ON THE STRUCTURE OF THE KNEADING SPACE OF BIMODAL DEGREE ONE CIRCLE MAPS

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For continuous maps on the interval with finitely many monotonicity intervals, the kneading theory developed by Milnor and Thurston [6] gives a symbolic description of the dynamics of these maps. This description is given in terms of the kneading invariants which essentially consist on the symbolic orbits of the turning points of the map under consideration. Moreover, this theory also gives a classification of all such maps through these invariants. For continuous bimodal degree one circle maps similar invariants were introduced by Alsedà and Mañosas [3]. In that paper, the first part of the program just described was carried through, and relations between the circle maps invariants and the rotation interval were elucidated. Later on, in [1, Theorem A] the set of all these kneading invariants (the *kneading space*) was characterized. Our main goal is to give a description of the kneading space of the bimodal degree one circle maps using some self–similarity operators which allow us to identify certain subsets with known structure. This work is, in some sense, a continuation of [1] and we use heavily the notation and results from that paper. Moreover, in [2], all definitions and the proof of the main theorem will be done.

As it is usual, instead of working with the circle maps themselves we will rather use their liftings to the universal covering space $\mathbb{R}$. To this end, we introduce the following class $\mathcal{A}$ of maps. First we define $\mathcal{L}$ to be the class of all continuous maps F from $\mathbb{R}$ into itself such that $F(x+1) = F(x)+1$ for all $x \in \mathbb{R}$. That is, $\mathcal{L}$ is the class of all liftings of degree one circle maps. Then we will say that $F \in \mathcal{A}$ if (see Figure 1):

(1) $F \in \mathcal{L}$.

(2) There exists $c_F \in (0,1)$ such that F is strictly increasing in $[0, c_F]$ and strictly decreasing in $[c_F, 1]$.

We note that every map $F \in \mathcal{A}$ has a unique local maximum and a unique local minimum in $[0,1)$. To define the class $\mathcal{A}$ we restricted ourselves to the case in which F has the minimum at 0. Since each map from $\mathcal{L}$ is conjugate by a translation to a map from $\mathcal{L}$ having the minimum at 0, the fact that in (2) we fix that F has a minimum in 0 is not restrictive.

For a map $\bar{F} \in \mathcal{A}$ one can define the *kneading pair* denoted by $\mathcal{K}(\bar{F})$ (see[1, pp. 277–278]) which captures all dynamics of the map F (see [3, Proposition A]). The kneading space is a subset of the product space $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$ where both $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ are totally ordered spaces equipped with the order topology (see [4]). Also, the set of all kneading pairs will be called the *kneading space*. Now, we introduce the following index space. It will