ON THE MINIMUM POSITIVE ENTROPY FOR CYCLES ON TREES

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Abstract. Consider, for any $n \in \mathbb{N}$, the set Pos_n of all n-periodic tree patterns with positive topological entropy and the set $\mathrm{Irr}_n \subsetneq \mathrm{Pos}_n$ of all n-periodic irreducible tree patterns. The aim of this paper is to determine the elements of minimum entropy in the families Pos_n and Irr_n. Let λ_n be the unique real root of the polynomial $x^n - 2x - 1$ in $(1, +\infty)$. We explicitly construct an irreducible n-periodic tree pattern $\mathcal{Q}_n$ whose entropy is $\log(\lambda_n)$. For $n = m^k$, where m is a prime, we prove that this entropy is minimum in the set Pos_n. Since the pattern $\mathcal{Q}_n$ is irreducible, $\mathcal{Q}_n$ also minimizes the entropy in the family Irr_n.

1. Introduction

The notion of *pattern* plays a central role in the theory of topological and combinatorial dynamics. Consider a family $\mathcal{X}$ of topological spaces (such as the set of all trees, graphs, compact surfaces, closed intervals of the real line, etc) and the family $\mathcal{F}_\mathcal{X}$ of all maps $\{f\colon X \longrightarrow X : X \in \mathcal{X}\}$ satisfying a given restriction (continuous maps, homeomorphisms, etc). Given a map $f\colon X \longrightarrow X$ in $\mathcal{F}_\mathcal{X}$ which is known to exhibit a finite invariant set P, the *pattern of P in $\mathcal{F}_\mathcal{X}$* is the equivalence class $\mathcal{P}$ of all maps $g\colon Y \longrightarrow Y$ in $\mathcal{F}_\mathcal{X}$ having an invariant set $Q \subset Y$ that, at a combinatorial level, behaves like P. That is, the relative positions of the points of Q inside Y are the same as the relative positions of P inside X, and the way these positions are permuted under the action of g coincides with the way f acts on the points of P. In this case, it is said that every map g in the class *exhibits* the pattern $\mathcal{P}$. If in particular P is a periodic orbit of f, the pattern $\mathcal{P}$ is said to be *cyclic* or *periodic*.

When $\mathcal{F}_\mathcal{X}$ is the family of continuous self-maps of closed intervals, the points of an invariant set P of a map in $\mathcal{F}_\mathcal{X}$ are totally ordered and the pattern of P can be clearly identified with a permutation π in a natural way. The notion of pattern for interval maps has its roots in the well known Sharkovskii's Theorem [23], but it was formalized and developed in the early 1990s [10, 22].

As another important example, when $\mathcal{F}_\mathcal{X}$ is the family of surface homeomorphisms, the pattern (or *braid type*) of a cycle P of a map $f\colon M \longrightarrow M$ in $\mathcal{F}_\mathcal{X}$ is characterized by the isotopy class, up to conjugacy, of $f\big|_{M \setminus P}$ [16, 21].

Going back to one-dimensional spaces, recently there has been a growing interest in extending the notion of *pattern* from the interval case to more general spaces such as graphs [2, 8] or trees [6, 11, 12]. In this paper we deal with patterns of invariant sets of continuous maps defined on trees (simply connected graphs). From now on, such patterns will be called *tree patterns*.

Let us recall the notion of a tree pattern. If $f\colon T \longrightarrow T$ is a continuous map of a tree and $P \subset T$ is a finite invariant set of f, the triplet (T, P, f) will be called a *model*. Two points x, y of P will be called *consecutive* if the unique closed interval

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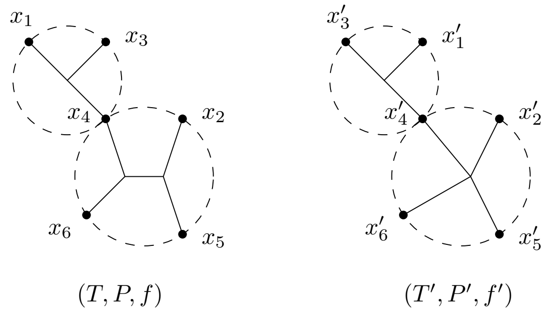

