c_1c_2c_{31}\big)\big)/(b_3)+b_3\cos^3\theta r^2((c_1^2d_1+c_2^2e_1+c_3^2f_1)(a_3c_1+c_3^2)+b_3c_1(c_1^2d_2+\\
& c_2^2e_2+c_3^2f_2))\Big)+\big(-a_3c_1-b_3c_2-c_3^2\big)^2(b_3c_1(c_2-f_2)+a_3c_1(c_1-f_1)+2c_1c_3(c_3+f_3)-c_1^2c_3-3c_3^2f_1)r^2\sin^2\theta\\
& \cos\theta b_3+(\sqrt{-a_3c_1-b_3c_2-c_3^2}\sin\theta W\big(\big(\big((c_1^3d_3-c_1^2c_3d_1+c_1c_3^2f_3-c_3^3f_1)(b_3^3c_2+b_3^2(a_3c_1+c_3^2))+b_3c_2a_3^2c_1^2\\
& (c_1e_3-2c_3e_1)+(c_1e_3-c_3e_1)\big((a_3c_1+c_3^2)^3+b_3c_2(4a_3c_1c_3^2+c_3^4)\big)\big)W-b_{11}c_3(a_3c_1+c_3^2)^2-b_3c_3(3c_1c_{31}(a_3c_1\\
& +b_3c_2+c_3^2)+b_{11}c_2(2a_3c_1+c_3^2))\big)\big)/(b_3)\Big)=F_1(\theta,r,W),
\end{aligned}
$$

For this differential system we compute the integrals (4) and we obtain

$$
f_1(r,W)=-\frac{r\big(T_5W+N_5\big)}{2c_1\big(-a_3c_1-b_3c_2-c_3^2\big)^{3/2}},\qquad f_2(r,W)=-\frac{D_5W^2-R_5r^2+C_5W}{2c_1\big(-a_3c_1-b_3c_2-c_3^2\big)^{3/2}}.
$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$
(r^*,W^*)=\left(\frac{1}{T_5}\sqrt{\frac{D_5N_5^2-C_5N_5T_5}{R_5}},-\frac{N_5}{T_5}\right),
$$

if $T_5>0$, $R_5\neq 0$ and $R_5(D_5N_5^2-C_5N_5T_5)>0$. The Jacobian (5) at (r^*,W^*) is $N_5\big(C_5T_5-D_5N_5\big)/(2\,T_5\big(a_3c_1+b_3c_2+c_3^2\big)^3c_1^2)$, where

$$
\begin{aligned}
D_5 = \; & \big(2b_3^2\big(c_1^2(c_1d_2-c_2d_1)+c_3^2(c_1f_2-c_2f_1)\big)+(c_1e_2-c_2e_1)\big(2a_3c_1\big(a_3c_1+2c_3^2\big)+2c_3^4\big)\big),\\
N_5 = \; & \big(a_3c_1+c_3^2\big)(b_{11}c_2-c_1c_{31})+b_3b_{21}c_1c_2,\\
R_5 = \; & b_3^2\big(c_2c_1(b_3f_2-c_2e_2)+a_3c_1(c_1f_2-c_2f_1)-c_1^2(c_1d_2-c_2d_1)-c_2^2(b_3f_1-c_2e_1)\big),\\
T_5 = \; & \Big(b_3c_1\big(c_1c_3-c_3^2-2c_3f_3-2c_1d_1+2c_2e_2\big)\big(a_3c_1+c_3^2\big)+2c_2e_1\big(a_3c_1+c_3^2\big)^2-b_3^2\big(2c_1^3d_2-c_3c_2\big(c_1^2+2c_3f_1\big)\\
& +c_3c_1(c_2c_3+2c_2f_3+2c_3f_2)\big)\Big),\;\; C_5=\big(2\big(a_3c_1+c_3^2\big)(b_{21}c_1-b_{11}c_2)-2b_3c_1c_2c_{31}\big),
\end{aligned}
$$

Finally from Theorem 3 for $\varepsilon>0$ sufficiently small system (23) has a periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ which tends to the equilibrium $(1,1,1)$ when $\varepsilon\to 0$. So it is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon=0$. This completes the proof of statement (v) of Theorem 2.

Example 5. Consider the Lotka-Volterra system

$$
\dot x=x\big(-x-1+2z+(x-1)^2+(y-1)^2+2(z-1)^2\big),\;\dot y=y\big(-2x-2+4z\big),\;\dot z=z\Big(-2x+\frac{1}{2}+\frac{1}{2}y+z\Big). \tag{25}
$$

This system in the variables (X,Y,Z) writes

$$
\dot X=(X+1)\big((Y-2X)\varepsilon+X^2+Y^2+2Z^2-X+2Z\big),\;\dot Y=(Y+1)\big((Y-2X)\varepsilon-2X+4Z\big),\;\dot Z=(Z+1)\Big(Z\varepsilon-2X+\frac{1}{2}Y+Z\Big).
$$

The corresponding system associated to system (24) is

$$
\begin{aligned}
F_1(\theta,r,W)= \; & -\frac{1}{2}(31W+2)r\cos\theta\sin\theta+4r^2\cos^2\theta\sin\theta-\frac{1}{2}r(93W+2)\cos^2\theta-\frac{1}{2}r(-3W-2)-\cos\theta W\\
& \Big(-\frac{63}{2}W+2\Big)+\frac{33}{2}\cos\theta^3r^2-\frac{1}{2}r^2\sin^2\theta\cos\theta+\sin\theta W\Big(-\frac{11}{2}W+8\Big),\\
F_2(\theta,r,W)= \; & -2\sin\theta Wr+21W^2-W-\frac{1}{8}r^2\big(-80\cos^2\theta-8\big)-30\cos\theta rW.
\end{aligned}
$$

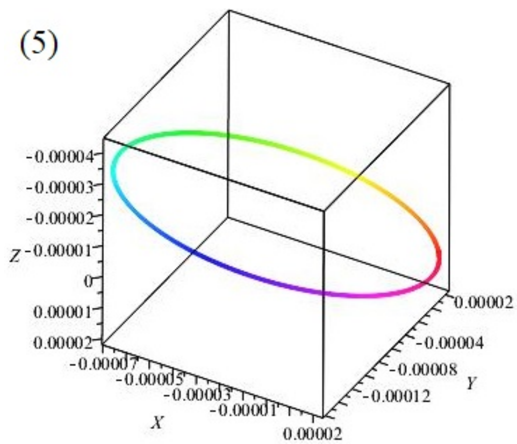
To look for the limit cycles we must solve the system

$$f_1(r,W) = -\frac{1}{4}r\big(87W-2\big),\quad f_2(r,W) = 21W^2+6r^2-W.$$

This system has the solutions

$$(0,0),\quad (0,1/21),\quad (\sqrt{15}/87, 2/87),\quad (-\sqrt{15}/87, 2/87).$$

The unique admissible root is $(\sqrt{15}/87, 2/87)$. The determinant of the Jacobian matrix (f_1, f_2) is $15/29$. Hence system (25) has exactly one limit cycle bifurcating from the origin. We plot this limit cycle for $\varepsilon = 10^{-3}$ in the following figure.



The eigenvalues of the Jacobian matrix of (f_1, f_2) at $(\sqrt{15}/87, 2/87)$ are $(-1 \pm \sqrt{1739}\, i)/58$. So this limit cycle is stable. This completes the proof of Example 5.

Now we perturb the Lotka-Volterra system (1) with the parameters given in statement (vi) of Proposition 1 satisfying the conditions of Theorem 2(vi). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates and system (1) becomes

$$\begin{aligned}
\dot{X} &= (X+1)\big((-2b_{21}-2c_{31})X\varepsilon + (-b_2-c_3)X + b_1Y + c_1Z + d_1X^2 + e_1Y^2 + f_1Z^2\big),\\
\dot{Y} &= (Y+1)\left(\left(\frac{b_{21}Y + b_{21}c_1Z}{b_1}\right)\varepsilon + a_2X + b_2Y + \frac{b_2c_1Z}{b_1} + d_2X^2 + e_2Y^2 + f_2Z^2\right),\\
\dot{Z} &= (Z+1)\left(\left(\frac{2b_1c_{31}Y}{c_1} + c_{31}Z\right)\varepsilon + a_3X + \frac{b_1c_3Y}{c_1} + c_3Z + d_3X^2 + e_3Y^2 + f_3Z^2\right).
\end{aligned}\tag{26}$$

