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PATTERNS AND MINIMAL DYNAMICS FOR GRAPH MAPS

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1. *Introduction*

In this paper, we shall study the phenomenon of rigidity of the dynamics of graph maps. The notion of rigidity is often associated with the existence of a canonical representative within a well-defined class of objects. This is the case, for example, in hyperbolic geometry (Mostow [**23**]), and for surface homeomorphisms (Nielsen and Thurston [**26**, **13**]). In each of these cases, there exists a unique (up to conjugacy) canonical representative which satisfies many extremal dynamical properties, such as minimisation of the growth rate (Besson, Courtois and Gallot [**8**] for hyperbolic manifolds and Fathi and Shub [**13**] for pseudo-Anosov homeomorphisms), and minimisation of the number of closed geodesics for hyperbolic manifolds and of periodic orbits for pseudo-Anosov homeomorphisms (Asimov and Franks [**5**] and T. Hall [**14**]) in their respective classes.

With the aim of comparing periodic orbits of different maps, Misiurewicz [**21**] proposed a general approach to the notion of *pattern* using the ideas of some previous works ([**6**, **2**, **7**] for instance). According to this point of view, the notion of pattern was introduced for each of the following important classes of maps:

- continuous maps of the interval, of the circle, and of 'fixed' graphs (where the notion of pattern is termed *action*) [**6**, **3**, **4**],
- continuous maps of (finite) trees [**1**],
- surface homeomorphisms [**10**, **20**, **18**] (where the notion of pattern is usually termed *braid type*).

The basic phenomenon that these notions of pattern are designed to encapsulate is that of *coexistence* or *forcing* of periodic orbits. The original motivation for this stemmed from Sharkovskiĭ's theorem in 1964 for interval maps [**25**], which roughly speaking, states that the existence of a single periodic orbit P of a given period n is enough to imply the existence of other periodic orbits and often of infinitely many orbits. This result may be refined by considering the permutation $\sigma \in S_n$ induced by the map on the points of P, the points being ordered by the natural ordering of the interval. Each permutation σ may be interpreted as a subset of $\mathcal{C}(I, I)$, namely those continuous maps of the interval I which admit a periodic orbit whose associated permutation is σ. This subset is essentially (up to homeomorphism) a relative homotopy equivalence class in $\mathcal{C}(I, I)$, relative to the periodic orbit in question. It possesses a unique (up to homeomorphism) canonical representative (the piecewise linear or 'connect-the-dots' map) which minimises the topological entropy as well as the set of periodic orbits [**3**]. As we pointed out above, a canonical representative with analogous minimisation properties

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