The center problem for $\mathbb{Z}_2$-symmetric nilpotent vector fields

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Abstract

We say that a polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ having the origin as a singular point is $\mathbb{Z}_2$-symmetric if $P(-x, -y) = -P(x, y)$ and $Q(-x, -y) = -Q(x, y)$.

It is known that there are nilpotent centers having a local analytic first integral, and others which only have a C^∞ first integral. But up to know there are no characterized these two kinks of nilpotent centers.

Here we prove that the origin of any $\mathbb{Z}_2$-symmetric is a nilpotent center if, and only if, there is a local analytic first integral of the form $H(x, y) = y^2 + \cdots$, where the dots denote terms of degree higher than two.

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1 Introduction and statement of the main results

We consider a planar differential system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1.1}$$

with P and Q defined and analytic in a neighborhood of the origin where the origin is an isolated singular point. The local phase portrait near an isolated singular point can be determined by the Hartman-Grobman theorem except for the case of a monodromic singularity. We recall that a singular point is monodromic when nearby orbits rotate around it. For analytic differential systems it is known that the unique monodromic singularities are centers and foci. We recall that a center is a singular point for which there exists a punctured neighborhood filled of periodic orbits, and a focus has a punctured neighborhood filled of spiraling orbits. The *center problem* consists in distinguishing between a center or a focus at a monodromic singular point. If the linear part has pure imaginary eigenvalues or has zero eigenvalues but the linear part is not identically zero then there exist algorithms to find the necessary conditions to have a center, see [10, 15, 25]. However, the characterization when the linear part is totally zero is an open problem, see [17, 18, 19, 20, 22] for some partial results.

In this work we focus on nilpotent singularities, that is, the case when the linear part has two zero eigenvalues but the linear part is nonzero. For such singularities, unlike the case of pure imaginary eigenvalues, does not exist, in general, an analytic first integral in a neighborhood of the origin when the singular point is a center, see [11]. Nevertheless an interesting question is: What nilpotent centers still have an analytic first integral like the linear type centers? Of course the nilpotent Hamiltonian systems is a big family which have this property. But the question is if there exist other big families with this characteristic. It is well known that all the centers, and in particular the nilpotent centers always have a C^∞ first integral, see [24].

In [9] it is considered the following differential system

$$\dot{x} = y + X_{2n+1}(x, y), \qquad \dot{y} = Y_{2n+1}(x, y), \tag{1.2}$$

where X_{2n+1} and Y_{2n+1} are homogeneous polynomials of degree $2n+1$ and the origin is a monodromic singular point. The change of variables $x = x_1 - \alpha(-\beta)^{-1/2}y_1$, $y = (-\beta)^{-1/2}y_1$ and $dt = (-\beta)^{-1/2}\mathtt{d}\tau$ where $\alpha = X_{2n+1}(1, 0)$ and $\beta = Y_{2n+1}(1, 0)$ transforms system (1.2) into the system

$$\dot{x} = y + P_{2n+1}(x, y), \qquad \dot{y} = Q_{2n+1}(x, y), \tag{1.3}$$

where P_{2n+1} and Q_{2n+1} are homogeneous polynomials of degree $2n+1$ with $P_{2n+1}(1, 0)$ $= 0$, and $Q_{2n+1}(1, 0) = -1$. From [8] it is known that for system (1.3) there exists a formal series of the form

$$U = (n + 1)y^2 + \sum_{k=1}^{\infty} P_{2(kn+1)}(x, y), \tag{1.4}$$

where $P_{2(kn+1)}$ are homogeneous polynomials of degree $2(kn+1)$ and $P_{2(n+1)}(1, 0) = 1$, such that its derivative along the trajectories of system (1.3) takes the form

$$\frac{dU}{dt} = \sum_{k=1}^{\infty} f_k\, x^{2(k+1)n+2},$$

where f_k are the focus quantities at the origin of system (1.3). In fact if $P_{2(kn+1)}(0, 1) =$ 0, then the formal series (1.4) is uniquely determined. In [8] it is proved that the origin of system (1.3) is a center if and only if $f_k = 0$ for all k and if $f_k = 0$ for $k = 1, \ldots, m-1$ but $f_m \neq 0$ then the origin is a focus of order m.

This result for system (1.3) was generalized in [1] to the following analytic family of planar vector fields

$$\mathcal{X} = \sum_{i=0}^{\infty} \mathcal{X}_{q-p+2is}, \tag{1.5}$$

where $\mathcal{X}_k$ denotes a (p, q)-quasi-homogeneous vector field of weighted degree k (see definition in the next section) and satisfying the following three conditions:

(i) p and q are positive odd integers without common factors and $p \leq q$;

(ii) $s = np - q \geq 1$ for some integer $n \geq 2$;

(iii) $\mathcal{X}_{q-p} = y\partial_x$ and $\mathcal{X}_{q-p+2s} = \mathcal{X}_{(2n-1)p-q} = A(x, y)\partial_x + B(x, y)\partial_y$ with $B(1, 0) < 0$.

Without loss of generality we can take $B(1,0) = -1$, which means that the monomial $-x^{2n-1}$ is always present in $B(x, y)$. Family (1.5) contains the systems of the form (1.3). For such a big family in [1] was obtained the same result that for system (1.3). In [1] it was conjectured that all systems of family (1.5) have an analytic first integral defined in a neighborhood of the origin. This claim is, in fact, a straightforward consequence of the results given in [23]. Hence our first result is the following:

Theorem 1.1 *The origin of system* (1.5) *is a nilpotent center if, and only if, there is a local analytic first integral which can be expanded as $H(x, y) = y^2 + \cdots$, where the dots denote terms of degree higher than two.*

Moreover in [14] it is studied the cyclicity of system (1.5). Now we give the following definition in order to establish the main result of this work.

Definition 1.2 *System* (1.1) *is $\mathbb{Z}_2$-symmetric (with respect to the origin) if it is invariant under the involution $(x, y) \to (-x, -y)$, that is $P(-x, -y) = -P(x, y)$ and $Q(-x, -y) = -Q(x, y)$.*

Note that system (1.5) and its particular case system (1.3) are $\mathbb{Z}_2$-symmetric.
The main result of this work is the following.

Theorem 1.3 *The origin of any $\mathbb{Z}_2$-symmetric is a nilpotent center if, and only if, there is a local analytic first integral of the form $H(x, y) = y^2 + \cdots$, where the dots denote terms of degree higher than two.*

Theorem 1.3 gives the first great family of systems, of course apart from the Hamiltonians ones, having a nilpotent center with a local analytic first integral around the singular point.

2 Preliminaries results and proof of the main result

As usual we define the set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. A scalar polynomial f is quasi-homogeneous of type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$ and degree k if $f(\varepsilon^{t_1} x, \varepsilon^{t_2} y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous scalar polynomials of type $\mathbf{t}$ and degree k is denoted by $\mathcal{P}_k^{\mathbf{t}}$. A polynomial vector field $\mathbf{F} = (P, Q)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. The vector space of quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k is denoted by $\mathcal{Q}_k^{\mathbf{t}}$. Given an analytic vector field $\mathbf{F}$, we can write it as a quasi-homogeneous expansion corresponding to a fixed type $\mathbf{t}$:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots = \sum_{j \geq r} \mathbf{F}_j, \tag{2.6}$$

where $\mathbf{x} \in \mathbb{R}^2$, $r \in \mathbb{Z}^+$ and $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ i.e., each term $\mathbf{F}_j$ is a quasi-homogeneous vector field of type $\mathbf{t}$ and degree j. Any $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ can be uniquely written as

$$\mathbf{F}_j = \mathbf{X}_{h_j} + \mu_j \mathbf{D}_0, \tag{2.7}$$

where $\mu_j = \frac{1}{r+|\mathbf{t}|} \operatorname{div}(\mathbf{F}_j) \in \mathcal{P}_j^{\mathbf{t}}$, $h_j = \frac{1}{r+|\mathbf{t}|} \mathbf{D}_0 \wedge \mathbf{F}_j \in \mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$, $\mathbf{D}_0 = (t_1 x, t_2 y)^T$, and $\mathbf{X}_{h_j} = (-\partial h_j / \partial y, \partial h_j / \partial x)^T$ is the Hamiltonian vector field with Hamiltonian function h_j, see [2, Prop.2.7] for more details of this decomposition.