We write the linear part of system (26) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\sqrt{A_1} & 0\\ \sqrt{A_1} & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

where $A_1 = -a_2b_1 - a_3c_1 - b_2^2 - 2b_2c_3 - c_3^2$. For this we do the change of variables

$$\begin{pmatrix} u\\ v\\ w\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0\\ \dfrac{b_2+c_3}{\sqrt{A_1}} & -\dfrac{b_1}{\sqrt{A_1}} & -\dfrac{c_1}{\sqrt{A_1}}\\ \dfrac{a_2b_1c_3 - a_3b_2c_1}{A_1c_1} & \dfrac{b_1(a_3c_1+b_2c_3+c_3^2)}{A_1c_1} & -\dfrac{a_2b_1+b_2^2+b_2c_3}{A_1}\end{pmatrix}\begin{pmatrix} X\\ Y\\ Z\end{pmatrix},$$

and by following the same steps as the case (i) we get the system

(27)

$$\dot{r} = -\frac{\varepsilon}{b_1^2c_1^2\left(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2\right)^{5/2}}\Big(c_1\sin\theta\cos\theta r\big(\big(b_1^3\big(c_1(2a_3(a_2f_1-c_3f_2)-a_2c_3(b_2+c_3+2f_3))$$
$$+a_2c_1^2(a_2-a_3)+2c_3(b_2+c_3)(2a_2f_1-b_2f_2-c_3f_2)\big)-b_1c_1^2\big(2a_2e_1\big(a_3c_1+c_3^2\big)+2a_2b_2(3b_2e_1-c_1e_3+4c_3e_1)$$
$$-b_2(b_2+c_3+2e_2)\big((b_2+c_3)^2+a_3c_1\big)\big)-b_1^2\big(-a_2c_1^2\big(b_2(2b_2+3c_3+2e_2)+c_3^2\big)-a_3c_1(b_2c_1c_3(8f_1-3)$$
$$-\big(a_3c_1+b_2^2\big)(c_1-2f_1)-2c_3^2(c_1-3f_1)+c_1(a_2c_1-2c_3f_3)\big)+b_2^3c_3(c_1-4f_1)+b_2^2c_3(3c_1c_3+2c_1f_3-12c_3f_1)$$
$$+b_2c_3^2(3c_1c_3+4c_1f_3-12c_3f_1)+c_1c_3^3(c_3+2f_3)+2a_2^2c_1^2e_1-4c_3^4f_1\big)-2f_2b_1^4c_3a_2-2b_2c_1^2(2b_2e_1-c_1e_3+2c_3e_1)$$
$$(a_3c_1+b_2^2+2b_2c_3+c_3^2)\big)W-b_1^2c_1(3b_2b_{21}+4b_2c_{31}+3b_{21}c_3+3c_3c_{31})(a_2b_1+a_3c_1+b_2^2+2b_2c_3+c_3^2)\big)+$$
$$\sqrt{-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2}\Big(\cos\theta\sin^2\theta r^2\big(b_1^3\big(c_3^2(a_2c_1+2b_2f_2+2c_3f_2)+a_2(a_2-b_2-c_3)c_1^2+2a_3c_1$$
$$c_3f_2\big)-b_1^2\big(a_3c_1\big(b_2c_1^2-2b_2c_1c_3+4b_2c_3f_1+c_1^2c_3-3c_1c_3^2-2c_1c_3f_3+4c_3^2f_1\big)-b_2(2a_2c_1^2c_3+2a_2c_1^2e_2-3c_1^2c_3^2$$
$$+4c_1c_3^3+2c_1c_3^2f_3-6c_3^3f_1)-b_2^2(3a_2c_1^2-3c_1^2c_3+2c_1c_3^2-3c_3^2f_1)+c_1^2(a_2^2e_1+a_3^2f_1+b_2^3+c_3^3)-c_3^2(2c_1c_3+2c_1f_3$$
$$-3c_3f_1)-a_3^2c_1^3\big)-b_1b_2c_1^2(b_2(4a_2e_1-a_3c_1-2c_3^2-2c_3e_2)-2b_2^2(b_2+2c_3+e_2)-2a_2(c_1e_3-2c_3e_1))-b_2^2c_1^2$$
$$(b_2+c_3)(3b_2e_1-2c_1e_3+3c_3e_1)\big)+c_1\sin^2\theta r((b_1^3(a_2c_1c_3+2a_3c_1f_2+2b_2c_3f_2+2c_3^2f_2)+b_1c_1^2(b_2(2a_2e_1-a_3c_1$$
$$-c_3^2-2c_3e_2)-b_2^2(b_2+2c_3+2e_2)-2a_2(c_1e_3-c_3e_1))-b_1^2(c_1^2(a_2b_2+2a_2e_2-a_3c_3-2a_3f_3)+c_1(b_2+c_3)$$
$$(2a_3f_1-b_2c_3-c_3^2-2c_3f_3)+2c_3f_1(b_2+c_3)^2)+2b_2c_1^2(b_2+c_3)(b_2e_1-c_1e_3+c_3e_1))W+b_1^2c_1(a_2b_1+a_3c_1+b_2^2$$
$$+2b_2c_3+c_3^2)(b_{21}+c_{31})+b_1^2c_1c_{31}(a_2b_1+b_2^2+b_2c_3))\Big)+\sin\theta\cos^2\theta r^2\big(b_1b_2c_1^2\big(b_2^2(5a_2e_1-a_3c_1-3c_3^2-2c_3e_2)$$
$$-b_2(a_2c_1e_3-7a_2c_3e_1+a_3c_1c_3+a_3c_1e_2+c_3^3+c_3^2e_2)-b_2^3(b_2+3c_3+e_2)+2a_2e_1(a_3c_1+c_3^2)\big)-b_1^3\big(c_1^2(-(b_2$$
$$+c_3)^2(2a_2-d_2)+a_2^2b_2+(a_3c_3-b_2d_1-c_3d_1)a_2)+c_3^2(a_2(b_2c_1-3b_2f_1+c_1f_3)+a_3c_1f_2+b_2^2f_2)+c_3^3(a_2c_1$$
$$-3a_2f_1+2b_2f_2)-c_1^3(2a_2a_3-a_2d_3-a_3d_2)+f_2c_3^4-2a_2a_3c_1c_3f_1\big)+b_1^2\big(-b_2^2(c_1^2(c_3(3a_2+a_3-6c_3-3d_1)$$
$$+a_2e_2)-c_1c_3(2a_3f_1-3c_3^2-c_3f_3)-c_1^3(2a_3-d_3)-9c_3^3f_1)+c_1^2(b_2(2a_2^2e_1-c_3^2(a_2+3a_3-4c_3-3d_1))-b_2^3$$
$$(2a_2-b_2-4c_3-d_1)+c_3a_3(2a_3f_1-2c_3^2-c_3f_3)+c_3^4+c_3^3d_1)-b_2(a_3(a_2c_1^3-7c_3^2f_1-c_1^3(4c_3+d_1))+3c_3^4$$
$$(c_1-3f_1)+2c_1c_3(c_1^2d_3+c_3^2f_3))+c_3^2c_1^3(2a_3-d_3)+a_3^2c_1^3(c_1-c_3)+(b_2^3c_3^2+c_3^5)(3f_1-c_1)+c_1c_3^3(5a_3f_1-c_3f_3)$$
$$-a_3c_1^3(c_1d_3-c_3d_1)\big)+a_2b_1^4\big(a_2c_1^2-c_1^2d_2-c_3^2f_2\big)+b_2^2c_1^2(3b_2e_1-c_1e_3+3c_3e_1)(a_3c_1+b_2^2+2b_2c_3+c_3^2)\big)+\sin^3\theta$$
$$r^2\big(b_1^3\big(c_1^2(a_2^2b_2+a_2^2e_2+a_2a_3c_3+a_3^2f_2)+c_3(b_2+c_3)(c_1(a_2c_3+2a_3f_2)+c_3f_2(b_2+c_3))\big)-b_1b_2c_1^2(b_2+c_3)\big(b_2$$
$$(2a_2e_1-a_3c_1-c_3^2-c_3e_2)-b_2^2(b_2+2c_3+e_2)-2a_2(c_1e_3-c_3e_1)\big)-b_1^2\big(c_1^3(-a_2^2e_3-a_2a_3b_2-a_3^2c_3-a_3^2f_3)+c_1^2$$
$$(b_2+c_3)(a_2^2e_1+a_3^2f_1-a_3c_3(b_2+2c_3+2f_3)-a_2b_2(2b_2+c_3+2e_2))+c_1c_3(b_2+c_3)^2(2a_3f_1-c_3(b_2+c_3+f_3))$$
$$+c_3^2f_1(b_2+c_3)^3\big)-b_2^2c_1^2(b_2+c_3)^2(b_2e_1-c_1e_3+c_3e_1)\big)+c_1^2(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2)((b_1^2(b_1f_2-b_2f_1$$
$$+c_1f_3-c_3f_1)+c_1^2(b_1e_2-b_2e_1+c_1e_3-c_3e_1))W-b_1^2c_1c_{31})W\sin\theta-2\cos^2\theta r(Wb_1^2c_3f_1-Wb_2c_1^2e_1-b_1^2b_{21}c_1$$
$$-b_1^2c_1c_{31})c_1\big(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2\big)^{5/2}-\cos\theta W^2(b_1^2f_1+c_1^2e_1)c_1^2(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2)^{5/2}$$
$$-\cos^3\theta r^2(b_1^2c_1^2d_1+b_1^2c_3^2f_1+b_2^2c_1^2e_1)\big(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2\big)^{5/2}+\sin\theta\cos\theta ra_3b_2c_3f_1(7\cos\theta$$
$$rc_1c_3-8Wc_1^3-7\cos\theta rc_3+8Wc_1^2)b_1^2\Big)=F_1(\theta,r,W),$$

$$\dot{W} = -\frac{\varepsilon}{b_1^2c_1^3\left(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2\right)^{5/2}}\Big(\sqrt{-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2}\,r^2\cos\theta\sin\theta\Big(b_1^3\Big(a_2^2c_1\big(c_3c_1$$
$$(b_1+b_2)+a_3c_1^2-b_1c_3^2+c_1c_3^2\big)-a_2\big(a_3c_1\big(c_1c_3\big(2b_2+3c_3+2f_3\big)+(b_2-c_3)c_1^2-2c_3^2f_1\big)-b_2^2c_1c_3(c_1-3c_3)$$
$$+a_3^2c_1^3-c_3^2(c_1(2c_1-5c_3-2f_3)+2c_3f_1)(b_2+c_3)+c_3^3c_1(c_1-3c_3)\big)+2c_3f_2\big(a_3c_1+b_2c_3+c_3^2\big)^2\Big)-b_1c_1^2b_2^2$$
$$\big(2a_2(e_1\big(a_3c_1-b_2c_3-c_3^2\big)+2c_1e_3(b_2+c_3)\big)-a_3c_1(b_2+c_3)(2b_2+3c_3+2e_2)-2b_2^2c_3(3c_3+e_2+b_2)-2c_3^2$$
$$(c_3(3b_2+c_3+e_2)+2b_2e_2)-a_3^2c_1^2\big)-b_1^2b_2c_1\big(c_1^2\big(2a_2^2e_3+a_3^2(b_2+c_1+c_3)-a_3\big(a_2(3b_2+2c_3+2e_2)-(b_2+$$
$$c_3)^2\big)\big)-c_1c_3\big((b_2+c_3)(a_2(3b_2+2c_3+2e_2)-a_3(2b_2+3c_3+2f_3))+2a_2^2e_1-2a_3^2f_1\big)+2c_3^2(b_2+c_3)(b_2(b_2+$$
$$2c_3+f_3)+a_3f_1+c_3^2+c_3f_3))-2b_2^3c_1^3(b_2+c_3)(a_3e_1+b_2e_3+c_3e_3)\Big)-c_1\sqrt{-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2}$$