FIGURE 1. Set $P = \{x_i\}_{i=1}^6$ and $P' = \{x'_i\}_{i=1}^6$. If $f\colon T \longrightarrow T$ and $f'\colon T' \longrightarrow T'$ are continuous maps such that $f(x_i) = x_{i+1}$ and $f'(x'_i) = x'_{i+1}$ for $1 \leq i \leq 5$, $f(x_6) = x_1$ and $f'(x'_6) = x'_1$, then the models (T, P, f) and (T', P', f') belong to the same pattern $\mathcal{P}$.

of T whose endpoints are x and y contains no other points of P. Any maximal subset of P consisting only of pairwise consecutive points will be called a *discrete component*. In this setting, a *pattern* can be identified with the conjugacy class of all models with a fixed distribution of discrete components and images of points in P. For instance, in Figure 1 we show two different models (T, P, f) and (T', P', f') of a 6-periodic pattern $\mathcal{P}$ with two discrete components. Observe that two points x_i, x_j of P are consecutive in T if and only if the corresponding points x'_i, x'_j of P' are consecutive in T'.

Measuring the dynamical richness of a pattern is a classical problem in the theory of discrete dynamical systems. The question can be posed as follows. Given a pattern $\mathcal{P}$ in $\mathcal{F}_\mathcal{X}$, is it possible to establish, in terms only of the combinatorial data enclosed in $\mathcal{P}$, a lower bound for the dynamical complexity that will be present in any map in $\mathcal{F}_\mathcal{X}$ exhibiting $\mathcal{P}$? In order for this question to make sense, we have to be precise on how we understand the dynamical complexity of a map. A classical way of measuring it is in terms of the *topological entropy*, a notion first introduced in 1965 [1]. The topological entropy of a continuous map $f\colon X \longrightarrow X$ of a compact metric space is a non-negative real number (or infinity) that measures how the iterates of the map mix the points of X. It will be denoted by $h(f)$. It is known that an interval map with positive entropy is *chaotic* in the sense of Li and Yorke [20]. For general compact metric spaces, the same result has been recently obtained in [13]. It is also well known that the topological entropy of f is closely related to the number of different periodic orbits exhibited by f and the sizes of such orbits. On the other hand, a map with zero topological entropy can be viewed as dynamically trivial.

Recovering the question posed in the previous paragraph, it is natural to define the *topological entropy of a pattern* $\mathcal{P}$ *in* $\mathcal{F}_\mathcal{X}$, denoted from now on by $h(\mathcal{P})$, as the infimum of the topological entropies of all maps in $\mathcal{F}_\mathcal{X}$ exhibiting $\mathcal{P}$.

Although computing the entropy of a continuous map is difficult in general, in some cases the computation of the entropy of a pattern $\mathcal{P}$ in $\mathcal{F}_\mathcal{X}$ can be easily performed thanks to the existence of the so called *canonical models*. Roughly speaking, a *canonical model* of a pattern $\mathcal{P}$ in $\mathcal{F}_\mathcal{X}$ is a map $f \in \mathcal{F}_\mathcal{X}$ that exhibits $\mathcal{P}$ and satisfies at least the following properties:

(a) f is essentially unique and is determined by the combinatorial data of $\mathcal{P}$
(b) f minimizes the entropy in the set of all maps exhibiting the pattern $\mathcal{P}$
(c) The dynamics of f (in particular, its entropy) can be completely described and easily computed using some algebraic tools.