As we have said, in this work we are interested in the center problem for $\mathbb{Z}_2$-symmetric analytic nilpotent differential systems in the plane, i.e., differential systems of the form

$$\dot{x} = y + P(x, y), \qquad \dot{y} = Q(x, y), \tag{2.8}$$

where P, Q are analytic function in a neighborhood of the origin without constants and linear terms with $P(-x,-y) = -P(x,y)$ and $Q(-x,-y) = -Q(x,y)$.

The following result provides the first quasi-homogeneous component of a monodromic $\mathbb{Z}_2$-symmetric nilpotent vector field.

Proposition 2.4 (Monodromic normal preform) *Consider system (2.8) that now we write as $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$. If the origin of system (2.8) is monodromic then there exist a $\mathbb{Z}_2$-symmetric polynomial change Φ and a type $\mathbf{t}$ such that $\widetilde{\mathbf{F}} := \Phi_*\mathbf{F}$ is a $\mathbb{Z}_2$-symmetric vector field, $\widetilde{\mathbf{F}} = \widetilde{\mathbf{F}}_r + \cdots$, $\widetilde{\mathbf{F}}_r \in \mathcal{Q}_r^t$, where the dots are quasi-homogeneous terms of type $\mathbf{t}$ and degree greater than r, and the first quasi-homogeneous component respect to the type t, $\widetilde{\mathbf{F}}_r$ is of one of the following two forms:*

(A) $\widetilde{\mathbf{F}}_r = (y, -x^{4n-1})^T \in \mathcal{Q}_{2n-1}^{(1,2n)}$, *i.e.*, $\mathbf{t} = (1, 2n)$ *and* $r = 2n-1$, $n \in \mathbb{N}$.

(B) $\widetilde{\mathbf{F}}_r = (y + dx^{2n+1}, -x^{4n+1} + (2n+1)dx^{2n}y)^T \in \mathcal{Q}_{2n}^{(1,2n+1)}$, *i.e.*, $\mathbf{t} = (1, 2n+1)$ *and* $r = 2n$, $n \in \mathbb{N}$.

Proof. System (2.8) can be written as

$$\begin{array}{rcl} \dot{x} & = & y + x\widetilde{f}_1(x^2) + yx^2\widetilde{f}_2(x^2,y^2) + y^3\widetilde{f}_3(y^2), \\ \dot{y} & = & x\widetilde{g}_1(x^2) + y\widetilde{g}_2(x^2) + xy^2\widetilde{g}_3(x^2,y^2) + y^3\widetilde{g}_4(y^2). \end{array} \tag{2.9}$$

with $\widetilde{f}_1(0) = \widetilde{g}_1(0) = \widetilde{f}_3(0) = \widetilde{g}_2(0) = \widetilde{g}_4(0) = 0$ and $\widetilde{f}_2(0,0) = \widetilde{g}_3(0,0) = 0$. Let us denote by M the lowest-degree in the Taylor expansion of $\widetilde{g}_1(x)$; and N is the minimum of the lowest-degrees of the Taylor expansions for $\widetilde{f}_1(x)$ and $\widetilde{g}_2(x)$. Hence, $M = \infty$ arises if $\widetilde{g}_1(x) \equiv 0$ and $N = \infty$ corresponds to $\widetilde{f}_1(x) \equiv \widetilde{g}_2(x) \equiv 0$. Then, we can write the nilpotent system (2.9) as

$$\begin{array}{rcl} \dot{x} & = & y + x^{2N+1}\hat{f}_1(x^2) + x^2y\widetilde{f}_2(x^2,y^2) + y^3\widetilde{f}_3(y^2), \\ \dot{y} & = & x^{2N}y\hat{g}_2(x^2) + x^{2M+1}\hat{g}_1(x^2) + xy^2\widetilde{g}_3(x^2,y^2) + y^3\widetilde{g}_4(y^2), \end{array} \tag{2.10}$$

where $M \in \mathbb{N} \cup \{\infty\}$, $M \geq 1$, $N \in \mathbb{N} \cup \{\infty\}$, and $\hat{f}_1(x) = a + \mathcal{O}(x)$, $\hat{g}_1(x) = b + \mathcal{O}(x)$, $\hat{g}_2(x) = c + \mathcal{O}(x)$, $\widetilde{f}_2(x,y) = \mathcal{O}(x,y)$, $\widetilde{f}_3(y) = \mathcal{O}(y)$, $\widetilde{g}_3(x,y) = \mathcal{O}(x,y)$, $\widetilde{g}_4(y) = \mathcal{O}(y)$ with $(a^2 + c^2)b \neq 0$.

- If $M = \infty$, then the line $y = 0$ is filled up of singular points, the origin is not monodromic and we must exclude this case.
- If $M < 2N$ then the Newton diagram of (2.10) has two exterior vertices $V_1 = (0,2)$ associated to the vector field $(y,0)^T$ and $V_2 = (2(M+1), 0)$ associated to the vector field $(0, cx^{2M+1})^T$, with $c \neq 0$, and a unique compact edge of type $(1, M+1)$ whose vector field associated is $\mathbf{F}_M = (y, cx^{2M+1})^T$ where $\mathbf{F}_M = \mathbf{X}_h$ with $h = -(cx^{2(M+1)} - (M+1)y^2)/(2(M+1))$. We look at different cases in function of the discriminant of h, $\Delta = 4(M+1)c$
  - If $c > 0$ applying statement (2) of [3, Theorem 3] we have that the origin of system (2.8) is not monodromic.
  - If $c < 0$ then h has not any real factor. Then by [3, Proposition 6] the origin of system(2.8) is monodromic. Applying the rescaling $x = u(-1/c)^{1/(2M)}$, $y = v(-1/c)^{1/(2M)}$ we obtain the case **(A)** when M is odd taking $2n = M+1$, or the case **(B)** when M es even taking $d = 0$ and $2n = M$.
- If $2N < M < \infty$ then the Newton diagram of (2.10) has two exterior vertices $V_1 = (0,2)$ associated to the vector field $(y,0)^T$ and $V_3 = (2(M+1), 0)$ associated to the vector field $(0, cx^{2M+1})^T$ with $c \neq 0$ and the inner vertex $V_2 = (2N+1, 1)$ associated to the vector field $(ax^{2N+1}, bx^{2N}y)^T$, with $a^2 + b^2 \neq 0$, where this last vertex has no even coordinates. Applying [3, Theorem 3, statement **(1)**], the origin of system (2.10) is not monodromic.

- If $2N = M < \infty$ then the Newton diagram of (2.10) has two exterior vertices $V_1 = (0,2)$ associated to the vector field $(y,0)^T$ and $V_2 = (2(2N+1),0)$ associated to the vector field $(0, cx^{4N+1})^T$ with $c \neq 0$, and a unique compact edge of type $(1, 2N+1)$ whose vector field associated is $\mathbf{F}_{2N} = (y + ax^{2N+1}, cx^{4N+1} + bx^{2N}y)^T$ with $c(a^2+b^2) \neq 0$.

  It is a simple task to perform the splitting (2.7) for this case. We obtain $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$, with $r = 2N$, $r + |\mathbf{t}| = 2(2N+1)$, and

$$
\begin{aligned}
h(x,y) &= \tfrac{c}{2(2N+1)} x^{2(2N+1)} + \left(\tfrac{b}{2(2N+1)} - \tfrac{1}{2}a\right) x^{2N+1} y - \tfrac{1}{2} y^2 \\
&= -\tfrac{1}{2}\left(y - \left(\tfrac{b}{2(2N+1)} - \tfrac{a}{2}\right) x^{2N+1}\right)^2 - \frac{\Delta}{2(2N+1)} x^{2(2N+1)}, \\
\mu(x,y) &= \tfrac{b+(2N+1)a}{2(2N+1)} x^{2N},
\end{aligned}
$$

  where $\Delta = (b - (2N+1)a)^2 + 4(2N+1)c$ is the discriminant of h. We see the different cases in function of the sign of Δ.