$$\Big(\Big(b_1^3\big(a_2^2b_1c_1c_3-2f_2\big(a_3c_1+b_2c_3+c_3^2\big)^2+a_2(2(c_1f_3-c_3f_1)\big(a_3c_1+b_2c_3+c_3^2\big)+c_1c_3(a_3c_1+2b_2^2+3b_2c_3+$$
$$c_3^2))\Big)-b_1^2c_1\big(a_2c_1^2(2a_2e_3-a_3b_2-2a_3e_2)-c_1c_3\big(b_2^2(a_2+a_3)+2a_2(a_2e_1+c_3e_2)+b_2(a_2c_3+2a_2e_2+a_3c_3)\big)$$
$$-b_2c_3(b_2+c_3)^3-2b_2(a_3f_1+b_2f_3+c_3f_3)\big(a_3c_1+b_2c_3+c_3^2\big)\big)-b_1c_1^2b_2\big(-a_3^2c_1^2-c_3(b_2+c_3)^3-2e_2(b_2+c_3)$$
$$\big(a_3c_1+b_2c_3+c_3^2\big)+4a_2c_1e_3(b_2+c_3)+2a_2e_1\big(a_3c_1-b_2c_3-c_3^2\big)-a_3c_1(b_2+2c_3)(b_2+c_3)\big)-2b_2^2c_1^3(b_2+c_3)$$
$$(a_3e_1+b_2e_3+c_3e_3)\big)W+c_1b_1^2\Big(b_2^2(c_{31}(4a_2b_1+a_3c_1+2b_2^2+5b_2c_3+4c_3^2)-b_{21}\big(a_3c_1+b_2c_3+3c_3^2\big)\big)-b_2c_3$$
$$\big(b_{21}(a_2b_1+3a_3c_1+3c_3^2)-c_{31}(5a_2b_1+a_3c_1+c_3^2)\big)-a_2b_1(b_{21}-c_{31})(a_3c_1+c_3^2)-b_{21}\big(a_3c_1+c_3^2\big)^2+2a_2^2b_1^2c_{31}$$
$$\Big)\Big)r\sin\theta-c_1^2\big(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2\big)\big(\big(b_1c_1^2a_2(c_1e_3-c_3e_1)+b_1^3a_2(c_1f_3-c_3f_1)-b_1(a_3c_1+b_2c_3$$
$$+c_3^2)(b_1^2f_2+c_1^2e_2)+b_1^2b_2c_1(a_3f_1+b_2f_3+c_3f_3)+b_2c_1^3(a_3e_1+b_2e_3+c_3e_3)\big)W-b_1^2c_1c_{31}\big(a_2b_1+b_2^2+b_2c_3\big)\big)W$$
$$-c_1\big(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2\big)\big(\big(b_1^3\big(a_2\big(c_3(b_2+c_3+2f_3)c_1+a_3c_1^2-2c_3^2f_1\big)-2c_3f_2\big(a_3c_1+b_2c_3+c_3^2\big)\big)$$
$$+b_1^2c_1\big(c_1(b_2+c_3)(a_2c_3+a_3b_2)+b_2c_3\big(b_2(b_2+2c_3+2f_3)+c_3^2+2c_3f_3+2a_3f_1\big)+a_2a_3c_1^2\big)-b_1b_2c_1^2\big(-2e_2$$
$$\big(a_3c_1+b_2c_3+c_3^2\big)+c_1(2a_2e_3-a_3(b_2+c_3))-2a_2c_3e_1-c_3^2(2b_2+c_3)\big)-b_2^2c_1^2(2c_1e_3(b_2+c_3)+2a_3c_1e_1-b_1$$
$$b_2c_3)\big)W+b_1^2c_1\big(c_{31}\big((2b_2+c_3)\big(a_2b_1+b_2^2+b_2c_3\big)+2a_2b_1c_3-2a_3b_2c_1\big)+b_{21}\big(2a_2b_1c_3-a_3c_1(3b_2+c_3)-c_3(b_2$$
$$+c_3)^2\big)\big)\big)r\cos\theta+r^2\big(a_2b_1^4\big(a_2\big(c_1^2c_3(2a_3-d_1)+c_1c_3^2(2b_2+2c_3+f_3)+c_1^3d_3-c_3^3f_1\big)-c_3^3f_2(b_2+c_3)-c_1c_3^2(a_3f_2$$
$$+c_1d_2)-c_1^2d_2(a_3c_1+b_2c_3)\big)+b_1c_1^2b_2^2\big(b_2(a_2(3c_1(a_3e_1+3c_3e_3)-4c_3^2e_1\big)-a_3c_1(2a_3c_1+10c_3^2+5c_3e_2)-8c_3^4-$$
$$6c_3^3e_2)+b_2^2\big(a_2(5c_1e_3-2c_3e_1)-2a_3c_1(4c_3+e_2)-6c_3^2(2c_3+e_2)\big)-2b_2^3(a_3c_1+c_3(b_2+4c_3+e_2))\big)+a_3c_1^2(a_2e_3$$
$$-2a_3c_3-a_3e_2)+c_1c_3(a_2(a_3e_1+4c_3e_3)-a_3c_3(4c_3+3e_2))-2c_3^3\big(a_2e_1+c_3^2+c_3e_2\big)\big)+b_1^3\big(-2c_3^3(b_2+c_3)^2(a_2f_1$$
$$+b_2f_2+c_3f_2)-c_1^2c_3(a_2^2(2b_2+e_2)(b_2+c_3)-a_2(a_3(4b_2^2+8b_2c_3+2b_2f_3+4c_3^2+3c_3f_3)-d_1(b_2+c_3)^2)+a_3^2f_2$$
$$(3b_2+4c_3)+a_2^3e_1+d_2(b_2+c_3)^3)+c_1^3(a_2(2a_3^2c_3+a_3^2f_3+a_3b_2d_1-a_3c_3d_1+2b_2^2d_3+3b_2c_3d_3+c_3^2d_3)-a_2^2a_3$$
$$(2b_2+e_2)+a_2^3e_3-a_3^3f_2-a_3d_2(b_2+2c_3)(b_2+c_3))+a_3c_1^4(a_2d_3-a_3d_2)-c_1c_3^2(a_2((a_3f_1-8c_3^2-5c_3f_3)b_2-b_2^2$$
$$(4b_2+10c_3+3f_3)+3a_3c_3f_1-2c_3^2f_3)+f_2a_3(4b_2+5c_3)(b_2+c_3)-2c_3^3)\big)+b_1^2b_2c_1\big(c_1^2(a_2^2(a_3e_1+4b_2e_3+3c_3e_3)$$
$$-a_2a_3(4b_2^2+8b_2c_3+3b_2e_2+4c_3^2+2c_3e_2)+a_3^2(2c_3+f_3)(b_2+c_3)+a_3d_1(b_2+c_3)^2+d_3(b_2+c_3)^3+a_3^3f_1)-c_1$$
$$c_3(a_2^2e_1(3b_2+2c_3)-a_3^2f_1(2b_2+3c_3)+(b_2+c_3)(2b_2^2(2a_2-a_3)+3a_2b_2(2c_3+e_2)+2a_2c_3(c_3+e_2)-a_3(2b_2(3c_3$$
$$+f_3)+4c_3^2+3c_3f_3)))+2c_3^2(6a_2b_2^2c_3^2+b_2^4+4b_2^3c_3+4b_2c_3^3+c_3^4)-a_3c_1^3(2a_2a_3-a_3d_1-b_2d_3-c_3d_3)+2c_3^2(b_2+$$
$$c_3)^2(a_3f_1+b_2f_3+c_3f_3)\big)+b_2^3c_1^3(a_3e_1+b_2e_3+c_3e_3)(a_3c_1+2b_2^2+4b_2c_3+2c_3^2)-b_1^2c_1c_3\big(a_2\big(b_1\big(a_3^2c_1f_1-2c_3^4\big)$$
$$+12b_2^3c_3^3\big)+2c_3^3\big(-6b_2^3+b_1c_3\big)\big)\big)\cos^2\theta+r^2\big(2b_1^3c_1^2c_3(b_2+c_3)^2(-a_2a_3+1)-b_1c_1^2b_2^2(b_2+c_3)\big(a_2(e_1(2a_3c_1-b_2$$
$$c_3-c_3^2)+3c_1e_3(b_2+c_3))-b_2^2c_3(3c_3+e_2+b_2)-b_2c_3^2(3c_3+2e_2)-a_3c_1(b_2+c_3)(b_2+2c_3+e_2)-c_3^3(c_3+e_2)$$
$$-a_3^2c_1^2\big)-b_1^3\big(-c_1^2c_3((b_2+c_3)(a_2^2(b_2+e_2)-2a_2a_3f_3+3a_3^2f_2)-2(b_2+c_3)^2+a_2^3e_1+a_2a_3^2f_1)-c_1c_3^2(b_2+c_3)$$
$$(a_2(2a_3f_1-2b_2^2-3b_2c_3-c_3^2-f_3(b_2+c_3))+3f_2a_3(b_2+c_3))+c_1^3(a_2^3e_3-a_3^3f_2-a_3(b_2+e_2)a_2^2+a_3^2(c_3+f_3)a_2)$$
$$-c_3^3(b_2+c_3)^2(a_2f_1+b_2f_2+c_3f_2)\big)-c_1b_1^2b_2\big(c_1^2(a_2^2(a_3e_1+3e_3(b_2+c_3))-a_3(b_2+c_3)(2a_2(b_2+c_3+e_2)-a_3(c_3$$
$$+f_3))+a_3^3f_1)-c_1c_3(b_2+c_3)((b_2+c_3)(a_2(2b_2+c_3+2e_2)-a_3(b_2+2c_3+2f_3))+2a_2^2e_1-2a_3^2f_1)+c_3^2(b_2+$$
$$c_3)^2(a_3f_1+b_2f_3+c_3f_3)+c_3^2(b_2+c_3)^4-c_1^3a_2a_3^2\big)-b_2^3c_1^3(b_2+c_3)^2(a_3e_1+b_2e_3+c_3e_3)-a_2^2b_1^4c_1c_3\big(a_3c_1+b_2$$
$$c_3+c_3^2\big)\big)\Big)=F_2(\theta,r,W).$$

