ON THE INTEGRABILITY OF POLYNOMIAL VECTOR FIELDS IN THE PLANE BY MEANS OF PICARD-VESSIOT THEORY

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Abstract. We study the integrability of polynomial vector fields using Galois theory of linear differential equations when the associated foliations is reduced to a Riccati type foliation. In particular we obtain integrability results for some families of quadratic vector fields, Liénard equations and equations related with special functions such as Hypergeometric and Heun ones. The Poincaré problem for some families is also approached.

Introduction

Given a *polynomial differential system* in $\mathbb{C}^2$,

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

with $P, Q \in \mathbb{C}[x,y]$, we consider its associated differential vector field

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \tag{2}$$

whose integral curves are intimately related to the solutions of system (1). These solutions, taken as curves on the plane and leaving for a while its time-dependence, constitute its so-called *foliation* and satisfy the first order differential equation

$$y' = \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}. \tag{3}$$

This expression (3) is often written as a Pfaff equation

$$\Omega = 0, \tag{4}$$

where $\Omega = Q(x,y)dx - P(x,y)dy$ is the corresponding differential 1-form. The connection between integral curves of the vector field X and solutions of $\Omega = 0$ is clear:

- geometrically, it is given by $\Omega \cdot X = 0$, which means that the vector field X is tangent to the leaves of the foliation (the orbits) defined by (4);

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