  (i) If $\Delta > 0$, then h is decomposed into a product of two simple factors. Applying [3, Theorem 3, statement **(4)**], the origin of system (2.8) is not monodromic.

  (ii) If $\Delta \leq 0$ taking $\Psi_0(x,y) = \left(x, y - \left(\tfrac{b}{2(2N+1)} - \tfrac{a}{2}\right) x^{2N+1}\right)^T$ we get $(\Psi_0)_* \mathbf{F}_r = (y + \widetilde{d}x^{2N+1}, \tfrac{\Delta}{4(2N+1)} x^{4N+1} + (2N+1)\widetilde{d}x^{2N} y)^T$ with $\widetilde{d} = \tfrac{b+(2N+1)a}{2(2N+1)}$. We must take into account that as the change Ψ_0 is $\mathbb{Z}_2$-symmetric this change transforms system (2.10) into another $\mathbb{Z}_2$-symmetric one.

  (ii.1) If $\Delta < 0$ the rescaling $x = u(-4(2N+1)/\Delta)^{1/(4N)}$, $y = v(-4(2N+1)/\Delta)^{1/(4N)}$ transforms the system into another one whose first quasi-homogeneous term is $\mathbf{F}_r = (y + dx^{2N+1}, -x^{4N+1} + (2N+1)dx^{2N}y)^T$ with $d = (-4(2N+1)/\Delta)^{1/2}\widetilde{d}$. This corresponds to the case **(B)** for $n = N$.

  (ii.2) If $\Delta = 0$ we have a system (2.10) with new values of N and M, and we repeat the previous arguments.

■

Next Lemma is a technical result that will be used later on.

Lemma 2.5 *Let* $\mathbf{F}$ *be the vector field defined by* $\mathbf{F} := -\mathbf{X}_h + \sum_{j=k}^{2k} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{2k} \alpha_j^{(l)} x^j h^l \mathbf{D}_0$, *where* $k \in \mathbb{N}$, $h = \frac{1}{2}y^2 + \frac{1}{2k+2}x^{2k+2} \in \mathcal{P}_{2k+2}^{\mathbf{t}}$, $\mathbf{D}_0 = (x, (k+1)y)^T \in \mathcal{Q}_0^{\mathbf{t}}$ *and* $\mathbf{t} = (1, (k+1))^T$ *and consider the change of coordinates to generalized polar coordinates and the scaling of time given by*

$$
(x,y) = (u\mathrm{Cs}(\theta), u^{k+1}\mathrm{Sn}(\theta)), \quad dt = \tfrac{1}{u^k} d\tau, \tag{2.11}
$$

where $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$ *are the solutions to the initial value problem* $(dx/d\theta, dy/d\theta) = \mathbf{X}_h$ *with* $x(0) = 1$, $y(0) = 0$. *Then system* $(\dot{x}, \dot{y})^T = \mathbf{F}(x,y)$, *doing the change of variables (2.11), is transformed into system*

$$
\begin{aligned}
\frac{du}{d\tau} &= u\left[\sum_{j=k}^{2k} \alpha_j^{(0)} \mathrm{Cs}^j(\theta) u^{j-k} + \sum_{l=1}^{\infty}\sum_{j=0}^{2k} \alpha_j^{(l)} \mathrm{Cs}^j(\theta) u^{2(k+1)l+j-k}\right], \\
\frac{d\theta}{d\tau} &= 1.
\end{aligned}
$$

Proof. It is a simple matter to show that the functions $\mathrm{Cs}(\theta), \mathrm{Sn}(\theta)$ have the following properties:

(i) They satisfy $\frac{1}{2}\mathrm{Sn}^2(\theta) + \frac{1}{2k+2}\mathrm{Cs}^{2k+2}(\theta) = 1$ for all θ. Therefore $h(x,y) = u^{2k+2}$ and $\nabla h \cdot \mathbf{D}_0 = (2k+2)h(x,y) = (2k+2)u^{2k+2}$.

(ii) They are periodic functions with the same minimal period T, and satisfy $\mathrm{Cs}(T) = 1$ and $\mathrm{Sn}(T) = 0$.

Additional properties of these functions can be found in [13].

Differentiating $x = u\mathrm{Cs}(\theta)$, $y = u^{k+1}\mathrm{Sn}(\theta)$ with respect to the time, and denoting $\mathbf{x} = (x,y)^T$, we get $\dot{\mathbf{x}} = \frac{1}{u}\mathbf{D}_0\dot{u} + \frac{1}{u^k}\mathbf{X}_h\dot{\theta}$. From this we obtain:

$$
\begin{aligned}
\dot{\mathbf{x}} \wedge \mathbf{X}_h &= \tfrac{1}{u}\mathbf{D}_0 \wedge \mathbf{X}_h\dot{u} = \tfrac{1}{u}\nabla h \cdot \mathbf{D}_0\dot{u} = \tfrac{1}{u}(2k+2)h(x,y)\dot{u} = (2k+2)u^{2k+1}\dot{u},\\
\mathbf{D}_0 \wedge \dot{\mathbf{x}} &= \tfrac{1}{u^k}\mathbf{D}_0 \wedge (-\mathbf{X}_h)\,\dot{\theta} = -\tfrac{1}{u^k}\nabla h \cdot \mathbf{D}_0\dot{\theta} = -\tfrac{2k+2}{u^k}h(x,y)\dot{\theta} = -(2k+2)u^{k+2}\dot{\theta}.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
\dot{\mathbf{x}} \wedge \mathbf{X}_h &= \left[\sum_{j=k}^{2k}\alpha_j^{(0)}x^j + \sum_{l=1}^{\infty}\sum_{j=0}^{2k}\alpha_j^{(l)}x^jh^l\right]\mathbf{D}_0 \wedge \mathbf{X}_h\\
&= \left[\sum_{j=k}^{2k}\alpha_j^{(0)}\cos^j(\theta)u^j + \sum_{l=1}^{\infty}\sum_{j=0}^{2k}\alpha_j^{(l)}\cos^j(\theta)u^{2(k+1)l+j}\right]\nabla h \cdot \mathbf{D}_0\\
&= (2k+2)u^{2k+2}\left[\sum_{j=k}^{2k}\alpha_j^{(0)}\cos^j(\theta)u^j + \sum_{l=1}^{\infty}\sum_{j=0}^{2k}\alpha_j^{(l)}\cos^j(\theta)u^{2(k+1)l+j}\right],\\
\mathbf{D}_0 \wedge \dot{\mathbf{x}} &= \mathbf{D}_0 \wedge (-\mathbf{X}_h) = -\nabla h \cdot \mathbf{D}_0 = -(2k+2)h(x,y) = -(2k+2)u^{2k+2}.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
\dot{u} &= u\left[\sum_{j=k}^{2k}\alpha_j^{(0)}\cos^j(\theta)u^j + \sum_{l=1}^{\infty}\sum_{j=0}^{2k}\alpha_j^{(l)}\cos^j(\theta)u^{2(k+1)l+j}\right],\\
\dot{\theta} &= u^k,
\end{aligned}
$$

and applying the rescaling of time $dt = u^{-k}d\tau$ we obtain the result. ∎

Hence in order to study the centers at the origin of systems (2.8), it is enough to study the systems whose first quasi-homogeneous component are of type **(A)** or **(B)** according to Proposition 2.4.

The following result shows that in case **(B)** the first quasi-homogeneous component can be simplified.