Computing the integrals (4) we obtain

$$f_1(r,W)=-\frac{r\big(T_6W+N_6\big)}{2c_1b_1^2\big(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2\big)^{3/2}},\quad f_2(r,W)=\frac{D_6W^2-R_6r^2+C_6W}{2b_1^2c_1^3\big(-a_2b_1-a_3c_1-b_2^2-2b_2c_3-c_3^2\big)^{5/2}}.$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$(r^*,W^*)=\left(\frac{1}{T_6}\sqrt{\frac{D_6N_6^2-C_6N_6T_6}{R_6}},-\frac{N_6}{T_6}\right),$$

if $T_6 > 0$, $R_6 \neq 0$ and $R_6(D_6N_6^2 - C_6N_6T_6) > 0$. The Jacobian (5) at (r^*, W^*) is $N_6\big(C_6T_6 - D_6N_6\big)/\big(2\,T_6 b_1^4 c_1^4 \big(a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2\big)^4\big)$, where

$$\begin{aligned}
D_6 = \;& 2c_1^2\big(a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2\big)\big(b_1^2b_2c_1(a_3f_1 + b_2f_3 + c_3f_3) + b_1^3\big(a_2c_1f_3 - a_2c_3f_1 - f_2\big(a_3c_1 + b_2c_3 + c_3^2\big)\big) \\
& + b_1c_1^2\big(a_2c_1e_3 - a_2c_3e_1 - e_2\big(a_3c_1 + b_2c_3 + c_3^2\big)\big) + b_2c_1^3\big(a_3e_1 + b_2e_3 + c_3e_3\big)\big), \\
T_6 = \;& 2b_1^2\big(a_3c_1 + b_2c_3 + c_3^2\big)(b_1f_2 + c_1f_3) - 2c_1^2(b_1e_2 + c_1e_3)\big(a_2b_1 + b_2^2 + b_2c_3\big) + 2\big(b_1^2f_1 + c_1^2e_1\big)(a_2b_1c_3 - a_3b_2c_1) \\
& + b_1c_1(b_1c_3 - b_2c_1)\big(a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2\big), \\
R_6 = \;& \Big(- a_2b_1^4\big(a_2(c_1^3d_3 - c_1^2c_3d_1 + c_1c_3^2f_3 - c_3^3f_1) - (c_1^2d_2 + c_3^2f_2)(a_3c_1 + b_2c_3 + c_3^2)\big) + b_1^3c_1\big(a_2^3c_1(c_1e_3 - c_3e_1) - \\
& a_2^2c_1e_2(a_3c_1 + b_2c_3 + c_3^2) - a_2a_3c_3f_1(a_3c_1 + 3b_2c_3 + c_3^2) - a_2c_1d_1\big((b_2 - c_3)(a_3c_1 - b_2c_3) - 3b_2c_3^2 - c_3^3\big) + a_2f_3 \\
& \big(a_3c_1(a_3c_1 + 2b_2c_3 + c_3^2) - b_2^2c_3^2 - b_2c_3^3\big) - a_2c_1^2d_3\big(a_3c_1 + 2b_2^2 + 3b_2c_3 + c_3^2\big) - \big(a_3c_1 + b_2c_3 + c_3^2\big)\big((a_3c_1 + 2b_2 \\
& c_3 + c_3^2)(a_3f_2 - c_1d_2) - b_2^2c_1d_2\big)\big) + b_1b_2^2c_1^3\big(a_2(a_3e_1(b_2 + 3c_3) - e_3(a_3c_1 - b_2^2 - 3b_2c_3 - 2c_3^2)) + a_3e_2\big(a_3c_1 + b_2 \\
& c_3 + c_3^2\big)\big) + c_1^2b_1^2b_2\big(+ a_3^3c_1f_1 + a_2^2(e_1(a_3c_1 - b_2c_3 - 2c_3^2) + c_1e_3(2b_2 + 3c_3)) - a_2e_2(b_2 + 2c_3)(a_3c_1 + b_2c_3 + c_3^2) \\
& + a_3^2(f_3(b_2 + c_3)c_1 - c_1^2d_1 + c_3f_1(2b_2 + c_3)) - a_3(b_2 + c_3)(c_1^2d_3 + c_1d_1(b_2 + c_3) - c_3f_3(2b_2 + c_3)) - c_1d_3(b_2 + \\
& c_3)^3\big) - a_3b_2^3c_1^4(a_3e_1 + b_2e_3 + c_3e_3)\Big), \\
N_6 = \;& -b_1^2b_{21}c_1(a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2) - b_1^2c_1c_{31}(a_3c_1 + b_2c_3 + c_3^2), \\
C_6 = \;& \big(- 2b_1^2c_1^3c_{31}\big(a_2b_1 + b_2^2 + b_2c_3\big)\big(a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2\big)\big).
\end{aligned}$$

Finally we apply Theorem 3 with $\varepsilon > 0$ sufficiently small and system (26) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ which tends to the equilibrium $(1,1,1)$ when $\varepsilon \to 0$. Therefore it is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon = 0$. This completes the proof of statement (vi) of Theorem 2.

Example 6. Consider the Lotka-Volterra system

(28)

$$\dot{x} = x\big(y - 2x + z + (x-1)^2 + (y-1)^2 + 2(z-1)^2\big),\ \dot{y} = y\big(y - 2x + z - (x-1)^2 + 3(y-1)^2 + (z-1)^2\big),\ \dot{z} = z\big(1 - 3x + y + z + (z-1)^2\big).$$

This system in the new variables (X, Y, Z) writes

$$\begin{aligned}
\dot{X} = \;& (X+1)\big(X^2 - 4X\varepsilon + Y^2 + 2Z^2 - 2X + Y + Z\big), \\
\dot{Y} = \;& (Y+1)\big((Y+Z)\varepsilon - 2X + Y + Z - X^2 + 3Y^2 + Z^2\big), \\
\dot{Z} = \;& (Z+1)\big((2Y+Z)\varepsilon - 3X + Y + Z + Z^2\big).
\end{aligned}$$

The corresponding system associated to system (27) is

$$\begin{aligned}
F_1(\theta, r, W) = \;& -\sin\theta\cos\theta r(-3W + 13) - 3\cos\theta\sin^2\theta r^2 - \sin^2\theta r(4W - 2) + \sin^3\theta r^2 - (-W - 1)W\sin\theta \\
& + 2\cos^2\theta r(W - 2) + 3\cos\theta W^2 + 4\cos^3\theta r^2, \\
F_2(\theta, r, W) = \;& -2r^2\cos\theta\sin\theta - (3W - 1)r\sin\theta - W^2 - (-6W + 6)r\cos\theta + r^2.
\end{aligned}$$

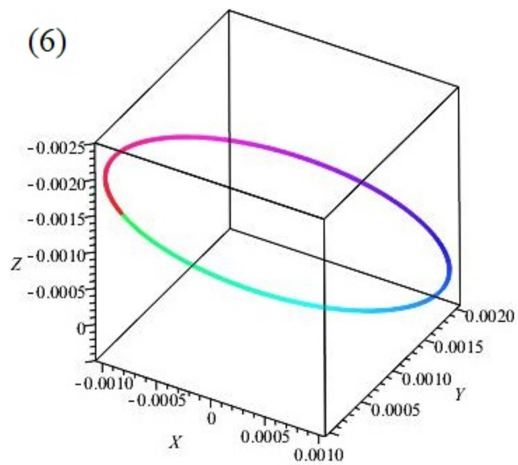
To look for the limit cycles we must solve the system

$$f_1(r, W) = -\frac{1}{2}(2W + 2)r, \quad f_2(r, W) = -W^2 + r^2.$$

This system has the solutions $/r, W)$ given by

$$(0,0), \quad (-1,-1), \quad (1,-1).$$

The unique admissible root is $(1,-1)$. The determinant of the Jacobian matrix of (f_1, f_2) is 2. Hence system (28) has exactly one limit cycle bifurcating from the origin. We plot this limit cycle for $\varepsilon = 10^{-3}$ in the next figure.



The eigenvalues of the Jacobian matrix of (f_1, f_2) at $(1,-1)$ are $1 \pm i$. So this limit cycle is a repeller. This completes the proof of Example 6.

