PERIODIC ORBITS OF THE PLANAR ANISOTROPIC KEPLER PROBLEM

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Abstract. In this paper we prove that at every energy level the anisotropic problem with small anisotropy has two periodic orbits which bifurcate from elliptic orbits of the Kepler problem with high eccentricity. Moreover we provide approximate analytic expressions for these periodic orbits. The tool for proving this result is the averaging theory.

1. Introduction

The anisotropic Kepler problem is a modified model of the Kepler problem. This model can be used to describe the motion of two-body in an anisotropic configuration plane under a mutual gravitational attraction described by Newtonian's universal law of gravitation.

The anisotropic Kepler problem comes originally from quantum mechanics, in this model the flat space is replaced with an anisotropic one, it was introduced by Gutzwiller in [8, 9, 10, 11], and after studied by other authors see for instance Devaney [3, 4, 5], and Casasayas and Llibre [2].

Gutzwiller in his researches wanted to find an approximation of the quantum mechanical energy levels for a chaotic system. He chooses to study the anisotropic Kepler problem because it is a chaotic system and it is considered one of the most suitable models to study physical phenomena faced in semiconductors.

1.1. **The planar anisotropic Kepler problem.**

The equations of motion of the planar anisotropic problem in Hamiltonian formulation are

$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -\frac{\partial H}{\partial q_i}, \quad \text{for } i = 1, 2, \tag{1}$$

with

$$H = H(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2),$$

where the potential is

$$V(q_1, q_2) = -\frac{1}{\sqrt{(1+\varepsilon)q_1^2 + q_2^2}}.$$

Here $|\varepsilon| > 0$ denotes a small parameter and the prime denotes derivative with respect to the time variable t.

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We change these equations of motion to the McGehee coordinates (r, θ, v, u), more precisely we choose polar coordinates (r, θ) in the configuration space, together with scaled versions (u, v) of the radial and angular components of velocity as follows

$$\begin{aligned} (q_1, q_2) = &\quad r(\cos\theta, \sin\theta), \\ r^{-1/2} v = &\quad (p_1, p_2) \cdot (\cos\theta, \sin\theta), \\ r^{-1/2} u = &\quad (p_1, p_2) \cdot (-\sin\theta, \cos\theta), \end{aligned}$$

for more details in these coordinates see [4, 5, 14]. So in the McGehee coordinates the equations of motion (1) write

$$\begin{aligned} r' = &\quad r^{-1/2} v, \\ \theta' = &\quad r^{-3/2} u, \\ v' = &\quad r^{-3/2} \left(u^2 + \frac{1}{2} v^2 + V(\theta) \right), \\ u' = &\quad r^{-3/2} \left(-\frac{1}{2} vu - \frac{dV(\theta)}{d\theta} \right), \end{aligned} \tag{2}$$

where

$$V(\theta) = -\frac{1}{\sqrt{(1+\varepsilon)\cos^2\theta + \sin^2\theta}}$$

and, the energy level $H = h$ becomes

$$\frac{1}{2}(v^2 + u^2) + V(\theta) = rh. \tag{3}$$

The equations of motion (2) have a collision singularity at $r = 0$, which can be removed with the following change in the independent variable

$$\frac{dt}{d\tau} = r^{3/2}.$$

The equations of motion (2) in the new time τ goes over to

$$\begin{aligned} \dot{r} = &\quad rv, \\ \dot{\theta} = &\quad u, \\ \dot{v} = &\quad u^2 + \frac{1}{2} v^2 + V(\theta), \\ \dot{u} = &\quad -\frac{1}{2} vu - \frac{dV(\theta)}{d\theta}, \end{aligned} \tag{4}$$

here the dot denotes derivative with respect to τ.

In short, we have that if $\varepsilon = 0$ the equations of motion (4) corresponds to the planar Kepler problem. If $|\varepsilon| > 0$ and small, system (4) provides the equations of motion of the planar anisotropic Kepler problem.

1.2. **Our main result.** In this paper we study analytically the periodic orbits of the planar anisotropic Kepler problem in its energy levels using the averaging theory.