Theorem 2.6 *If the vector field* $\mathbf{F} = \widetilde{\mathbf{F}}_{2n} + \cdots$, *with* $\widetilde{\mathbf{F}}_{2n} = (y + dx^{2n+1}, -x^{4n+1} + (2n+1)dx^{2n}y)^T \in \mathcal{Q}_{2n}^{(1,2n+1)}$ *has a center at the origin then* $d = 0$.

Proof. If the origin of system $(\dot{x},\dot{y})^T = \widetilde{\mathbf{F}}_{2n} + \cdots$ is a center, by [5, Theorem 5] also is a center the origin of system $(\dot{x},\dot{y})^T = (y + dx^{2n+1}, -x^{4n+1} + (2n+1)dx^{2n}y)^T$. Applying the change of variables (2.11) for $k = 2n$ we obtain system $(u',\theta')^T = (d\mathrm{Cs}^{2n}(\theta)u, 1)^T$ whose solutions are given by $\theta(\tau) = \theta_0$ a constant and $u(\tau) = u_0e^{d\mathrm{Cs}^{2n}(\theta_0)\tau}$. Therefore, we have $d = 0$, otherwise the origin of $(\dot{x},\dot{y})^T = \widetilde{\mathbf{F}}_{2n}$ is a focus. ∎

Consequently to study the center problem for system (2.8), without lost of generality, we can assume that the first quasi-homogeneous component of the vector field respect to the type $\mathbf{t} = (1, n+1)$ is $\mathbf{F}_n = (y, -x^{2n+1})^T \in \mathcal{Q}_n^{(1,n+1)}$. Notice that this case includes cases **(A)** and **(B)** of Proposition 2.4.

The following result provides a normal form of these vector fields.

Proposition 2.7 *Let* $\mathbf{F}$ *be the* $\mathbb{Z}_2$*-symmetric vector field,* $\mathbf{F} := \sum_{j\geq n} \mathbf{F}_j$ *where* $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$*,* $\mathbf{t} = (1, n+1)$ *and* $\mathbf{F}_n = (y, -x^{2n+1})^T$*. Then the vector field* $\mathbf{F}$ *is orbitally equivalent to*

$$\mathbf{G} := \mathbf{F}_n + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} \alpha_j x^{2j} \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{n} \alpha_j^{(l)} x^{2j} h^l \mathbf{D}_0,$$

where $h = \frac{1}{2}y^2 + \frac{1}{2(n+1)}x^{2n+2}$*, and* $\mathbf{D}_0 = (x, (n+1)y) \in \mathcal{Q}_0^{\mathbf{t}}$*.*

Proof. By [4, Theorem 16] we have that the vector field $\mathbf{F}_n + \cdots$ is orbitally equivalent to $\mathbf{F}_n + \sum_{j=n+1}^{2n} \alpha_j x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{2n} \alpha_j^{(l)} x^j h^l \mathbf{D}_0$. On the other hand the vector field $\mathbf{F}$ is $\mathbb{Z}_2$-symmetric. If μ is a scalar function sum of homogeneous monomial of degree even then $\mu\mathbf{F}$ is a $\mathbb{Z}_2$-symmetric vector field, and if $\mathbf{G}$ is a $\mathbb{Z}_2$-symmetric vector field then $[\mathbf{F}, \mathbf{G}]$ is also a $\mathbb{Z}_2$-symmetric vector field. So applying changes of variables that are $\mathbb{Z}_2$-symmetric and the rescaling of time μ with $\mu(-x, -y) = \mu(x, y)$, we obtain a $\mathbb{Z}_2$-symmetric normal form. Following the ideas [4, Section 2] it is possible to prove that a normal form $\mathbb{Z}_2$-symmetric of $\mathbf{F}$ is the projection of the normal form shown above over the $\mathbb{Z}_2$-symmetric vector fields, i.e.

$$\mathbf{F}_n + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} \alpha_j x^{2j} \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{n} \alpha_j^{(l)} x^{2j} h^l \mathbf{D}_0,$$

■

Theorem 2.8 *The origin of system (2.8) is a center if, and only if, there exists* $n \in \mathbb{N}$ *such that it is orbitally equivalent to* $(\dot{x}, \dot{y})^T = (y, -x^{2n+1})^T$*.*

Proof. The sufficient condition is trivial since the origin of $(\dot{x}, \dot{y})^T = (y, -x^{2n+1})^T$ is a center.

Now we see the necessary condition. If the origin of system (2.8) is a center then by Proposition 2.4 and Theorem 2.6, we can affirm that system (2.8) is conjugate to system $(\dot{x}, \dot{y})^T = \mathbf{F}_n + \cdots$ being $\mathbf{F}_n = -\mathbf{X}_h \in \mathcal{Q}_n^{\mathbf{t}}$ with $h = \frac{1}{2}y^2 + \frac{1}{2n+2}x^{2n+2} \in \mathcal{P}_{2n+2}^{\mathbf{t}}$, $\mathbf{t} = (1, n+1)$. Applying Proposition 2.7, we get that system (2.8) is orbitally equivalent to the system $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ where

$$\mathbf{G} := \mathbf{F}_n + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} \alpha_j^{(0)} x^{2j} \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{n} \alpha_j^{(l)} x^{2j} h^l \mathbf{D}_0.$$

By Lemma 2.5, applying now the change (2.11) for $k = n$, we obtain the differential equation

$$\frac{du}{d\theta} = u \left[\sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} \alpha_j^{(0)} \mathrm{Cs}^{2j}(\theta) u^{2j-n} + \sum_{l=1}^{\infty} \sum_{j=0}^{n} \alpha_j^{(l)} \mathrm{Cs}^{2j}(\theta) u^{2(n+1)l+2j-n} \right].$$

If all $\alpha_j^{(l)} = 0$ the result is proved. Otherwise we define

$$l_0 = \min \left\{ l \in \mathbb{N} \cup \{0\} : \alpha_j^{(l)} \neq 0 \text{ for some j} \right\}, j_0 = \min \left\{ j : \alpha_j^{(l_0)} \neq 0 \right\},$$

in this case the differential equation takes the form

$$\frac{du}{d\theta} = \alpha_{j_0}^{(l_0)} \mathrm{Cs}^{2j_0}(\theta) u^{2(n+1)l_0+2j_0-n} (1 + \mathcal{O}(u, \theta)). \tag{2.12}$$

We write the solution of (2.12) starting at $u = u_0$ when $\theta = 0$ as

$$u(\theta, u_0) = \sum_{i=1}^{\infty} a_i(\theta) u_0^i + f(\theta, u_0), \tag{2.13}$$

where $a_1(0) = 1$, $a_i(0) = 0$ for $i \geq 2$ and $f(0, u_0) = 0$ with f flat at $u_0 = 0$. Hence the Poincaré return map from the section $\{(u, \theta) = (u_0, 0), u_0 > 0\}$ to itself is given by the power series $P(u_0) = a_1(T)u_0 + a_2(T)u_0^2 + \cdots$.

By replacing (2.13) into the differential equation (2.12) we get $a_1(\theta) \equiv 1$, $a_i(\theta) \equiv 0$, for $i = 2, \cdots, 2(n+1)l_0 + 2j_0 - n - 1$ and

$$a_{2(n+1)l_0+2j_0-n}(T) = \alpha_{j_0}^{(l_0)} \int_0^T \mathrm{Cs}^{2j_0}(\theta) d\theta \neq 0.$$

Hence the origin of system $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ would be a focus, which is a contradiction. ∎

Next result relates the center problem with the integrability of the $\mathbb{Z}_2$-symmetric nilpotent vector fields.

Theorem 2.9 *If the origin of system (2.8) is monodromic, then the origin of system (2.8) is a center if, and only if, system (2.8) is analytically integrable.*

Proof. The sufficient condition is trivial because if system(2.8) is analytically integrable and the origin is monodromic then the origin is a center.