Now we perturb the Lotka-Volterra system (1) with the parameters given in statement (vii) of Proposition 1 satisfying the conditions of Theorem 2(vii). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates and system (1) becomes

(29)

$$
\begin{aligned}
\dot{X} &= (X+1)\big((-2b_{21}-2c_{31})X\varepsilon + (-b_2-c_3)X + b_1Y + c_1Z + d_1X^2 + e_1Y^2 + f_1Z^2\big),\\
\dot{Y} &= (Y+1)\left(\left(\left(-\frac{b_2c_{31}}{b_1} - \frac{b_2(b_{21}+c_{31}) + b_{21}(b_2+c_3)}{b_1}\right)X + b_{21}Y + \frac{b_{21}c_1Z}{b_1}\right)\varepsilon - \frac{b_2(b_2+c_3)X}{b_1} + b_2Y + \frac{b_2c_1Z}{b_1}\right.\\
&\quad \left. + d_2X^2 + e_2Y^2 + f_2Z^2\right),\\
\dot{Z} &= (Z+1)\big(X^2d_3 + Y^2e_3 + Z^2f_3 + Z\varepsilon c_{31} + Xa_3 + Yb_3 + Zc_3\big).
\end{aligned}
$$

As usual we write the linear part of system (29) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$
\begin{pmatrix}
0 & -\dfrac{\sqrt{A_2}}{b_1} & 0\\
\dfrac{\sqrt{A_2}}{b_1} & 0 & 0\\
0 & 0 & 0
\end{pmatrix},
$$

where $A_2 = -b_1\left(a_3b_1c_1 + b_1c_3^2 + b_2b_3c_1\right)$. For this we do the change of variables

$$
\begin{pmatrix} u\\ v\\ w \end{pmatrix} =
\begin{pmatrix}
\dfrac{a_3c_1 + b_2c_3 + c_3^2}{A_2} & -\dfrac{b_1c_3 - b_3c_1}{A_2} & 0\\
-\dfrac{b_2+c_3}{b_1\sqrt{A_2}} & \dfrac{1}{\sqrt{A_2}} & \dfrac{c_1}{b_1\sqrt{A_2}}\\
\dfrac{b_2}{A_2} & -\dfrac{b_1}{A_2} & 0
\end{pmatrix}
\begin{pmatrix} X\\ Y\\ Z \end{pmatrix},
$$

and by following the same steps as the case (i) we get the system

(30)

$$
\begin{aligned}
\dot{r} = \; & \frac{\varepsilon}{c_1^2\left(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\right)\sqrt{-b_1\left(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\right)}}\Big(\sqrt{-b_1(a_3b_1c_1+b_1c_3^2+b_2b_3c_1)}\Big(-\sin\theta r\cos\theta \\
& c_1\big(\big(\big(a_3c_1+b_2c_3+c_3^2\big)b_1c_1((b_1-b_2)(b_1b_2+b_1c_3-b_3c_1)+b_1(b_1c_3-2b_2e_2-b_3c_3)+2b_2(b_2e_1-c_1e_3+c_3e_1)) \\
& -(b_1c_3-b_3c_1)b_1c_1\big((b_2+c_3)\big(b_1b_2+b_1c_3-2b_1d_1-b_2^2\big)+b_1(a_3c_3+2b_1d_2+2c_1d_3)\big)-(a_3b_1+b_2b_3+b_3c_3) \\
& b_1\big(2b_1\big(\big(a_3c_1+b_2c_3+c_3^2\big)f_1-(b_1c_3-b_3c_1)f_2-c_3(b_1f_2-b_2f_1+c_1c_3+c_1f_3-c_3f_1)\big)-c_1^2\big((a_3-b_2-c_3)b_1 \\
& +b_2^2+b_2b_3\big)\big)\big)W-3b_1c_1c_3c_{31}-b_{21}c_1(2b_1b_2+4b_1c_3-b_3c_1)+b_{21}b_1(b_2+c_3)c_1\big)+b_1^2\big(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\big) \\
& \sin^3\theta r^2(b_1f_2-b_2f_1+c_1c_3+c_1f_3-c_3f_1)+b_1\sin\theta r^2\cos^2\theta(b_1^2(c_1^3(a_3-d_3)+3c_3^2f_1(b_2+c_3)+c_1c_3(2a_3f_1 \\
& +2b_3f_2-c_3^2-c_3f_3)-c_1^2(c_3(a_3-b_2-c_3-d_1)-b_2d_1))-b_1^3\big(c_1^2d_2+3c_3^2f_2\big)-b_2c_1^2(b_1b_2(c_3+e_2)-b_2(b_2e_1+ \\
& b_3c_1-c_1e_3+c_3e_1)+b_1b_3c_3))-c_1^2\sin\theta\Big(\Big(b_1\Big(\big(a_3c_1+b_2c_3+c_3^2\big)^2(b_1b_2+b_1e_2-b_2e_1+c_1e_3-c_3e_1)+(b_1c_3 \\
& -b_3c_1)^2(b_1d_2+b_2(b_2+2c_3-d_1)+c_1d_3+c_3^2-c_3d_1)+(a_3b_1+b_2b_3+b_3c_3)^2(b_1f_2-b_2f_1+c_1c_3+c_1f_3-c_3f_1)\Big) \\
& -\big(a_3c_1+b_2c_3+c_3^2\big)b_1((b_1c_3-b_3c_1)(b_2+c_3)(b_2+b_1)+(a_3b_1+b_2b_3+b_3c_3)c_1(b_2+b_3))-(b_1c_3-b_3c_1)(a_3b_1+ \\
& b_2b_3+b_3c_3)b_1c_1(a_3-b_2-c_3)\Big)W^2+\big(b_{21}\big(\big(a_3c_1+b_2c_3+c_3^2\big)b_1+(b_2+2c_3)(b_1c_3-b_3c_1)-(a_3b_1+b_2b_3+b_3c_3) \\
& c_1\big)+c_{31}(2(b_1c_3-b_3c_1)c_3-(a_3b_1+b_2b_3+b_3c_3)c_1)-b_{21}(b_1c_3-b_3c_1)(b_2+c_3)\big)W\big)\big)+b_1^2(a_3b_1c_1+b_1c_3^2+b_2b_3 \\
& c_1)\cos\theta\sin^2\theta r^2(b_1f_1(a_3c_1+3b_2c_3+3c_3^2)-c_1^2\big((a_3-b_2-c_3)b_1+b_2^2+b_2b_3\big)-b_1(3b_1c_3f_2-c_1(b_3f_2-2c_3^2- \\
& 2c_3f_3)))+c_1^2\cos\theta\Big(\Big(-(a_3c_1+b_2c_3+c_3^2)(b_1c_3-b_3c_1)b_1((a_3c_1+b_2c_3+c_3^2)b_1(b_1-b_2-e_2)-(b_1c_3-b_3c_1) \\
& ((b_2+c_3-d_1)b_1-b_2^2-b_2c_3)-(a_3b_1+b_2b_3+b_3c_3)c_1(b_1-b_2))-b_1^2\Big(\big(a_3c_1+b_2c_3+c_3^2\big)^3e_1-(b_1c_3-b_3c_1)^3 \\
& d_2\Big)-(a_3b_1+b_2b_3+b_3c_3)^2b_1^2\big(\big(a_3c_1+b_2c_3+c_3^2\big)f_1-(b_1c_3-b_3c_1)f_2\big)\Big)W^2+\big((b_1c_3-b_3c_1)b_{21}((2a_3c_1+2b_2c_3 \\
& +3c_3^2)b_1-b_3c_1c_3)+2(b_1c_3-b_3c_1)c_{31}(a_3b_1c_1+b_1c_3^2+b_2b_3c_1)-b_{21}(b_1c_3-b_3c_1)^2(b_2+c_3))W\Big)+b_1^2\big((b_1c_3- \\
& b_3c_1)\big(b_1^2c_1^2d_2+b_1^2c_3^2f_2+b_2^2c_1^2e_2\big)-\big(a_3c_1+b_2c_3+c_3^2\big)\big(b_1^2c_1^2d_1+b_1^2c_3^2f_1+b_2^2c_1^2e_1\big)\big)\cos^3\theta r^2+b_1c_1r\Big(\Big(\Big(\Big(a_3c_1 \\
& +b_2c_3+c_3^2\Big)^2b_1c_1\big(b_1^2+2b_2e_1\big)-(b_1c_3-b_3c_1)^2c_1\big(2b_1^2d_2-b_2^3-b_2^2c_3\big)-\big(a_3c_1+b_2c_3+c_3^2\big)(b_1c_3-b_3c_1)b_1c_1 \\
& \big((b_2+c_3-2d_1)b_1+b_2^2+2b_2e_2\big)-(a_3b_1+b_2b_3+b_3c_3)((a_3c_1+b_2c_3+c_3^2)b_1^2\big(c_1^2+2c_3f_1\big)-(b_1c_3-b_3c_1)\big(2b_1^2c_3f_2 \\
& +b_2^2c_1^2\big)\big)\Big)W-b_{21}c_1\big(2\big(a_3c_1+b_2c_3+c_3^2\big)b_1+(b_1c_3-b_3c_1)c_3+a_3b_1c_1+b_1c_3^2+b_2b_3c_1\big)-c_1c_{31}(2(a_3c_1+b_2c_3 \\
& +c_3^2)b_1-2(b_1c_3-b_3c_1)b_2+a_3b_1c_1+b_1c_3^2+b_2b_3c_1)+b_{21}(b_1c_3-b_3c_1)(b_2+c_3)c_1)\cos^2\theta+b_1(a_3b_1c_1+b_1c_3^2 \\
& +b_2b_3c_1)((a_3c_1+b_2c_3+c_3^2)c_1(b_2+b_3)+(b_1c_3-b_3c_1)c_1(a_3-b_2-c_3)-2(a_3b_1+b_2b_3+b_3c_3)(b_1f_2-b_2f_1+ \\
& c_1c_3+c_1f_3-c_3f_1))W\sin^2\theta+\big(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\big)c_1(b_{21}+c_{31})\Big)\Big)=F_1(\theta,r,W),
\end{aligned}
$$