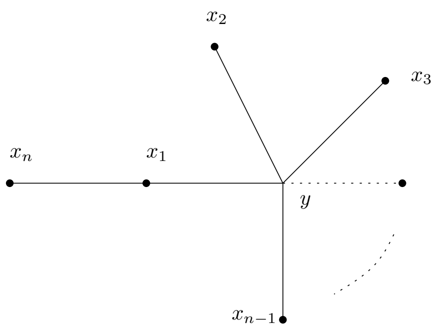

FIGURE 2. The canonical model (T, P, f) of the pattern $\mathcal{Q}_n$, for which $P = \{x_i\}_{i=1}^n$ is time labeled and $f(y) = y$.

It follows that $h(\mathcal{P})$, defined as an infimum of entropies of maps, is in fact a minimum and coincides with the entropy of the canonical model of $\mathcal{P}$, which can be easily computed. The existence of canonical models for patterns has been proved for continuous maps of closed intervals (see [7] for a list of references), homeomorphisms of compact surfaces [17, 25] and continuous maps on trees [6].

Once we have depicted the idea of tree pattern and established that for each tree pattern one can compute its entropy, we are ready to explain the aim of this paper. We will focus on periodic patterns. In this setting, and according to the topological entropy as a measure of dynamical complexity, several extremality questions arise. Fix $n \in \mathbb{N}$ and consider the (finite) set Pat_n of all n-periodic tree patterns. A first natural classification in Pat_n is given by the zero/positive entropy character of its elements. On one hand, the zero entropy tree patterns have been completely characterized [6, 5]. Let Pos_n denote the subset of Pat_n of all n-periodic tree patterns with positive entropy. Describing the patterns with maximal/minimal entropy in Pos_n is a natural problem. The entropy-maximal patterns in Pos_n are still unknown, although several advances have been recently reported [4]. It is worth noticing that the maximality problem is unsolved even in the particular case of interval patterns. Indeed, the maximal-entropy n-cyclic permutations, when n has the form $4k + 2$, are still unknown (a very recent paper [3] studies this case from a computational point of view and proposes a conjecture).

In this paper we deal with the opposite problem: the characterization of the patterns of minimal entropy in Pos_n. For interval maps, the description of the minimum entropy cyclic permutations is well known (see [7] for a review). In contrast, as far as we know, there is no literature about this problem in the setting of tree maps. We will define, for any $n \geq 3$, an n-periodic tree pattern $\mathcal{Q}_n$ that we conjecture has minimal entropy in the set Pos_n (observe that the problem makes no sense when $n = 1, 2$, since every periodic pattern of period 1 or 2 has entropy zero). See the canonical model of $\mathcal{Q}_n$ in Figure 2.

Conjecture. *Let $n \in \mathbb{N}$ with $n \geq 3$. The pattern $\mathcal{Q}_n$ has minimum entropy in the set of all n-periodic patterns with positive entropy.*

The main result of this paper (Theorem A) states that this conjecture is true when $n = m^k$, where $k \geq 1$ and m is prime.

A final remark has to be made. Another important classification of the periodic tree patterns is given by the reducible/irreducible character of each pattern. Roughly speaking, a pattern is *reducible* when the points of the invariant set can be partitioned into disjoint subtrees that are permuted under the action of the map. The notion of reducibility arose early in the very beginning of the study of interval maps and has been recently extended to the setting of tree patterns [5]. The irreducible tree patterns are closely related to pseudo-Anosov braid types of periodic orbits of orientation preserving disk homeomorphisms [18]. Since the zero-entropy tree patterns are reducible [5], the set Irr_n of all irreducible tree patterns is a proper subset of Pos_n. Thus, it makes sense to study the minimality of the entropy in the subclass Irr_n. Again we note that, remarkably, this problem remains unsolved for interval maps when n is even. Since our candidates $\mathcal{Q}_n$ turn out to be irreducible patterns, a corollary of the main result of our paper is that, when n is a power of a prime, the pattern $\mathcal{Q}_n$ also minimizes the topological entropy in the subclass Irr_n (Corollary B).