On the other hand if the origin of system (2.8) is a center by Theorem 2.8 there exists $n \in \mathbb{N}$ such that system (2.8) is orbitally equivalent to $(\dot{x}, \dot{y})^T = (y, -x^{2n+1})^T$. But this system is Hamiltonian, and therefore polynomially integrable. Undoing the change of variables we have that system (2.8) is formally integrable and by applying [23, Theorem A] we deduce that $\mathbf{F}$ is analytically integrable. ∎

Remark. We know that all the linear type centers are analytically integrable, see [21, 25]. This does not happen with nilpotent centers, in this case the characterization of a center is determined by the orbital reversibility, see [10]. Theorem 2.9 provides another large family of vector fields with this property, that is, the center problem is equivalent to the analytic integrability problem for these systems.

The following result provides an efficient algorithm to characterize and compute $\mathbb{Z}_2$-symmetric nilpotent centers.

Theorem 2.10 *Let* $\mathbf{F}$ *be the vector field of system (2.8). The following statements are satisfied.*

(i) *There exists a formal function $I(x, y) = y^2 + \sum_{j \geq 2} I_{2j}(x, y)$ with $I_{2j}(x, y)$ homogeneous polynomial of degree $2j$ and certain constants $\alpha_j \in \mathbb{R}$, $j \geq 3$, such that*

$$\nabla I \cdot \mathbf{F} = \sum_{j \geq 3} \alpha_j x^{2j} \tag{2.14}$$

Moreover, it is possible to choose $I_{2j}(0, y) \equiv 0$ for all $j \geq 3$ and in this case I is the unique formal function that satisfies (2.14).

(ii) *If the origin of system (2.8) is monodromic, it is a center if, and only if, $\alpha_j = 0$ for all $j \geq 3$.*

Proof. We write $\mathbf{F} = (y, 0)^T + \sum_{k \geq 1} \mathbf{F}_{2k+1}$ with $\mathbf{F}_{2k+1} = (P_{2k+1}, Q_{2k+1})^T$.

We prove that it is possible to choose I_{2j} with $j \geq 3$ satisfying statement (i). For $j = 2$, the expression $\nabla I \cdot \mathbf{F}$ of degree 4 is

$$\begin{aligned} (\nabla I \cdot \mathbf{F})_4 &= \nabla y^2 \cdot \mathbf{F}_3 + \nabla I_4 \cdot (y, 0)^T = 2yQ_3 + \frac{\partial I_4}{\partial x} y. \\ &= \left(\frac{\partial I_4}{\partial x} + 2Q_3 \right) y \end{aligned}$$

Choosing $I_4(x, y) = \beta_2 y^4 + xJ_3(x, y)$ with $\beta_2 \in \mathbb{R}$, and $J_3(x, y) = -\frac{1}{x} \int_0^x 2Q_3(u, y)du$ which is a homogeneous polynomial of degree 3, we get $(\nabla I \cdot \mathbf{F})_4 = 0$.

The expression $\nabla I \cdot \mathbf{F}$ of degree $2j$ with $j > 2$ is

$$\begin{aligned} (\nabla I \cdot \mathbf{F})_{2j} &= \nabla y^2 \cdot \mathbf{F}_{2j-1} + \nabla I_{2j} \cdot (y, 0)^T + \sum_{i=2}^{j-1} \nabla I_{2i} \cdot \mathbf{F}_{2(j-i)+1} \\ &= 2yQ_{2j-1} + \frac{\partial I_{2j}}{\partial x} y + \sum_{i=2}^{j-1} \nabla I_{2i} \cdot \mathbf{F}_{2(j-i)+1}. \end{aligned}$$

There exists an homogeneous polynomial R_{2j-1} of degree $2j - 1$ and a constant $\alpha_j \in \mathbb{R}$ such that

$$\sum_{i=2}^{j-1} \nabla I_{2i} \cdot \mathbf{F}_{2(j-i)+1} = R_{2j-1} y + \alpha_j x^{2j}. \tag{2.15}$$

and therefore

$$(\nabla I \cdot \mathbf{F})_{2j} = \left(2Q_{2j-1} + \frac{\partial I_{2j}}{\partial x} + R_{2j-1} \right) y + \alpha_j x^{2j}.$$

So we must to take $I_{2j}(x, y) = \beta_j y^{2j} + xJ_{2j-1}(x, y)$ where

$$J_{2j-1}(x, y) = -\frac{1}{x} \int_0^x \left(R_{2j-1}(u, y) + 2Q_{2j-1}(u, y) \right) du,$$

is a homogeneous polynomial of degree $2j - 1$ and $\beta_j \in \mathbb{R}$. If we choose $\beta_j = 0$ we obtain a unique polynomial I_{2j} with $I_{2j}(0, y) \equiv 0$.

Now we prove statement (ii). First, we see the sufficient condition. If $\alpha_j = 0$ for all $j \geq 3$, then system (2.8) is formally integrable, by applying [23, Theorem A] we deduce that it is analytically integrable. As the origin of the system is monodromic then it is a center. Next we see the necessary condition. If the origin of system (2.8) is a center, by Theorem 2.9 the system is analytically integrable and by [11, Theorem 1] we can affirm that there exists a first integral of the form $I = y^2 + \sum_{j>2} I_j$ where I_j is a homogeneous polynomial of degree j.

Taking into account that system (2.8) is $\mathbb{Z}_2$-symmetric, that is, invariant by the involution $(x, y) \to (-x, -y)$ its first integral inherits this property. Consequently $I_{2i+1} = 0$ for all $i \geq 1$.

On the other hand, the level curves $I = C$ where C is a constant are ovals in a neighborhood of the origin because I is a first integral of a center. We assume that not all α_j are zero, and we consider $j_0 = \min \{j \in \mathbb{N}, j \geq 3 : \alpha_j \neq 0\}$. Applying condition (2.14) we have

$$\nabla I \cdot \mathbf{F} = \alpha_{j_0} x^{2j_0} (1 + o(1)).$$

Hence $\nabla I \cdot \mathbf{F} \geq 0$ if $\alpha_{j_0} > 0$, or $\nabla I \cdot \mathbf{F} \leq 0$ if $\alpha_{j_0} < 0$, i.e. the orbits of the system (2.8) cross the ovals $I = C$ always outward or inward. Therefore the origin of system (2.8) is a focus, which is a contradiction. ■

The following results characterize the centers of some families of $\mathbb{Z}_2$-symmetric nilpotent systems of the form

$$\begin{array}{rcl} \dot{x} & = & y + P_{2n+1}(x,y), \\ \dot{y} & = & Q_{2m+1}(x,y), \end{array} \tag{2.16}$$

where P_{2n+1} and Q_{2m+1} are homogeneous polynomials of degree $2n+1$ and $2m+1$, respectively, and $Q_{2m+1}(1,0) < 0$, otherwise the origin of system (2.16) is not monodromic. Without loss of generality we can assume that $Q_{2m+1}(1,0) = -1$, that is, we can take $Q_{2m+1}(x,0) = -x^{2m+1}$.

Proposition 2.11 *If the origin of* (2.16) *is a center, then it is satisfied one of the following conditions:*

(a) $m < 2n$.

(b) $2n \leq m$ *and* $P_{2n+1}(1,0) = 0$.

Moreover, in this last case the first component of the vector field associated to system (2.16), respect to the type $\mathbf{t} = (1, m+1)$ *is* $\mathbf{F}_m = (y, -x^{2m+1})^T \in \mathcal{Q}_m^{\mathbf{t}}$.

Proof. If $P(1,0) = a_{2n+1} \neq 0$ and $2n < m$, then the Newton diagram of (2.16) has two exterior vertices $V_1 = (0,2)$ associated to the vector field $(y,0)^T$ and $V_3 = (2(m+1), 0)$ associated to the vector field $(0, -x^{2m+1})^T$, and the inner vertex $V_2 = (2n+1, 1)$ associated to the vector field $(a_{2n+1}x^{2n+1}, 0)^T$ where this last vertex has no even coordinates. Applying [3, Theorem 3, statement **(1)**], the origin of system (2.16) is not monodromic, therefore it is not a center.