$$
\begin{aligned}
\dot{W} = \; & \frac{\varepsilon}{c_1^2\left(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\right)\sqrt{-b_1\left(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\right)}}\Big(\sqrt{-b_1\big(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\big)}\big(c_1b_1\sin\theta r\big(Wb_1 \\
& \big(b_2c_1\big(a_3c_1-b_1c_3+b_2c_3+b_3c_1+c_3^2\big)-2(a_3b_1+b_2b_3+b_3c_3)(b_1f_2-b_2f_1)\big)+b_{21}c_1\big)-\cos\theta\sin\theta r^2b_1^2\big(2b_1^2 \\
& c_3f_2+b_2^2c_1^2-b_2\big(c_1^2+2c_3f_1\big)b_1\big)\big)+b_1^2c_1\cos\theta r\big(\big(\big(a_3c_1+b_2c_3+c_3^2\big)b_2c_1\big(b_1(b_1-b_2-2e_2)+2b_2e_1\big)-(b_1c_3 \\
& -b_3c_1)c_1(2b_1(b_1d_2-b_2d_1)+b_2(b_2+c_3)(b_1-b_2))+(a_3b_1+b_2b_3+b_3c_3)\big(2b_1^2c_3f_2+b_2^2c_1^2-b_2\big(c_1^2+2c_3f_1\big)b_1\big)\big) \\
& W-b_{21}c_1(2b_2+c_3)+b_{21}(b_2+c_3)c_1\big)+b_1c_1^2\Big(\Big(\big(a_3c_1+b_2c_3+c_3^2\big)^2b_1(b_1b_2+b_1e_2-b_2e_1)+(b_1c_3-b_3c_1)^2b_1 \\
& (b_1d_2+b_2(b_2+c_3-d_1))-\big(a_3c_1+b_2c_3+c_3^2\big)(b_1c_3-b_3c_1)b_1b_2(b_1+b_2+c_3)-(a_3b_1+b_2b_3+b_3c_3)b_1b_2c_1(a_3c_1 \\
& -b_1c_3+b_2c_3+b_3c_1+c_3^2)+(a_3b_1+b_2b_3+b_3c_3)^2b_1(b_1f_2-b_2f_1)\big)W^2+b_{21}\big(\big(a_3c_1+b_2c_3+c_3^2\big)b_1+(b_1c_3-b_3 \\
& c_1)b_2-(a_3b_1+b_2b_3+b_3c_3)c_1-(b_1c_3-b_3c_1)(b_2+c_3)\big)W\big)+r^2b_1^2\cos^2\theta\big(c_1^2\big(b_1^3d_2-b_2^3e_1\big)+b_1^3c_3^2f_2-b_1b_2\big(b_1 \\
& c_1^2d_1+b_1c_3^2f_1-b_2c_1^2e_2\big)\big)-b_1^3\big(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\big)r^2(b_1f_2-b_2f_1)\sin^2\theta\Big)=F_2(\theta,r,W).
\end{aligned}
$$

Now we compute the integrals (4) and we get

$$f_1(r,W) = -\frac{b_1^2 r\big(T_7 W + N_7\big)}{2c_1\big(-b_1(a_3b_1c_1+b_1c_3^2+b_2b_3c_1)\big)^{3/2}},\quad f_2(r,W) = \frac{b_1^2\Big(D_7W^2+R_7r^2+C_7W\Big)}{2c_1^2\big(-b_1(a_3b_1c_1+b_1c_3^2+b_2b_3c_1)\big)^{3/2}}.$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$(r^*,W^*) = \left(\frac{1}{T_7}\sqrt{\frac{C_7N_7T_7-D_7N_7^2}{R_7}}, -\frac{N_7}{T_7}\right),$$

if $T_7>0$, $R_7\neq 0$ and $R_7(C_7N_7T_7-D_7N_7^2)>0$. The Jacobian (5) at (r^*,W^*) is $-b_1N_7\big(C_7T_7-D_7N_7\big)/(2\,T_7c_1^3\Big(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\Big)^3)$, where

$$\begin{aligned}
T_7 = {} & \big(a_3c_1+b_2c_3+c_3^2\big)^2 b_1c_1\big(b_1^2+2b_2e_1\big) - \big(b_1c_3-b_3c_1\big)^2 c_1\big(2b_1^2d_2-b_2^3-b_2^2c_3\big) - \big(a_3c_1+b_2c_3+c_3^2\big)b_1\big((b_1c_3-b_3\\
& c_1)c_1\big((b_2+c_3-2d_1)b_1+b_2^2+2b_2e_2\big)+(a_3b_1+b_2b_3+b_3c_3)b_1\big(c_1^2+2c_3f_1\big)\big)+(b_1c_3-b_3c_1)(a_3b_1+b_2b_3+b_3\\
& c_3)\big(2b_1^2c_3f_2+b_2^2c_1^2\big)+b_1\big(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\big)\big(\big(a_3c_1+b_2c_3+c_3^2\big)c_1(b_2+b_3)+(b_1c_3-b_3c_1)c_1(a_3-b_2-c_3)\\
& -2(a_3b_1+b_2b_3+b_3c_3)(b_1f_2-b_2f_1+c_1c_3+c_1f_3-c_3f_1)\big),\\
D_7 = {} & 2b_1b_2c_1^2\big(\big(a_3c_1+b_2c_3+c_3^2\big)(b_1c_3-b_3c_1)(b_1+b_2+c_3)+(a_3b_1+b_2b_3+b_3c_3)c_1\big(a_3c_1-b_1c_3+b_2c_3+b_3c_1+c_3^2\big)\big)\\
& -2b_1c_1^2\Big(\big(a_3c_1+b_2c_3+c_3^2\big)^2(b_1b_2+b_1e_2-b_2e_1)+\big(b_1c_3-b_3c_1\big)^2(b_1d_2+b_2^2+b_2c_3-b_2d_1)+\big(a_3b_1+b_2b_3+b_3\\
& c_3\big)^2(b_1f_2-b_2f_1)\Big),\\
N_7 = {} & b_{21}c_1\big(-b_1\big(a_3c_1+2b_2c_3+2c_3^2\big)+b_2b_3c_1+b_3c_1c_3\big)-c_1c_{31}\big(a_3b_1c_1+b_1c_3^2+b_2b_3c_1\big)+\big(b_1b_2c_3+b_1c_3^2-b_2b_3c_1\\
& -b_3c_1c_3\big)b_{21}c_1,\\
R_7 = {} & b_1\big(b_1b_2(b_1c_1^2d_1+b_1c_3^2f_1-b_2c_1^2e_2)-b_1^3(c_1^2d_2+c_3^2f_2)+b_2^3c_1^2e_1+b_1(a_3b_1c_1+b_1c_3^2+b_2b_3c_1)(b_1f_2-b_2f_1)\big),\\
C_7 = {} & -2b_{21}c_1^2\big(2b_1b_2c_3+b_1c_3^2-2b_2b_3c_1-b_3c_1c_3-(b_2+c_3)(b_1c_3-b_3c_1)\big).
\end{aligned}$$

From Theorem 3 for $\varepsilon>0$ sufficiently small system (29) has a periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ which tends to the equilibrium $(1,1,1)$ when $\varepsilon\to 0$. Therefore it is a periodic solution starting at the zero-Hopf equilibrium $(1,1,1)$ when $\varepsilon=0$. This completes the proof of statement (vii) of Theorem 2.

Example 7. Consider the Lotka-Volterra system

(31) $$\dot{x} = x\big(-2+y+z+4(x-1)^2\big),\ \dot{y} = y\big(-2+y+z-3(x-1)^2+8(y-1)^2\big),\ \dot{z} = z\big(3-2y-z\big).$$

This system in the variables (X,Y,Z) writes

$$\dot{X} = (X+1)\big(4X^2+Y+Z\big),\ \dot{Y} = (Y+1)\big((X+Y+Z)\varepsilon+Y+Z-3X^2+8Y^2\big),\ \dot{Z} = (Z+1)\big(-Z\varepsilon-2Y-Z\big).$$

The corresponding system associated to system (34) is

$$\begin{aligned}
F_1(\theta,r,W) = {} & \sin\theta r\cos\theta(6W-3)-\sin^3\theta r^2+7\sin\theta r^2\cos^2\theta+\sin\theta(-3W^2+W)-\cos\theta\sin^2\theta r^2\\
& -\cos\theta(-3W^2+W)-5\cos^3\theta r^2-r(6W-1)\cos^2\theta,\\
F_2(\theta,r,W) = {} & -\sin\theta r(-W+1)-\cos\theta r(14W-1)+7W^2-W-r^2\cos^2\theta.
\end{aligned}$$

To look for the limit cycles we must solve the system

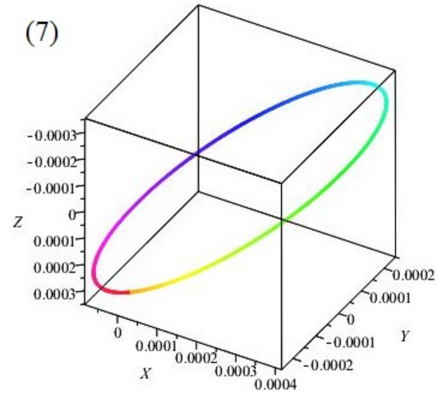
(32) $$f_1(r,W) = -\frac{1}{2}r\big(6W-1\big),\quad f_2(r,W) = 7W^2-\frac{1}{2}r^2-W.$$

This system has the solutions

$$(0,0),\quad (0,1/7),\quad (\sqrt{2}/6,1/6),\quad (-\sqrt{2}/6,1/6).$$

The unique admissible root is $(\sqrt{2}/6,1/6)$.The determinant of the Jacobian matrix of (f_1,f_2) at $(\sqrt{2}/6,1/6)$ is $-1/6$. So system (31) has exactly one limit cycle bifurcating from the origin. We plot this limit cycle for $\varepsilon=10^{-3}$ in the next figure.