2. Definitions and statement of the main results

A *tree* is a compact uniquely arcwise connected space which is a point or a union of a finite number of intervals (by an *interval* we mean any space homeomorphic to $[0, 1]$). Any continuous map $f : T \longrightarrow T$ from a tree T into itself will be called a *tree map*. A set $X \subset T$ is said to be *f-invariant* if $f(X) \subset X$. For each $x \in T$, we define the *valence* of x to be the number of connected components of $T \setminus \{x\}$. A point of valence different from 2 will be called a *vertex* of T and the set of vertices of T will be denoted by $V(T)$. Each point of valence 1 will be called an *endpoint* of T. The set of such points will be denoted by $\mathrm{En}(T)$. Also, the closure of a connected component of $T \setminus V(T)$ will be called an *edge of T*. Any tree which is a union of $r \geq 2$ intervals whose intersection is a unique point y of valence r will be called an *r-star*, and y will be called its *central point*.

Given any subset X of a topological space, we will denote by $\mathrm{Int}(X)$ and $\mathrm{Cl}(X)$ the interior and the closure of X, respectively. For a finite set P we will denote its cardinality by $|P|$.

A triplet (T, P, f) will be called a *model* if $f : T \longrightarrow T$ is a tree map and P is a finite f-invariant set such that $\mathrm{En}(T) \subset P$. In particular, if P is a periodic orbit of f and $|P| = n$ then (T, P, f) will be called an *n-periodic model*. Given $X \subset T$ we will define the *convex hull* of X, denoted by $\langle X\rangle_T$ or simply by $\langle X\rangle$, as the smallest closed connected subset of T containing X. When $X = \{x, y\}$ we will write $[x, y]$ to denote $\langle X\rangle$. The notations (x, y), $(x, y]$ and $[x, y)$ will be understood in the natural way.

Let T be a tree and let $P \subset T$ be a finite subset of T. The pair (T, P) will be called a *pointed tree*. Two points x, y of P will be called *consecutive* if $(x, y) \cap P = \emptyset$. Any maximal subset of P consisting only of pairwise consecutive points will be called a *discrete component* of (T, P). We say that two pointed trees (T, P) and (T', P') are *equivalent* if there exists a bijection $\phi\colon P \longrightarrow P'$ which preserves discrete components. The equivalence class of a pointed tree (T, P) will be denoted by $[T, P]$.

Let (T, P) and (T', P') be equivalent pointed trees, and let $\theta\colon P \longrightarrow P$ and $\theta'\colon P' \longrightarrow P'$ be maps. We will say that θ and θ' are *equivalent* if $\theta' = \phi \circ \theta \circ \phi^{-1}$ for a bijection $\phi\colon P \longrightarrow P'$ which preserves discrete components. The equivalence class of θ by this relation will be denoted by $[\theta]$. If $[T, P]$ is an equivalence class of pointed trees and $[\theta]$ is an equivalence class of maps then the pair $([T, P], [\theta])$ will be called a *pattern*. We will say that a model (T, P, f) *exhibits* a pattern $(\mathcal{T}, \Theta)$ if $\mathcal{T} = [\langle P\rangle_T, P]$ and $\Theta = [f\big|_P]$.

Despite of the fact that the notion of a discrete component is defined for pointed trees, by abuse of language we will use the expression *discrete component of a pattern*, which will be understood in the natural way since the number of discrete components and their relative positions are the same for all models of the pattern.

The topological entropy (see Section 1) of a map $f\colon T \longrightarrow T$ will be denoted by $h(f)$. Given a pattern $\mathcal{P}$, the topological entropy of $\mathcal{P}$ is defined to be

$$h(\mathcal{P}) := \inf\{h(f) : (T, P, f) \text{ is a model exhibiting } \mathcal{P}\}.$$