If $P(1,0) = a_{2n+1} \neq 0$ and $2n = m$, then the Newton diagram of (2.16) has two exterior vertices $V_1 = (0,2)$ associated to the vector field $(y,0)^T$ and $V_3 = (2(m+1), 0)$ associated to the vector field $(0, -x^{2m+1})^T$, and a unique compact edge of type $(1, m+1)$ whose vector field associated is $\mathbf{F}_m = (y + a_{2n+1}x^{m+1}, -x^{2m+1})^T$.

It is a simple task to perform the splitting (2.7) for this case. We obtain $\mathbf{F}_m = \mathbf{X}_h + \mu \mathbf{D}_0$, where $\mathbf{D}_0 = (x, (m+1)y)^T$, and

$$\begin{array}{rcl} h & = & -\frac{1}{2(m+1)} \left((m+1)y^2 + a_{2n+1}(m+1)x^{m+1}y + x^{2(m+1)} \right) \\ & = & -\frac{1}{2} \left(\left(y + \frac{a_{2n+1}}{2} x^{m+1} \right)^2 + \Delta x^{2(m+1)} \right), \\ \Delta & = & \frac{4-(m+1)a_{2n+1}^2}{4(m+1)}, \\ \mu & = & \frac{1}{2} a_{2n+1} x^m \not\equiv 0. \end{array}$$

(i) If $\Delta < 0$ then h has two simple factors. Applying [3, Theorem 3, item (4)] we deduce that the origin of system (2.16) is not monodromic, which is a contradiction.

(ii) If $\Delta = 0$ the unique invariant curve of $\mathbf{F}_r$ is $C = y + \frac{1}{2}a_{2n+1}x^{m+1}$, and consequently $\mathbf{F}_r$ is not polynomially integrable because $\nabla C \cdot \mathbf{F}_r = (m+1)\mu C \neq 0$. Therefore $\mathbf{F}$ is not integrable and by Theorem 2.9 the origin of system (2.16) is not a center.

(iii) If $\Delta > 0$ the unique invariant curve of $\mathbf{F}_m$ is h and consequently $\mathbf{F}_m$ is not polynomially integrable because $\nabla h \cdot \mathbf{F}_m = 2(m+1)\mu h \neq 0$. Therefore $\mathbf{F}$ is not integrable and by Theorem 2.9 the origin of system (2.16) is not a center.

Therefore $m < 2n$ or $m = 2n$ with $P_{2n+1}(1,0) = 0$, and in this case the first component of the vector field associated to system (2.16), respect to the type $\mathbf{t} = (1, m+1)$ is $\mathbf{F}_m = (y, -x^{2m+1})^T$. ■

Theorem 2.12 *Consider system (2.16) with $m < 2n$ or ($m = 2n$ and $P_{2n+1}(1,0) = 0$), and $n \neq km$, $1 \leq k \leq m+1$. Then the origin of (2.16) is a center if, and only if, the system (2.16) is R_x-reversible, i.e. invariant by the symmetry $(x, y, t) \to (-x, y, -t)$.*

Proof. By Proposition 2.11 if $m < 2n$ or $m = 2n$ and $P_{2n+1}(1,0) = 0$ we have that the first component of the vector field $\mathbf{F}$ associated to system (2.16), respect to the type $\mathbf{t} = (1, m+1)$ is $\mathbf{F}_m = (y, -x^{2m+1})^T$, i.e. $\mathbf{F} = \mathbf{F}_m + \cdots$. The origin of the system $\dot{\mathbf{x}} = \mathbf{F}_m(\mathbf{x})$ is monodromic because $\mathbf{F}_m = \mathbf{X}_h$ with

$$h = -\frac{1}{2}\,y^2 - \frac{1}{2m+2}\,x^{2m+2},$$

which is a negative defined function. Applying [5, Theorem 2] the origin of system (2.16) is monodromic.

The sufficient condition is trivial because if system (2.16) is R_x-reversible, as the origin of system (2.16) is monodromic, this implies that the origin of (2.16) is a center.

Now we are going to prove the necessary condition. If we assume that the origin of system (2.16) is a center by Theorem 2.10 there exists a unique formal function $I = y^2 + \sum_{j\geq 2} I_{2j}$ where I_{2j} is a homogeneous polynomial of degree $2j$ with $I_{2j}(0, y) \equiv 0$ such that $\nabla I \cdot \mathbf{F} = 0$.

We consider $\mathbf{F} = \widetilde{\mathbf{F}} + \overline{\mathbf{F}}$ where $\widetilde{\mathbf{F}}$ is sum of even monomials in x in the first component and odd in the second one, i.e. $\widetilde{\mathbf{F}}$ is R_x-reversible, and $\overline{\mathbf{F}}$ is a sum of odd monomials in x in the first component and even in the second one. If $\overline{\mathbf{F}} \equiv 0$, $\mathbf{F}$ is R_x-reversible and the result is proved, otherwise let p be the lowest quasi-homogeneous degree respect to the type $\mathbf{t} = (1, m+1)$ of the vector field $\overline{\mathbf{F}}$ such that $\overline{\mathbf{F}}_p \not\equiv 0$.

Working with respect of type $\mathbf{t} = (1, m+1)$ we have that

$$\widetilde{\mathbf{F}} = \mathbf{F}_m + \cdots$$

where dots indicate R_x-reversible quasi-homogeneous vector fields of degree greater than m. Therefore $\tilde{\mathbf{F}}$ is R_x-reversible and applying [5, Theorem 2] the origin of system $\dot{\mathbf{x}} = \tilde{\mathbf{F}}(\mathbf{x})$ is monodromic, that is, the origin of system $\dot{\mathbf{x}} = \tilde{\mathbf{F}}(\mathbf{x})$ is a center. Hence by Theorem 2.10 there exists a unique formal function $\widetilde{I} = y^2 + \sum_{j\geq 2} \widetilde{I}_{2j}$ where $\widetilde{I}_{2j}$ is a homogeneous polynomial of degree $2j$ with $\widetilde{I}_{2j}(0, y) \equiv 0$ such that $\nabla\widetilde{I} \cdot \widetilde{\mathbf{F}} = 0$.

As $\mathbf{F}_m$ is R_x-reversible, we have $m < p$. Considering the type $\mathbf{t} = (1, m+1)$, then $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ coincide up to degree j with $m \leq j \leq p-1$, and $\tilde{\mathbf{F}}_p \not\equiv \mathbf{F}_p$ because $\overline{\mathbf{F}}_p \not\equiv 0$. Considering also the type $\mathbf{t} = (1, m+1)$ we will have $I = \sum_{j\geq 2(m+1)} I_j$, $\widetilde{I}_j = \sum_{j\geq 2(m+1)} \widetilde{I}_j$, with $I_j, \widetilde{I}_j \in \mathcal{P}^{\mathbf{t}}_j$, $I_j = \widetilde{I}_j$ for $2(m+1) \leq j \leq m+1+p$, being $I_{2(m+1)} = (m+1)y^2 + x^{2(m+1)}$.

Consider $J = I - \widetilde{I}$ then the first quasi-homogeneous term of J is of order $m+2+p$ and satisfies:

$$0 \;=\; \nabla I \cdot \mathbf{F} = \nabla(\widetilde{I} + J) \cdot (\widetilde{\mathbf{F}} + \overline{\mathbf{F}}) = \nabla J \cdot \widetilde{\mathbf{F}} + \nabla I \cdot \overline{\mathbf{F}}.$$

Therefore

$$0 \;=\; (\nabla I \cdot \mathbf{F})_{2(m+1)+p} = \nabla J_{m+2+p} \cdot \widetilde{\mathbf{F}}_m + \nabla I_{2(m+1)} \cdot \overline{\mathbf{F}}_p. \qquad (2.17)$$

Since $\overline{\mathbf{F}}_p$ is a sum of odd monomials in x in the first component and even in the second, we have that $\nabla I_{2(m+1)} \cdot \overline{\mathbf{F}}_p$ is sum of even monomials in x and $\nabla J_{m+2+p} \cdot \tilde{\mathbf{F}}_m$ too. Therefore J_{m+2+p} is sum of odd monomials in x. Moreover J_{m+2+p} is sum of homogeneous monomials of degree even. Then it is sum of odd monomials in y. Let $s \in \mathbb{N}$ such that $(2s-1)(m+1) \leq m+2+p < (2s+1)(m+1)$. Therefore

$$J_{m+2+p} \;=\; \sum_{l=1}^{s} \alpha_l x^{A_l} y^{2l-1}, \text{ where } A_l = m+2+p-(2l-1)(m+1).$$

Then the following situations can happen:

(i) $\overline{\mathbf{F}}_p = (0, b_{2m+1-i_1}x^{2m+1-i_1}y^{i_1})^T$ with $i_1 = 2i-1$ odd, that is, $p = 2im$ with $1 \le i \le m+1$.