(7)



The eigenvalues of the Jacobian matrix of (f_1, f_2) at $(\sqrt{2}/6, 1/6)$ are $4+\pm\sqrt{22})/6$. So this limit cycle is unstable as the one of case (i). This completes the proof of Example 7.

Now we perturb the Lotka-Volterra system (1) with the parameters given in statement (viii) of Proposition 1 satisfying the conditions of Theorem 2(viii). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates and system (1) becomes

(33)

$$\begin{aligned}
\dot{X} &= (X+1)\big(-2c_{31}X\varepsilon + (-b_2-c_3)X + b_1Y + c_1Z + d_1X^2 + e_1Y^2 + f_1Z^2\big),\\
\dot{Y} &= (Y+1)\big(X^2d_2 + Y^2e_2 + Z^2f_2 + Xa_2 + Yb_2 + Zc_2\big),\\
\dot{Z} &= (Z+1)\left(\left(\frac{(a_2b_1c_{31} + (b_2+c_3)b_2c_{31} + c_{31}(b_2c_3 - b_3c_2)X)}{b_1c_2 - b_2c_1} + c_{31}Z\right)\varepsilon + \frac{(a_2(b_1c_3 - b_3c_1) + (b_2+c_3)(b_2c_3 - b_3c_2))X}{b_1c_2 - b_2c_1}\right.\\
&\quad \left. + b_3Y + c_3Z + d_3X^2 + e_3Y^2 + f_3Z^2\right).
\end{aligned}$$

We write the linear part of system (33) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form

$$\begin{pmatrix} 0 & -\dfrac{\sqrt{(b_1c_2 - b_2c_1)A}}{b_1c_2 - b_2c_1} & 0 \\ \dfrac{\sqrt{(b_1c_2 - b_2c_1)A}}{b_1c_2 - b_2c_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $A = -(-b_2^3c_1 + b_1b_2^2c_2 + b_2(-2b_3c_1c_2 + b_1c_2c_3) + c_2(b_1b_3c_2 - b_3c_1c_3 + b_1c_3^2) + a_2(b_1^2c_2 - b_3c_1^2 + b_1c_1(c_3 - b_2)))$. For this we do the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \dfrac{FL}{EA} & -\dfrac{FM}{EA} & -\dfrac{F}{A} \\ -\dfrac{K\sqrt{FA}}{EA} & \dfrac{F\sqrt{FA}}{EA} & 0 \\ -\dfrac{G}{A} & \dfrac{H}{A} & -\dfrac{F}{A} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

where

$$\begin{aligned}
A =\ & a_2b_1c_1c_3 - a_2b_3c_1^2 + b_1b_3c_2^2 + b_1c_2c_3^2 + b_2^2c_1c_3 - 2b_2b_3c_1c_2 - b_3c_1c_2c_3,\ E = a_2c_1^2 - b_1c_2^2 + 2b_2c_1c_2 + c_1c_2c_3,\\
L =\ & a_2b_1c_2 + a_2c_1c_3 + b_2^2c_2 + 2b_2c_2c_3 + c_2c_3^2,\ M = a_2b_1c_1 + b_1c_2c_3 + b_2^2c_1,\ K = a_2c_1 + b_2c_2 + c_2c_3,\\
P =\ & a_2b_1 + b_2^2 + b_2c_3,\ F = b_1c_2 - b_2c_1,\ G = b_2c_3 - b_3c_2,\ H = b_1c_3 - b_3c_1.
\end{aligned}$$

By following the same steps as the first case (i) we get the system

(34)

$$\begin{aligned}\dot r = {} & \frac{1}{FAE\sqrt{FA}}\Big(\big(\sin\theta\sqrt{FA}\big(\big(F^3d_2+F^2K(a_2+b_2+c_3-d_1)-F\big(K^2(b_1-b_2-e_2)-P((c_1-c_2)K+Pf_2)\big)\\
&-K^3e_1-KP^2f_1\big)W^2-2WFKc_{31}\big)-(\cos\theta\big(\big(FE(F^2d_3+K^2e_3-P(-Kb_3+Pc_3)-KPb_3+P^2(c_3+f_3)\big)\\
&+F^2(F(L(b_2+c_3-d_1)+Md_2)-K(Lb_1-Ma_2)+LPc_1)-F\big(K^2(Le_1-Mb_2-Me_2)+KMPc_2+P^2(Lf_1\\
&-Mf_2)\big)\big)W^2-c_{31}(EFG+2F^2L)W)\big)\big)*F(2)+\cos^2\theta\sin\theta\sqrt{FA}r^2\big(F^5d_2+F^4K(a_2+b_2+c_3-d_1)+F^2\\
&(S+Fc_3(c_3+2f_3)))-2FK^3e_1-FK(L(b_1c_1+2c_2e_1)-M(a_2c_1+(c_3+2e_2+2b_2)c_2))-F(S((c_1-c_2)K+\\
&2Pf_2)-P\big(\big(c_1^2+2c_3f_1\big)L-M\big(c_2^2+2c_3f_2\big)\big)\big)\big)+F^2(E(2Fc_1d_3-Kb_3c_3+Pc_3^2)+F(KP(c_1-c_2)-2K^2(b_1\\
&-b_2-e_2)-L(b_1c_2-c_1(2b_2+c_3-2d_1))+M(a_2c_2+2c_1d_2)\big)\big)+2F^4(Fd_2+K(a_2+b_2+c_3-d_1))+2FKP\\
&Sf_1)W-Fc_{31}\big(E(Hc_2+Pc_1)+2F^2K+2FLc_1)\big)\big)-F*A*\big(\cos\theta\sin^2\theta r^2(F^2K(c_1(a_2+2b_2+c_3-2d_1)\\
&-c_2(b_1-2b_2-c_3-2e_2))+F^3(a_2c_2+2c_1d_2)+E(F(c_1^2d_3+c_2^2e_3+c_3^2f_3)+Sc_3)-F\big(K^2(b_1c_1+2c_2e_1)+L(c_1^2\\
&d_1+c_2^2e_1+c_3^2f_1+Fc_1)-M\big((b_2+c_3+e_2)c_2^2+a_2c_1c_2+c_1^2d_2+c_3^2f_2)-S(c_2^2+2c_3f_2)\big)-KS(c_1^2+2c_3f_1)\big)-\\
&\big(\sin^3\theta\sqrt{FA}r^2\big(F\big((b_2+c_3+e_2)c_2^2+a_2c_1c_2+c_1^2d_2+c_3^2f_2)-K(c_1^2d_1+c_2^2e_1+c_3^2f_1+Fc_1)\big)\big)\big)\Big)+F\Big(r\Big(-2A\\
&FKc_1c_{31}+\big(\big(2F^4(Ed_3+Md_2+L(b_2+c_3-d_1))+EF^2\big(2K^2e_3-P(-Kb_3+Pc_3)-KPb_3\big)-F^3(2K(Lb_1-\\
&Ma_2)-LPc_1)-2F^2K^2(Le_1-M(b_2+e_2))-F^2KMPc_2+S\big(-PEF(c_3+2f_3)-F^2Lc_1+F(KMc_2+2P\\
&(Lf_1-Mf_2))\big)\big)W-Fc_{31}(EF(P+G)+2F^2L-2KAc_1+ES)\big)\cos^2\theta+FW(FK(K-F+c_1(b_2+c_3-2\\
&d_1)+c_2(b_2+2e_2))-FP(c_2^2+2c_3f_2)+F^2(a_2c_2+2c_1d_2)-\big((b_1c_1+2c_2e_1)K-P(c_1^2+2c_3f_1))K\big)A\sin\theta^2\big)\\
&-\big(\cos^3\theta r^2\big(F(KS(Eb_3+Mc_2)+F(K^2(Ee_3-Le_1+M(b_2+e_2))-LSc_1)+ES(-Kb_3+Pc_3)\big)+F^4(Ed_3\\
&+Md_2+L(b_2+c_3-d_1))-F^3K\big(Lb_1-Ma_2)+S^2(Ec_3+Ef_3-Lf_1+Mf_2)\big)\big)\Big)\Big)=F_1(\theta,r,W),\end{aligned}$$

