International Journal of Bifurcation and Chaos, Vol. 12, No. 2 (2002) 429–437
© World Scientific Publishing Company

ROTATION SETS FOR ORBITS OF DEGREE ONE CIRCLE MAPS

LLUÍS ALSEDÀ* and FRANCESC MAÑOSAS†
Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08913 Cerdanyola del Vallès, Barcelona, Spain
**alseda@mat.uab.es*
†manyosas@mat.uab.es

MOIRA CHAS
Department of Mathematics and Institute for Mathematical Sciences, SUNY at Stony Brook, NY 11794-3651, USA
moira@math.sunysb.edu

Received October 23, 2000; Revised May 23, 2001

Let F be the lifting of a circle map of degree one. In [Bamón *et al.*, 1984] a notion of F-rotation interval of a point $x \in \mathbb{S}^1$ was given. In this paper we define and study a new notion of a rotation set of a point which preserves more of the dynamical information contained in the sequences $\{F^n(y)\}_{n=0}^{\infty}$ than the one preserved from [Bamón *et al.*, 1984]. In particular, we characterize dynamically the endpoints of these sets and we obtain an analogous version of the Main Theorem of [Bamón *et al.*, 1984] in our settings.

1. Introduction and Statement of the Main Results

Let $\mathcal{C}_1(\mathbb{S}^1)$ be the class of all continuous maps of the circle into itself of degree one. Let $f \in \mathcal{C}_1(\mathbb{S}^1)$ and let $F : \mathbb{R} \to \mathbb{R}$ be a lifting of f. Denote by π the canonical projection from $\mathbb{R}$ to $\mathbb{S}^1$. It is well known that, when f is a homeomorphism, then $\lim_{n\to\infty}(F^n(y) - y)/n$ exists, it does not depend on y and is called the rotation number of f. In the general case of endomorphisms, this limit may not exist. Newhouse *et al.* introduced in [Newhouse *et al.*, 1983] the notion of *rotation set* of an endomorphism, L_F, by defining:

$$L_F = \mathrm{Cl}(\{\rho_F^+(x) : x \in \mathbb{S}^1\})$$

where $\mathrm{Cl}(\cdot)$ means topological closure and

$$\rho_F^+(x) = \limsup_{n\to\infty} \frac{F^n(y) - y}{n}$$

for any $y \in \pi^{-1}(x)$ (note that since f has degree one, $F(y+1) = F(y)+1$ for each $y \in \mathbb{R}$ and, hence, $\rho_F^+(x)$ does not depend on y). They also proved that L_F is an interval. Clearly, L_F is defined up to translations by integers. In [Ito, 1981] Ito proved that each $\alpha \in L_F$ is realized as the rotation number of some point in $\mathbb{S}^1$ in the sense that, for some $x \in \mathbb{S}^1$, $\alpha = \lim_{n\to\infty}(F^n(y) - y)/n$, where $y \in \pi^{-1}(x)$. In such a case we will say that α is the *rotation number of* x and we will denote it by $\rho_F(x)$. We note that each point in the orbit of x, that is in the

*Current address: Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain.