THE SOLUTION OF THE POINCARÉ PROBLEM ON THE RATIONAL FIRST INTEGRAL FOR THE LIÉNARD POLYNOMIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work we classify the polynomial Liénard differential equations $\ddot{x} + f(x)\dot{x} + x = 0$, having a rational first integral. Such classification was asked by Poincaré in 1891 for any general polynomial differential systems in the plane $\mathbb{R}^2$. As far as we know it is the first time that the complete classification is given for a relevant class of polynomial differential equations of arbitrary degree.

1. Introduction and statement of the main results

We consider a *polynomial differential system* that we can write as

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{x} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials in the variables x and y, and t is the independent variable. The degree of the polynomial differential system (1) is the maximum degree of the polynomials P and Q. The polynomial differential system (1) has associated the *polynomial vector field* $\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$.

Let U be an open subset of $\mathbb{R}^2$. A *first integral* is defined as a $\mathcal{C}^1$ non–locally constant function $H : U \to \mathbb{R}$ such that it is constant on the solutions $(x(t), y(t))$ of the polynomial differential system (1) contained in U, that is, satisfies $\mathcal{X}H = P(x, y)\partial H/\partial x + Q(x, y)\partial H/\partial y \equiv 0$ in U. We say that H is a *rational first integral* when the function H is rational.

Let $F(x, y)$ be a real polynomial in the variables x and y. The algebraic curve $F(x, y) = 0$ is an *invariant algebraic curve* of a polynomial differential system (1) if for some polynomial $K = K(x, y)$ the equation

$$\mathcal{X}F = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = KF, \tag{2}$$

is satisfied. The curve $F = 0$ is formed by trajectories of the vector field $\mathcal{X}$ because on the points of the algebraic curve $F = 0$ the gradient

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