$$\begin{aligned}\dot W = {} & -\frac{1}{FA\sqrt{FA}}\Big(\sin\theta\sqrt{FA}r\big(W\big(F^2\big(2Fc_1d_3-Kb_3c_3+Pc_3^2\big)+F\big(GF\big(F-c_1(b_2+c_3-2d_1)\big)-F\big(H(a_2c_2+\\
&2c_1d_2)-K(b_3c_3+2c_2e_3)\big)-P\big(b_1c_3(a_2c_1+2c_2c_3)+b_2c_1c_3(b_2-c_3)+2Fc_3f_3-b_3\big(a_2c_1^2-b_1c_2^2+c_1c_2(2b_2\\
&+c_3)\big)\big)\big)+GF\big(K(b_1c_1+2c_2e_1)-P(c_1^2+2c_3f_1)\big)-HF\big(K(a_2c_1+(2b_2+c_3+2e_2)c_2)-P\big(c_2^2+2c_3f_2\big)\big)\big)-\\
&c_{31}(F(Hc_2+Pc_1)-2Fc_1G)\big)+F\big(\big(F^4d_3-F^2\big(G\big(F(b_2+c_3-d_1)-Kb_1+Pc_1\big)+H(Fd_2+Ka_2)+P\\
&(-Kb_3+Pc_3)-K(Ke_3-Pb_3)-P^2(c_3+f_3)\big)+F\big(G\big(K^2e_1+P^2f_1\big)-HK^2(b_2+e_2)+HP(Kc_2-Pf_2)\\
&\big)\big)W^2+WGF^2c_{31}\big)+r^2\big(\big(F^2(KF(Gb_1-Ha_2+Ke_3)+S(-Kb_3+Pc_3))+F(K^2F(Ge_1-Hb_2-He_2)+\\
&SF(Gc_1+Kb_3)-HKSc_2+S^2(c_3+f_3)\big)-F^4((b_2+c_3-d_1)G+Hd_2-Fd_3)+S^2(Gf_1-Hf_2)\big)\cos^2\theta\\
&+FA\big(G\big(c_1^2d_1+c_2^2e_1+c_3^2f_1+Fc_1\big)-H(c_1^2d_2+c_2^2e_2+c_3^2f_2+Kc_2)+F(c_1^2d_3+c_2^2e_3+c_3^2f_3)+c_3^2(Fc_3+P\\
&c_1)-b_3c_3(-Fc_2+Kc_1)\big)\sin^2\theta\big)-(\cos\theta r(2F^5Wd_3-F^3(2F(GW(b_2+c_3-d_1)+HWd_2)+PW(-Kb_3\\
&+Pc_3)+c_{31}(P+G))+F^2\big(W\big(FG(2Kb_1-Pc_1)-2FHKa_2+2K^2(Fe_3+Ge_1-H(b_2+e_2)\big)+KP(-Fb_3\\
&+Hc_2)+S(Gc_1-P(c_3+2f_3))\big)+c_{31}(2FG-S)\big)-SWF(2GPf_1+H(Kc_2-2Pf_2))))-(\cos\theta\sin\theta\sqrt{FA}\\
&r^2(F^2(2Fc_1d_3+2Kc_2e_3+Pc_3^2+G(F-c_1(b_2+c_3-2d_1))-H(a_2c_2+2c_1d_2))+KF(G(b_1c_1+2c_2e_1)-H\\
&(a_2c_1+(2b_2+c_3+2e_2)c_2))+S\big(2Fc_3f_3-b_3(a_2c_1^2-b_1c_2^2+c_1c_2(2b_2+c_3)\big)+c_3(b_1(a_2c_1+2c_2c_3)+b_2c_1(b_2\\
&-c_3)))+S\big(\big(c_1^2+2c_3f_1)G-H(c_2^2+2c_3f_2)\big)\big)\big)\big)\Big)=F_2(\theta,r,W).\end{aligned}$$

Computing the integrals (4) we obtain

$$f_1(r,W)=\frac{rF\big(T_8W-N_8\big)}{2AE\sqrt{FA}},\ \ f_2(r,W)=-\frac{\big(D_8W^2+R_8r^2+C_8W\big)}{2FA\sqrt{FA}}.$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$(r^*,W^*)=\left(\frac{1}{T)_8}\sqrt{\frac{D_8N_8^2-C_8N_8T_8}{R_8}},\frac{N_8}{T_8}\right),$$

if $T_8 > 0$, $R_8 \neq 0$ and $R_8(D_8 N_8^2 - C_8 N_8 T_8) > 0$. The Jacobian (5) at (r^*, W^*) is $-N_8\big(C_8 T_8 + D_8 N_8\big)/(2\,T_8 A^3 F E)$, where

$$\begin{aligned}
T_8 =\ & 2F^3\big(Ed_3 + \big(b_2 + c_3 - d_1\big)L + Md_2\big) - F^2\big(2K(Lb_1 - Ma_2) - LPc_1 + A(K - a_2c_2 - 2c_1d_2)\big) + F\big(E\big(2K^2e_3 \\
& -P^2c_3\big) - K^2(2Le_1 - 2Mb_2 - 2Me_2 - A) - K(MPc_2 - A((b_2 + c_3 - 2d_1)c_1 + c_2(b_2 + 2e_2))) - PA\big(c_2^2 + \\
& 2c_3f_2\big) - LSc_1\big) - KA\big(K(b_1c_1 + 2c_2e_1) - P\big(c_1^2 + 2c_3f_1\big)\big) - S(EP(c_3 + 2f_3) - KMc_2 - 2P(Lf_1 - Mf_2)), \\
R_8 =\ & F^3K\big(Gb_1 - Ha_2 + Ke_3\big) - F\big(HKSc_2 - S^2(c_3 + f_3) - A\big(G\big(c_1^2d_1 + c_2^2e_1 + c_3^2f_1\big) - H\big(c_1^2d_2 + c_2^2e_2 + c_3^2f_2 + \\
& Kc_2\big) - c_1c_3(Kb_3 - Pc_3)\big)\big) + F^2\big(K^2(Ge_1 - Hb_2 - He_2) + \big(b_3c_2c_3 + c_3^3 + Gc_1 + c_1^2d_3 + c_2^2e_3 + c_3^2f_3\big)A + S \\
& (Gc_1 + Pc_3)\big) + F^5d_3 + S^2(Gf_1 - Hf_2) - F^4\big((b_2 + c_3 - d_1)G + Hd_2\big), \\
N_8 =\ & c_{31}\big(EF(G + P) + 2F^2L + 2KAc_1 + ES\big), \\
D_8 =\ & 2F^5d_3 - 2F^4\big((b_2 + c_3 - d_1)G + Hd_2\big) + 2F^3\big(K\big(Gb_1 - Ha_2 + Ke_3\big) - P\big(Gc_1 - Pf_3\big)\big) + 2F^2\big(G\big(K^2e_1 + \\
& P^2f_1\big) - HK^2(b_2 + e_2) + HP(Kc_2 - Pf_2)\big), \qquad C_8 = 2F^3Gc_{31}.
\end{aligned}$$

From Theorem 3 for $\varepsilon > 0$ sufficiently small system (33) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ which tends to the equilibrium $(1,1,1)$ when $\varepsilon \to 0$. Therefore it is a periodic solution starting at the zero-Hopf equilibrium $(1,1,1)$ when $\varepsilon = 0$. This completes the proof of statement (viii) of Theorem 2.

Example 8. Consider the Lotka-Volterra system

(35)
$$\dot{x} = x\big(x - 10 + 6y + 3z\big),\ \dot{y} = y\left(-\frac{1}{3}x + \frac{4}{3} - 2y + z - 2(z-1)^2\right),\ \dot{z} = z\big(x - 1 - y + z - (x-1)^2 - 3(y-1)^2 - (z-1)^2\big).$$

This system in the variables (X, Y, Z) writes

$$\begin{aligned}
\dot{X} =\ & (X+1)\big(-2X\varepsilon + X + 6Y + 3Z\big), \\
\dot{Y} =\ & (Y+1)\left(-2Z^2 - \frac{1}{3}X - 2Y + Z\right), \\
\dot{Z} =\ & (Z+1)\left(\left(-\frac{1}{12}X + Z\right)\varepsilon - \frac{1}{6}X - Y + Z - X^2 - 3Y^2 - Z^2\right).
\end{aligned}$$

The corresponding system associated to system (34) is

$$\begin{aligned}
F_1(\theta, r, W) =\ & -\frac{913}{6}\cos^2\theta\sin\theta r^2 + \frac{464}{3}\sin\theta\cos\theta rW + \frac{824}{3}\sqrt{2}\cos^3\theta r^2 + \frac{49}{6}\sin\theta\cos\theta r - \frac{4}{3}r^2\sin\theta \\
& -\frac{1808}{3}\sqrt{2}\cos^2\theta Wr - \frac{16}{3}\sin\theta W - \frac{5}{3}\sqrt{2}\cos^2\theta r + 312\sqrt{2}\cos\theta W^2 + 34\sqrt{2}\cos\theta r^2 + \frac{10}{3}\sqrt{2} \\
& \cos\theta W - \frac{56}{3}\sqrt{2}Wr - \frac{2}{3}\sqrt{2}r, \\
F_2(\theta, r, W) =\ & -\frac{1}{864}\big(6\sin\theta\sqrt{2}r(-10800W - 180) - 269568W^2 - 1728W + r^2(-267624\cos^2\theta - 11664\sin^2\theta) \\
& -\cos\theta r(-537840W - 864) + 64800\sin\theta\cos\theta\sqrt{2}r^2\big)\sqrt{2}.
\end{aligned}$$

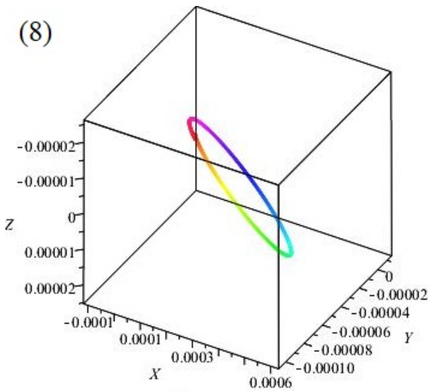
To look for the limit cycles we must solve the system

$$f_1(r, W) = -\frac{1}{216}r\big(69120W + 324\big)\sqrt{2}, \quad f_2(r, W) = -\frac{1}{1728}\big(-539136W^2 - 279288r^2 - 3456W\big)\sqrt{2}.$$

This system has the solutions

$$(0,0), \quad (0, -1/156), \quad (\sqrt{18533}/34480, -3/640), \quad (-\sqrt{18533}/34480, -3/640)$$

The unique admissible solution is $(\sqrt{18533}/34480, -3/640)$. The determinant of the Jacobian matrix of (f_1, f_2) at this solution is $129/40$. Hence system (35) has exactly one limit cycle bifurcating from the equilibrium $(1,1,1)$. We plot this limit cycle for $\varepsilon = 10^{-3}$ in the next figure.



The eigenvalues of the Jacobian matrix of (f_1, f_2) at $(\sqrt{18533}/34480, -3/640)$ are $-37\sqrt{2} \pm \sqrt{17902}\,i)/80$. So this limit cycle is stable. This complestes the proof of Example 8. ☐

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