Stable heteroclinic cycles and symbolic dynamics

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Let $S_0^1, S_1^1, \ldots, S_{n-1}^1$ be n circles. A rotation in n circles is a map $f: \cup_{i=0}^{n-1} S_i^1 \to \cup_{i=0}^{n-1} S_i^1$ which maps each circle onto another by a rotation. This particular type of interval exchange map arises naturally in bifurcation theory. In this paper we give a full description of the symbolic dynamics associated to such maps..

I. INTRODUCTION

In bifurcation theory, we often have to deal with a critical vector field in $\mathbf{R}^m$, $m \geq 2$, which satisfies the following properties: there exists a finite sequence of singularities, p_0, $p_1, \ldots, p_{n-1}$, of saddle type with only one unstable direction, and a sequence of $2n$ heteroclinic or homoclinic orbits, $\Gamma_{0,0}$, $\Gamma_{0,1}, \ldots, \Gamma_{i,0}$, $\Gamma_{i,1}, \ldots, \Gamma_{n-1,0}$, $\Gamma_{n-1,1}$, joining one of the p_i's to another. Such a configuration is called a *heteroclinic cycle* (see Fig. 1). It turns out that, in many situations, the set $\Gamma = \cup_{i=0}^{n-1}(\Gamma_{i,0} \cup \Gamma_{i,1})$ is an attractor. In such a case, we call it a *stable heteroclinic cycle.*

Stable heteroclinic cycles are obviously not structurally stable, and consequently the following question arises:

Question: Consider a $\mathscr{C}^1$ vector field in $\mathbf{R}^m$, $m \geq 2$, with a stable heteroclinic cycle Γ. What happens in a neighborhood of Γ when we perturb the vector field in the $\mathscr{C}^1$ topology?

This question has been completely answered, in the particular case called "gluing bifurcation," of a configuration involving only one singularity.[1–3]

However, so far, there is no answer in the general case, which remains very interesting for the following two reasons:

First, it is not an academic generalization of the gluing bifurcation: there exist some extra difficulties related to the increasing richness of the possible dynamical behavior.

Second, stable heteroclinic cycles (with more than one singularity) occur in problems of bifurcation coming from the PDEs world (for instance hydrodynamics[4]).

In this paper we focus our attention to a particular class of (stable) heteroclinic cycle. These cycles will be called *rotating (stable) heteroclinic cycles* (see Fig. 2) and correspond to the case when the two heteroclinic or homoclinic orbits $\Gamma_{i,0}$ and $\Gamma_{i,1}$, emanating from a singularity p_i, end at the same singularity p_j.

Consider now a $\mathscr{C}^1$ vector field X_0 in $\mathbf{R}^m$ with a rotating stable heteroclinic cycle Γ and assume that the linearized vector field $DX_0(p_i)$, at each singularity p_i, is such that its dominating stable eigenvalue is real. Then, generically, there exists a rigorous way to reduce the dynamics of any vector field X, $\mathscr{C}^1$-close to X_0 in a neighborhood of Γ, to the dynamics of a map $f_{[X]}$ on the interval with a finite number of discontinuities (see Fig. 3). Furthermore, this map is monotone and is a contraction of each interval of continuity. This reduction to a one-dimensional dynamics is the subject of Sec. II.

Some simple extra conditions on the vector field X_0 yields maps $f_{[X]}$ from the union of n intervals into itself, mapping one interval into another with a single discontinuity in each interval.

For a vector field X, $\mathscr{C}^1$-close to X_0, there is a natural way of coding the invariant curves of X which remain in a neighborhood of Γ.

Roughly speaking, we can code an invariant curve with a periodic sequence of $2n$ symbols corresponding to the fact that the curve has to follow successively some of the $2n$ heteroclinic or homoclinic orbits of X_0. In Sec. III, we show how this coding corresponds, on the interval, to the classical Milnor–Thurston coding for the periodic orbits of the map $f_{[X]}$.

We show also that each code of a periodic orbit of $f_{[X]}$ is also the code of the periodic orbit of a map $f_{[X]}^*$ which also has one discontinuity in each interval, the same monotonicity type of $f_{[X]}$ but which is an interval exchange.

When this interval exchange preserves the orientation it can be seen as a map from the union of n circles into itself, which maps each circle onto another by a rotation. We call this type of map a *composition of rotations in n circles* (see Fig. 4 for an example of these maps). Sections IV–VIII are devoted to the study of the symbolic dynamics of these maps.

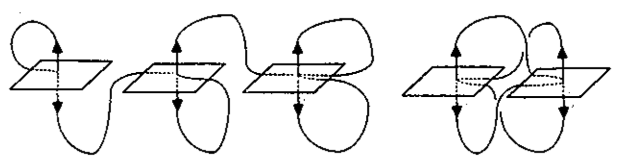

FIG. 1. Some heteroclinic cycles.