Theorem 1. *In every energy level* (3) *the anisotropic Kepler problem* (4) *has two periodic orbits bifurcating from two elliptic orbits of the Kepler problem with the same eccentricity* 0.95502193.. *and with the arguments of the pericenter given by* $\theta_0 = 0.90077616..$ *and* $2\pi - \theta_0$ *radians.*

The values of the eccentricity and of the argument of the pericenter which appear in the statement of Theorem 1 are zeros of explicit functions given in the proof of this theorem in section 2

We note that the existence of periodic orbits for the spatial anisotropic Kepler problem has been studied topologically by Devaney in [4] and Casasayas and Llibre in [2]. As far as we know this is the unique article which studies the periodic orbits of the anisotropic Kepler problem analytically. Gutzwiller [8] has studied numerically some periodic solutions of the anisotropic Kepler problem.

Note that our definition of the anisotropic Kepler problem coincides with the one of Gutzwiller, Devaney, Casasayas and Llibre. There are other different problems also called anisotropic Kepler problem, as the ones studied by Diacu, Pérez-Chavela and Santoprete [6], Escalona-Buendía and Pérez-Chavela [7], and López, Martínez and Vera [12]. These last authors also study analytically some periodic orbits of a different anisotropic Kepler problem using Delaunay variables.

2. Proof of Theorem 1

In what follows we shall study the periodic orbits of the anisotropic Kepler problem in the energy levels $H = h \in \mathbb{R}$. So in every energy level $H = h$ we compute $r = r(\theta, v, u, h)$, and we get

$$r = \frac{u^2+v^2-2}{2h} + \varepsilon \frac{1+\cos 2\theta}{4h} + O(\varepsilon^2).$$

Then the equations of motion (4) in the energy level $H = h$ taking as independent variable the variable θ become

$$\begin{aligned} \dot{v} &= \frac{dv}{d\theta} = \frac{2u^2+v^2-2}{2u} + \varepsilon \frac{\cos 2\theta}{2u} + O(\varepsilon^2), \\ \dot{u} &= \frac{du}{d\theta} = -\frac{1}{2}v + \varepsilon \frac{\cos\theta \, \sin\theta}{u} + O(\varepsilon^2). \end{aligned} \tag{5}$$

The unperturbed system when $\varepsilon = 0$ is

$$\dot{v} = \frac{2u^2+v^2-2}{2u}, \qquad \dot{u} = -\frac{1}{2}v,$$

which has the general solution

$$(v(\theta; e, \theta_0), u(\theta; e, \theta_0)) = \left(\frac{e\,\sin(\theta-\theta_0)}{\sqrt{1+e\,\cos(\theta-\theta_0)}}, \sqrt{1+e\,\cos(\theta-\theta_0)} \right). \tag{6}$$

This is the solution of the planar Kepler problem with eccentricity e and being θ_0 the angle of the direction of the pericenter, i.e. the argument of the pericenter. Of course, if $e = 0$ the solution is circular, if $e \in (0, 1)$ the solution is elliptic, if $e = 1$ the solution is parabolic, and if $e > 1$ the solution is hyperbolic.

We are interested in knowing what are the periodic solutions of the Kepler problem which can be extended to periodic solutions of the anisotropic Kepler problem, i.e. what solutions (6) with eccentricity $e \in [0,1)$ can be extended. For doing this study we shall use the averaging theory of first order described in the appendix.

In order to apply the averaging theory to the differential system (5) we identify our variables with the ones of averaging theory given in the appendix, and we obtain

$$\mathbf{x} = \begin{pmatrix} v \\ u \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} e \\ \theta_0 \end{pmatrix}, \quad \begin{pmatrix} t \\ T \end{pmatrix} = \begin{pmatrix} \theta \\ 2\pi \end{pmatrix},$$

$$F_0 = \begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix}, \quad F_1 = \begin{pmatrix} F_{11} \\ F_{12} \end{pmatrix}, \quad \mathbf{x}(t;\mathbf{z},0) = \begin{pmatrix} v(\theta;e,\theta_0) \\ u(\theta;e,\theta_0) \end{pmatrix},$$

where

$$F_{01} = \frac{2u^2+v^2-2}{2u}, \quad F_{11} = \frac{\cos^2\theta}{4u},$$
$$F_{02} = -\frac{1}{2}v, \qquad F_{12} = \frac{\cos\theta\,\sin\theta}{u}.$$