(ii) $\overline{\mathbf{F}}_p = (a_{2n+1-j_1}x^{2n+1-j_1}y^{j_1})^T$ with $j_1 = 2j$ even, that is, $p = 2n+2jm$ with $0 \le j \le n$.

(iii) $\overline{\mathbf{F}}_p = (a_{2n+1-j_1}x^{2n+1-j_1}y^{j_1}, b_{2m+1-i_1}x^{2m+1-i_1}y^{i_1})^T$ with $i_1 = 2i-1$ odd and $j_1 = 2j$ even, that is, $p = 2im = 2n+2jm$ with $0 \le i \le m+1$, $0 \le j \le n$, which implies $n = (i-j)m$ with $1 \le i-j \le m+1$. This case is excluded by the hypothesis $n \ne km$, $0 \le k \le m+1$.

(i) Case $p = 2im$ with $n \ne km$, for $0 \le k \le m+1$. In this case $\overline{\mathbf{F}}_p = \overline{\mathbf{F}}_{2im} = (0, b_{2(m+1-i)}x^{2(m+1-i)}y^{2i-1})^T$, and $m+2+p = (2i-1)(m+1)+2(m+1-i)+1$. Therefore $s = i$ and

$$
\begin{aligned}
J_{m+2+p} &= \sum_{l=1}^{i} \alpha_l x^{A_l} y^{2l-1}, \text{ where}\\
A_l &= 2(i-l)(m+1)+2(m+1-i)+1, \text{ and}\\
\nabla I_{2(m+1)} \cdot \overline{\mathbf{F}}_p &= 2(m+1)b_{2(m+1-i)}x^{2(m+1-i)}y^{2i}.
\end{aligned}
$$

From equation (2.17) we get

$$
\begin{aligned}
-\alpha_1 &= 0,\\
A_l\alpha_l - (2l+1)\alpha_{l+1} &= 0, \ 1 \le l \le i-1,\\
A_i\alpha_i &= -2(m+1)b_{2(m+1-i)} \ne 0.
\end{aligned}
$$

Consequently equation (2.17) provides a contradiction.

(ii) Case $p = 2n+2jm$ with $n \ne km$, for $0 \le k \le m+1$. In this case $\overline{\mathbf{F}}_p = \overline{\mathbf{F}}_{2n+2jm} = (a_{2(n-j)+1}x^{2(n-j)+1}y^{2j}, 0)^T$, and $m+2+p = (2j+1)(m+1)+2(n-j)+1$. Therefore $s = j+1$ and

$$
\begin{aligned}
J_{m+2+p} &= \sum_{l=1}^{j+1} \alpha_l x^{A_l} y^{2l-1}, \text{ where}\\
A_l &= 2(j+1-l)(m+1)+2(n-j)+1, \text{ and}\\
\nabla I_{2(m+1)} \cdot \overline{\mathbf{F}}_p &= 2(m+1)a_{2(n-j)+1}x^{2(m+n-j)+1}y^{2j}.
\end{aligned}
$$

Again from equation (2.17) we obtain

$$
\begin{aligned}
-\alpha_1 &= 0,\\
A_l\alpha_l - (2l+1)\alpha_{l+1} &= 0, \ 1 \le l \le j-1,\\
\alpha_j A_j - (2j+1)\alpha_{j+1} &= -2(m+1)a_{2(n-j)+1} \ne 0,\\
A_{j+1}\alpha_{j+1} &= 0.
\end{aligned}
$$

Consequently equation (2.17) provides also a contradiction.

■

3 Applications

We consider the differential system

$$
\begin{aligned}
\dot{x} &= y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{32}x^3y^2 + a_{23}x^2y^3 + a_{14}xy^4 + a_{05}y^5,\\
\dot{y} &= -x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.
\end{aligned}
\tag{3.18}
$$

Theorem 3.13 *The origin of system* (3.18) *is a center if, and only if, one of the following conditions holds.*

(a) $b_{21} = a_{32} = a_{12} = b_{03} = 0$,

(b) $b_{21} = a_{32} = a_{14} = a_{12} + 3b_{03} = a_{21} + b_{12} = 0$, $b_{03} \neq 0$,

(c) $b_{21} = a_{14} = a_{05} = a_{03} = a_{23} = a_{12} + 3b_{03} = a_{32} + 6b_{03}b_{12} = 0$, $b_{03}b_{12} \neq 0$.

Proof. The origin of system (3.18) is monodromic. Just see that the first quasi-homogeneous component respect to the type $\mathbf{t} = (1, 2)$ is $\mathbf{F}_1 = (y, -x^3)$ and apply Proposition 2.4. We will use the scalar algorithm of Theorem 2.10 and we will impose that the constants $\alpha_6, \alpha_8, \cdots$, must be null. When we apply the algorithm the value of the first constants modulo the annulation of the previous ones are:

$$\begin{aligned}
\alpha_6 &= b_{21}, \\
\alpha_8 &= a_{12} + 3b_{03}, \\
\alpha_{10} &= 2a_{21}b_{03} + 2b_{03}b_{12} + a_{32}, \\
\alpha_{12} &= 3a_{14} - 2b_{03}(a_{21} + b_{12})(a_{21} - 2b_{12}), \\
\alpha_{14} &= b_{03}\left[(a_{21} + b_{12})(a_{21}b_{12} - 2b_{12}^2 + 3a_{03}) - 3a_{23}\right].
\end{aligned}$$

(i) If $b_{03} = 0$ then we have $a_{14} = a_{32} = a_{12} = b_{21} = 0$, and we obtain the case **(a)**. In this case the vector field $\mathbf{F} = (y + a_{03}y^3 + a_{21}x^2y + a_{05}y^5 + a_{23}x^2y^3, -x^3 + b_{12}xy^2)^T$ is R_x reversible and therefore the origin is a center.

(ii) If $b_{03} \neq 0$ then $a_{23} = (a_{21} + b_{12})(a_{21}b_{12} - 2b_{12}^2 + 3a_{03})/3$ and the next constant is

$$\alpha_{16} = b_{03}(a_{21} + b_{12})^2(a_{21}b_{12} - 2b_{12}^2 + 3a_{03}).$$

(ii.1) If $a_{21} + b_{12} = 0$, then $a_{23} = a_{14} = a_{32} = b_{21} = 0$, with $a_{12} = -3b_{03} \neq 0$ that is the case **(b)**. The vector field in this case is $\mathbf{F} = (y - b_{12}x^2y - 3b_{03}xy^2 + a_{03}y^3 + a_{05}y^5, -x^3 + b_{12}xy^2 + b_{03}y^3)^T$ which is Hamiltonian and hence the origin is a center.

