DARBOUX INTEGRALS FOR SCHRÖDINGER PLANAR VECTOR FIELDS VIA DARBOUX TRANSFORMATIONS

PRIMITIVO B. ACOSTA-HUMÁNEZ AND CHARA PANTAZI

Abstract. In this paper we study the Darboux transformations of planar vector fields of Schrödinger type. Using the isogaloisian property of Darboux transformation we prove the "invariance" of the elements of the "Darboux Theory of Integrability". In particular, we also show how the shape invariance property of the potential is important in order to preserve the structure of the transformed vector field. Free particle, square well, harmonic oscillator, three dimensional harmonic oscillator and Coulomb potential, are presented as natural examples coming from supersymmetric quantum mechanics.

Introduction

In this paper we connect different results of Darboux: We deal with some generalization of the Darboux theory of integrability for planar vector fields and the Darboux transformation of the associated equation.

Darboux published in 1882 the paper [15] in where he presents as a proposition in a general way, which in particular case the history proved to be a notable theorem today known as *Darboux transformation*, see Theorem 12 and Corollary 13. Darboux had shown that whenever one knows to integrate the equation

$$\frac{d^2y}{dx^2} = (\Phi(x) + m)y,$$

for all the values of the constant m, one can obtain an infinite set of equations, displaying the variable parameter in the same way, which are integrable for any value of the parameter. This proposition also can be found in his book [16, p. 210]. Curiosly Darboux transformation was forgotten for a long time until that was recovered as an exercise in 1926 by Ince (see exercises 5, 6 and 7 [20, p. 132]) following closely the formulation of Darboux given in [15, 16]. In 1930, P. Dirac publishes *The Principles of Quantum Mechanics*, in where he gave a mathematically rigorous formulation of quantum

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