The first variational equation (11) of the unperturbed system (5) along the periodic solution (6) with $e \in [0,1)$ is

$$\begin{pmatrix} y_1' & y_2' \\ y_3' & y_4' \end{pmatrix} = J \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, \tag{7}$$

where

$$J = D_x F_0(t,\mathbf{x}(t;\mathbf{z},0)) = \left.\begin{pmatrix} \dfrac{\partial F_{01}}{\partial v} & \dfrac{\partial F_{01}}{\partial u} \\[2ex] \dfrac{\partial F_{02}}{\partial v} & \dfrac{\partial F_{02}}{\partial u} \end{pmatrix}\right|_{(v,u)=(v(\theta;e,\theta_0),u(\theta;e,\theta_0))},$$

and consequently

$$J = \begin{pmatrix} \dfrac{e\,\sin(\theta-\theta_0)}{1+e\,\cos(\theta-\theta_0)} & \dfrac{3}{2} + \dfrac{1-e^2}{2(1+e\,\cos(\theta-\theta_0))^2} \\[2ex] -\dfrac{1}{2} & 0 \end{pmatrix}.$$

The fundamental matrix $M_{\mathbf{z}}(\theta)$ of the differential system (7) such that $M_{\mathbf{z}}(0)$ is the identity matrix is

$$M_{\mathbf{z}}(\theta) = \begin{pmatrix} y_1(\theta;e,\theta_0) & y_2(\theta;e,\theta_0) \\ y_3(\theta;e,\theta_0) & y_4(\theta;e,\theta_0) \end{pmatrix},$$

where

$$y_1(\theta;e,\theta_0) = \frac{\sqrt{1+e\,\cos\theta_0}\,\left[4\cos\theta + e\left[\cos(2\theta-\theta_0)+3\cos\theta_0\right]\right]}{4\left[1+e\,\cos(\theta-\theta_0)\right]^{3/2}},$$

$$y_2(\theta;e,\theta_0) = \frac{q(\theta)}{4\left[1+e\,\cos(\theta-\theta_0)\right]^{3/2}\sqrt{1+e\,\cos\theta_0}},$$

$$y_3(\theta;e,\theta_0) = -\frac{\sin\theta\sqrt{1+e\,\cos\theta_0}}{2\sqrt{1+e\cos(\theta-\theta_0)}},$$

$$y_4(\theta;e,\theta_0) = \frac{e\,\sin\theta\sin\theta_0 + 2\cos\theta\left[1+e\,\cos\theta_0\right]}{2\sqrt{(1+e\,\cos\theta_0)(1+e\,\cos(\theta-\theta_0))}},$$

with

$$q(\theta) = 8\sin\theta + 8e(2+\cos\theta)\cos\frac{\theta-2\theta_0}{2}\sin\frac{\theta}{2} + e^2(\cos\theta + 3\cos(\theta-2\theta_0)\sin\theta).$$

Now we compute the integrand $(G_1, G_2) = M_{\mathbf{z}}^{-1}(\theta)F_1(\theta, \mathbf{x}(t;\mathbf{z},0))$ of the integral (12), and we obtain

$$G_1 = \frac{\cos\theta\sin\theta\, p(\theta)}{4(1+e\cos(\theta-\theta_0)(1+e\cos\theta_0)^{3/2}},$$
$$G_2 = \frac{(10\cos\theta + 3e\cos(2\theta-\theta_0) + 7e\cos\theta_0)\sin(2\theta)}{16(1+e\cos(\theta-\theta_0))\sqrt{1+e\cos\theta_0}},$$

where

$$\begin{aligned} p(\theta) = \ & -\Big(8 + e\big(e(\cos\theta + 3\cos(\theta-2\theta_0)) + 4(\cos\theta+2)\cos(\theta_0)\big)\Big)\sin\theta \\ & +2e(2\cos\theta + \cos(2\theta) - 3)\sin\theta_0. \end{aligned}$$

