# ON THE INTEGRABILITY OF THE GEODESIC FLOW ON HOMOGENEOUS SURFACES

JAUME LLIBRE[1], RAFAEL RAMÍREZ[2] AND VALENTÍN RAMÍREZ[3]

ABSTRACT. The three dimensional Euclidean space $\mathbb{E}^3$ is the configuration space of a particle which is constrained to move on the surface $f(x, y, z) = c$. If there is no external force acting on the particle, then it moves with a velocity of constant norm, and with acceleration orthogonal to the surface. The trajectory of this particle is called a geodesic curve. In this paper we study the geodesic flow when $f$ is a homogenous function of degree $m$.

We provide a new easier expression for the Gaussian curvature of any homogenous surface $f = c$. We put special attention to the Gaussian curvature of the homogenous surfaces $f = c \neq 0$ satisfying

$$||f_{\mathbf{x}}||^2 := \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = g(r, f),$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and $f_{\mathbf{x}} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ and $\mathbf{x} = (x, y, z)$. We prove that the geodesic flow of the particle on this surface is described by the flow of the differential equation

$$\mathbf{x}'' = -\sigma\, K\, f_{\mathbf{x}}, \quad < f_{\mathbf{x}}, \mathbf{x}' >= 0,$$

where $K = \dfrac{1}{\sigma} \dfrac{mf}{2rg^2(r, f)} \dfrac{\partial g(r, f)}{\partial r}$ is the Gaussian curvature of the given surface, $\sigma = \sigma(x, y, z)$ is a convenient nonzero function, and $< \, , \, >$ is the inner product of $\mathbb{E}^3$. Moreover we show that the geodesic flow on these homogeneous surfaces is integrable.

The obtained results are illustrated with some examples. In particular we apply our results to the surface $f = \sqrt{x^2 + y^2 + z^2} + b_1 x + b_2 y + b_3 z = c$, that appears in the Kepler problem, where $\mathbf{b} = (b_1, b_2, b_3)$ is the Laplace-Runge-Lenz vector.

## 1. Introduction and statement of the main results

Let $\mathbb{E}^3$ be the configuration space with coordinates $\mathbf{x} = (x, y, z)$ of a particle which is constrained to move on a surface $f(x, y, z) = c$. If there is no external force acting on the particle, then the particle will move with a velocity of constant norm, and with acceleration orthogonal to the surface.

The motion of the particle is described by the second-order differential equation

$$\mathbf{x}'' = \mu\, f_{\mathbf{x}}, \qquad < \mathbf{x}', f_{\mathbf{x}} >= 0, \tag{1}$$

---

2010 *Mathematics Subject Classification.* Primary 53A05, 53C30. Secondary 53C80, 53Z05.

*Key words and phrases.* geodesic flow, first integral, homogeneous surface, vector field, integrability, Gaussian curvature, developable surface, Kepler problem.