(ii.2) If $b_{03}(a_{21} + b_{12}) \neq 0$ then the vanishing of α_{16} implies $a_{03} = -\frac{1}{3}b_{12}(a_{21} - 2b_{12})$, and the following constant is

$$\alpha_{18} = b_{03}a_{05}(a_{21} + b_{12}).$$

Since $b_{03}(a_{21} + b_{12}) \neq 0$, the unique possibility to vanish α_{18} is to take $a_{05} = 0$, Imposing this condition the following constant value is

$$\alpha_{20} = b_{03}^3(a_{21} + b_{12})^2(a_{21} - 2b_{12}).$$

Imposing $a_{21} = 2b_{12}$, we get $a_{05} = a_{03} = a_{23} = a_{14} = b_{21} = 0$, $a_{12} = -3b_{03} \neq 0$, $a_{32} = -6b_{03}b_{12} \neq 0$, which determines the case **c)**. In this case the vector field associated is $\mathbf{F} = (y - 3b_{03}xy^2 + 2b_{12}x^2y - 6b_{03}b_{12}x^3y^2, -x^3 + b_{03}y^3 + b_{12}xy^2)^T$. This vector field has the inverse integrating factor $V = (1 + 2b_{12}x^2)^{3/2}$, $V(\mathbf{0}) \neq 0$, consequently $\mathbf{F}$ is analytic integrable, and since the origin is monodromic, it is center.

■

Now we consider the differential system

$$\begin{aligned}
\dot{x} &= y + a_{90}x^9 + a_{81}x^8y + a_{72}x^7y^2 + a_{63}x^6y^3 + a_{54}x^5y^4, \\
\dot{y} &= -x^5 + b_{41}x^4y + b_{23}x^2y^3 + b_{14}xy^4 + b_{05}y^5.
\end{aligned} \tag{3.19}$$

Theorem 3.14 *Then origin of system* (3.19) *is a center if, and only if, one of the following conditions holds.*

(a) $a_{90} = a_{72} = a_{54} = b_{41} = b_{23} = b_{05} = 0$.

(b) $a_{81} = a_{72} = a_{63} = a_{54} = b_{41} = b_{14} + b_{05} = b_{23} + 9a_{90} = 0$, $a_{90} \neq 0$.

Proof. The origin of system (3.19) is monodromic. This can easily be seen taking into account that the first quasi-homogeneous component respect to the type $\mathbf{t} = (1, 3)$ is $\mathbf{F}_1 = (y, -x^5)$ and applying Proposition 2.4. Now, as before, we will use the scalar algorithm of Theorem 2.10 imposing that the first constants $\alpha_6, \alpha_8, \cdots$, must be null. The value of the first constants modulo the annulation of the previous ones is:

$$\begin{aligned}
\alpha_{10} &= b_{41}, \\
\alpha_{14} &= b_{23} + 9a_{90}, \\
\alpha_{18} &= 7a_{72} + 15b_{05}, \\
\alpha_{22} &= -61a_{81}a_{90} - 21a_{90}b_{14} + 3a_{54}, \\
\alpha_{26} &= -567a_{90}a_{63} - a_{72}(195a_{81} + 11b_{14}), \\
\alpha_{30} &= 1280a_{90}^2a_{72} + (39a_{81}^2 - 337a_{81}b_{14} - 96b_{14}^2)a_{90} - 37a_{72}a_{63}.
\end{aligned}$$

In fact, we have compute the constants α_{30+4i} for $1 \leq i \leq 4$. In order to find the irreducible decomposition of the ideal generated by these constants we have used the routine `minAssGTZ` [12] of the computer algebra system SINGULAR [16]. As a result we have obtained the necessary conditions **(a)** and **(b)** of the theorem.

Now we will see the sufficiency of these two conditions.

In the case **(a)** system (3.19) is R_x-reversible and since the origin is monodromic then it is a center.

In the case **(b)** system (3.19) is a sum of two quasi-homogeneous vector field of type $\mathbf{t} = (1, 3)$. More specifically $\dot{\mathbf{x}} = \mathbf{F} := \mathbf{F}_2 + \mathbf{F}_8$ con $\mathbf{F}_2 = \mathbf{X}_h \in \mathcal{Q}_2^{\mathbf{t}}$ with $h = -y^2/2 - x^6/6$ and $\mathbf{F}_8 = a_{90}(x^9, -9x^2y^3)^T \in \mathcal{Q}_8^{\mathbf{t}}$. Moreover $\mathbf{F}$ has the inverse integrating factor $V = h(1 - 3a_{90}x^3y)$. Hence by [7, Theorem 1.3] system (3.19) is analytically integrable and as the origin is monodromic, see Proposition 2.4, then the origin is a center. ∎

Remark. Statement **(b)** of Theorem 3.14 proves that the hypothesis $n \neq km$ with $1 \leq k \leq m+1$ of Theorem 2.12 is necessary. Actually in this case $n = 4$, $m = 2$, $k = 2$ there exist centers which are not axis-reversible.

Finally, we consider the differential system

$$\begin{aligned}
\dot{x} &= y + a_{50}x^5 + a_{41}x^4y + a_{32}x^3y^2 + a_{23}x^2y^3 + a_{14}xy^4 + a_{05}y^5, \\
\dot{y} &= -x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.
\end{aligned} \tag{3.20}$$

Theorem 3.15 *The origin of system (3.20) is a center if, and only if, system (3.20) is R_x-reversible.*

Proof. By applying Proposition 2.4, the origin of system (3.20) is monodromic. From the algorithm of Theorem 2.10, the value of the first constants modulo the annulation of the previous ones are:

$$\begin{aligned}
\alpha_6 &= b_{21}, \\
\alpha_8 &= 5a_{50} + 3b_{03}, \\
\alpha_{10} &= 5a_{32} + b_{03}b_{12}, \\
\alpha_{12} &= 15a_{14} + b_{03}(45a_{41} + 8b_{12}^2), \\
\alpha_{14} &= b_{03}\left[345a_{23} - 288b_{03}^2 + 200b_{12}^3 + 1005b_{12}a_{41}\right].
\end{aligned}$$

i) If $b_{03} = 0$ then we have $a_{14} = a_{32} = a_{50} = b_{21} = 0$ and we obtain the vector field $\mathbf{F} = (y + a_{41}x^4y + a_{23}x^2y^3 + a_{05}y^5, -x^3 + b_{12}xy^2)^T$, that it is R_x reversible and therefore the origin is a center.

ii) If $b_{03} \neq 0$ then $a_{23} = 96b_{03}^2/115 - 40b_{12}^3/69 - 67b_{12}a_{41}/23$ and the next constant is

$$\alpha_{16} = b_{03}(77625a_{05} + 39312b_{03}^2b_{12} + 5175a_{41}^2 - 48000b_{12}^2a_{41} - 11200b_{12}^4).$$

Then the vanishing of α_{16} implies

$$a_{05} = -\frac{1456}{2875}b_{03}^2b_{12} - \frac{1}{15}a_{41}^2 + \frac{128}{207}b_{12}^2a_{41} + \frac{448}{3105}b_{12}^4,$$

and the next constants are:

$$\begin{aligned}
\alpha_{18} &= b_{03}\left[154350b_{12}a_{41}^2 + 75(6140b_{12}^3 + 48033b_{03}^2)a_{41} + 4b_{12}^2(24500b_{12}^3 + 72981b_{03}^2)\right],\\
\alpha_{20} &= b_{03}\left[-137473875a_{41}^3 + 14694097950b_{12}^2a_{41}^2 + 45b_{12}(383059300b_{12}^3\right.\\
&\quad \left.+1129106061b_{03}^2)a_{41} + 3184216000b_{12}^6 - 951034932b_{03}^2b_{12}^3 + 7784075376b_{03}^4\right],\\
\alpha_{22} &= b_{03}\left[-3203784643500b_{12}a_{41}^3 + 1350(65743832456b_{12}^3 - 44293983786b_{03}^2)a_{41}^2\right.\\
&\quad +45b_{12}^2(261072714500b_{12}^3 + 28971358803b_{03}^2)a_{41} + 4b_{12}(547617815500b_{12}^6\\
&\quad \left.-1232261979255b_{03}^2b_{12}^3 + 2341088776656b_{03}^4)\right].
\end{aligned}$$

In this case, there are no values such that $\alpha_{18} = \alpha_{20} = \alpha_{22} = 0$ except that all the remaining parameters be zero.

∎

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