We compute the integral $\mathcal{F}(\mathbf{z})$ of (12) and we obtain

$$\mathcal{F}(\mathbf{z}) = \mathcal{F}(e,\theta_0) = \begin{pmatrix} f_1(e,\theta_0) \\ \\ f_2(e,\theta_0) \end{pmatrix},$$

where

(8)

$$\begin{aligned} f_1(e,\theta_0) = \ & \frac{g(e,\theta_0)}{16e^3(1+e\cos\theta_0)^{3/2}}, \\ f_2(e,\theta_0) = \ & \frac{\left(\sqrt{1-e^2}-1\right)\sin\theta_0 e^2 + \left(\left(\sqrt{1-e^2}-3\right)e^2 - 4\sqrt{1-e^2}+4\right)\sin(3\theta_0)}{8e^3\sqrt{1+e\cos\theta_0}}, \end{aligned}$$

with

$$\begin{aligned} g(e,\theta_0) = \ & -\left(\sqrt{1-e^2}-1\right)e^3 - 4\left(\sqrt{1-e^2}-1\right)\cos\theta_0 e^2 \\ & -6\left(e^2+2\sqrt{1-e^2}-2\right)\cos(2\theta_0)e \\ & +\left(\left(\sqrt{1-e^2}-3\right)e^2 - 4\sqrt{1-e^2}+4\right)\left(4\cos(3\theta_0) + e\cos(4\theta_0)\right). \end{aligned}$$

Now we have to study the solutions (e,θ_0) of the system $f_1(e,\theta_0) = 0$, $f_1(e,\theta_0) = 0$ with $e \in (0,1)$. Note that since in the expressions of the functions $f_1(e,\theta_0)$ and $f_2(e,\theta_0)$ the e appears in the denominators we cannot consider $e = 0$, i.e. with the averaging theory of first order we cannot study if some circular periodic solutions of the Kepler problem can be extended to a periodic solution of the anisotropic Kepler problem.

For solving the system $f_1(e,\theta_0) = 0$, $f_2(e,\theta_0) = 0$ with $e \in (0,1)$, we first solve the equation $f_2(e,\theta_0) = 0$ with respect to e, and we obtain

$$e_1 = 0, \quad e_2 = -\frac{2\sqrt{\sin\theta_0\sin(3\theta_0)}}{|\sin\theta_0 + \sin(3\theta_0)\,|}, \quad e_3 = \frac{2\sqrt{\sin\theta_0\sin(3\theta_0)}}{|\sin\theta_0 + \sin(3\theta_0)\,|}.$$

Since $e \in (0,1)$, then the first two zeros e_1 and e_2 are not acceptable, and the third zero e_3 is only valid for the values of θ_0 satisfying the conditions

$$\sin(3\theta_0) > \sin\theta_0 \quad \text{and} \quad \sin(3\theta_0)\sin\theta_0 > 0,$$

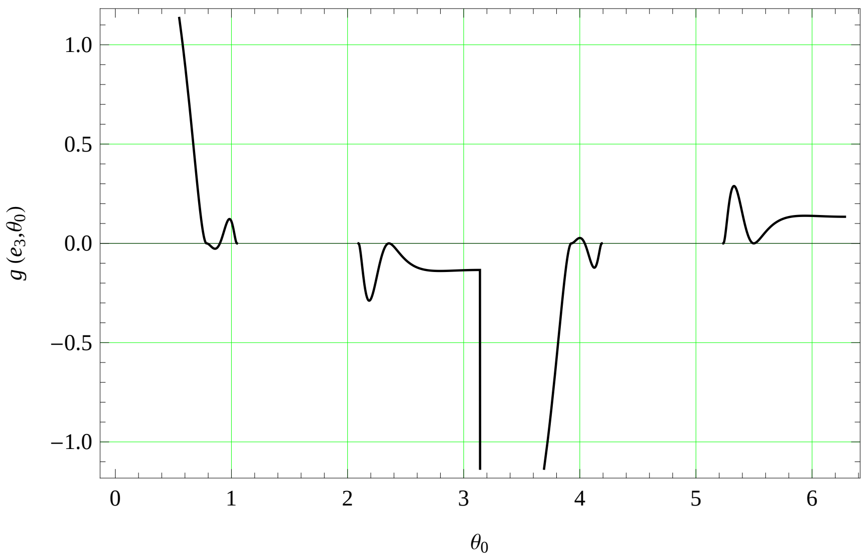

FIGURE 1. The graphic of the function $g(e_3, \theta_0)$ for $\theta_0 \in [0, 2\pi)$.

which are equivalent to say that $e_3 \in (0, 1)$.

Now let $g(e_3, \theta_0)$ be the numerator of the function $f_1(e, \theta_0)$ evaluated at $e = e_3$. This function has ten zeros in the interval $\theta_0 \in [0, 2\pi)$ as it is shown in Figure 1. These zeros are $\theta_0^1, \theta_0^2, ..., \theta_0^{10}$ and the corresponding values of the eccentricity $e_0^1, e_0^2, ..., e_0^{10}$ related with these zeros are given in Table 1. It is noted that there are only two values of $e_3 \in (0, 1)$, namely e_0^2 and e_0^7, these two values coincide, i.e. $e_0^2 = e_0^7 = 0.95502193..$, while $\sin\theta_0^7 = -\sin\theta_0^2$, and this means that $\theta_0^7 = 2\pi - \theta_0^2$.

TABLE 1. The solutions (θ_0^k, e_0^k) of system (8).

k	θ_0^k	e_0^k	case
1	0.78539818..	1.00000000..	Parabolic
2	0.90077616..	0.95502193..	Elliptic
3	1.04719755..	0.00000000..	Circular
4	2.09439510..	0.00000000..	Circular
5	2.35619449..	1.00000000..	Parabolic
6	3.92699081..	1.00000000..	Parabolic
7	4.04236881..	0.95502193..	Elliptic
8	4.18879020..	0.00000000..	Circular
9	5.23598775..	0.00000000..	Circular
10	5.49778714..	1.00000000..	Parabolic

Since the Jacobian (13) at the solutions (e_0^2, θ_0^2) and (e_0^7, θ_0^7) is $1/3 \neq 0$, by Theorem 2 for $|\varepsilon|$ sufficiently small we have at every energy level two periodic solutions which when $\varepsilon \to 0$ tend to two elliptic periodic orbits of the Kepler problem with the same eccentricity but with arguments of perigee given by $\theta_0 = 0.90077616..$ and $2\pi - \theta_0$ radians. This completes the proof of Theorem 1.

Appendix: Basic results on averaging theory

We want to provide sufficient conditions for the existence of T–periodic solutions for the differential systems

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{9}$$

when ε is sufficiently small. The functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the variable t, being Ω an open subset of $\mathbb{R}^n$. We assume that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \tag{10}$$

has a submanifold M of dimension n filled of periodic solutions of period T.

Let $\mathbf{x}(t; \mathbf{z}, 0)$ be the solution of system (10) satisfying $\mathbf{x}(0; \mathbf{z}, 0) = \mathbf{z}$. The first variational equation of the unperturbed system along the periodic solution $\mathbf{x}(t; \mathbf{z}, 0)$ is

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t; \mathbf{z}, 0))\mathbf{y}. \tag{11}$$

As usual we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (11).

Suppose that there exists an open set V with $\mathrm{Cl}(V) \subset M$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t; \mathbf{z}, 0)$ is T–periodic. Then the next result provides sufficient conditions in order that the some periodic solutions of V persist for values of $\varepsilon \neq 0$ sufficiently small.

Theorem 2 (Perturbations of an isochronous set)**.** *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V) \subset M$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t; \mathbf{z}, 0)$ is T–periodic, then we define the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$ as*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{T} \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t; \mathbf{z}, 0)) dt. \tag{12}$$

Assume that there exists $\mathbf{z_0} \in V$ with $\mathcal{F}(\mathbf{z}) = 0$ and

$$\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(\mathbf{z_0})\right) \neq 0. \tag{13}$$

Then the following statements hold.

- (a) *There exists a T–periodic solution $\varphi(t, \varepsilon)$ of system (9) such that $\varphi(0, \varepsilon) \to \mathbf{z}_0$ as $\varepsilon \to 0$.*
- (b) *The kind of stability or instability of the limit cycle $\varphi(t, \varepsilon)$ is given by the eigenvalues of the Jacobian matrix $\left(d\mathcal{F}/d\mathbf{z}\right)(\mathbf{z_0})$.*

Theorem 2 goes back to Malkin [13] and Roseau [15], for a shorter proof see [1].

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