The simplest models exhibiting a given pattern are the monotone ones, according to the following definition. Let S and T be trees and let $f\colon T \longrightarrow S$ be a map. Given $a, b \in T$ we say that $f\big|_{[a,b]}$ is *monotone* if $f([a, b])$ is either an interval or a point and $f\big|_{[a,b]}$ is monotone as an interval map. Let (T, P, f) be a model. A pair $\{a, b\} \subset P$ will be called a *basic path of* (T, P) if it is contained in a single discrete component of (T, P). We will say that f is P*-monotone* if $f\big|_{[a,b]}$ is monotone for any basic path $\{a, b\}$. The model (T, P, f) will be called *monotone.* In such case, Proposition 4.2 of [6] states that the set $P \cup V(T)$ is f-invariant. Hence, the map f is also $(P \cup V(T))$-monotone.

Theorem 2.1 (Theorem A of [6])**.** *Let $\mathcal{P}$ be a pattern. Then the following statements hold.*

(a) There exists a monotone model of $\mathcal{P}$.
(b) Every monotone model (T, P, f) of $\mathcal{P}$ satisfies $h(f) = h(\mathcal{P})$.

The monotone models from Theorem 2.1 are essentially unique in the following sense. Let (T, P, f) be a monotone model and let S be a non-empty union of edges disjoint from P. We will say that S is an *invariant forest of* (T, P, f) if either $f^i(S) \cap P = \emptyset$ for every $i \geq 0$ or there exists $n > 0$ such that $f^i(S) \cap P = \emptyset$ for every $i = 0, 1, \ldots, n-1$ and $f^n(S)$ degenerates to a point of P. We will say that a monotone model (T, P, f) is a *canonical model* of the pattern $([T, P], [f\big|_P])$ if it has no invariant forests. From [6, Theorem B] it follows that every pattern has a canonical model. Moreover, given two canonical models (T, P, f) and (T', P', f') of the same pattern there exists a homeomorphism $\phi\colon T \longrightarrow T'$ such that $\phi(P) = P'$, and $f' \circ \phi\big|_P = \phi \circ f\big|_P$. Hence, the canonical model of a pattern is essentially unique.

It is worth noticing that the proof of Theorem 2.1 is constructive and gives a finite algorithm to construct the canonical model of any pattern.

Let $P = \{x_i\}_{i=1}^n$ be an n-periodic orbit of a map θ. We will say that P is *time labeled* if $\theta(x_i) = x_{i+1}$ for $1 \leq i < n$ and $\theta(x_n) = x_1$.

An n-periodic pattern $\mathcal{P}$ will be called *trivial* if it has only one discrete component. In this case, for $n \geq 2$, let (T, P) be a pointed tree such that T is an n-star with $\mathrm{En}(T) = P = \{x_1, x_2, \ldots, x_n\}$ and let y be its central point. Consider a rigid rotation on T. That is, a model (T, P, f) such that $f(y) = y$ and f maps bijectively $[y, x_i]$ onto $[y, x_{i+1}]$ for $1 \leq i < n$ and $[y, x_n]$ onto $[y, x_1]$. Clearly, (T, P, f) is a monotone model with no invariant forests. In consequence, (T, P, f) is the canonical model of $\mathcal{P}$. Therefore, it easily follows that every trivial pattern has entropy 0.

The topological entropy of every map being Q-monotone with respect to a set Q containing the vertices of the tree can be easily computed as the logarithm of the spectral radius of the associated *Markov matrix.* Let us recall such technique.

A *combinatorial directed graph* is a pair $\mathcal{G} = (V, U)$ where $V = \{v_1, v_2, \ldots, v_k\}$ is a finite set and $U \subset V \times V$. The elements of V are called the *vertices* of $\mathcal{G}$ and each element (v_i, v_j) in U is called an *arrow* (from v_i to v_j) in $\mathcal{G}$. Such an arrow is usually denoted by $v_i \to v_j$. The notions of *path* and *loop* in $\mathcal{G}$ are defined as usual. The *length* of a path is defined as the number of arrows in the path. The *transition matrix* of $\mathcal{G}$ is a $k \times k$ binary matrix $(m_{ij})_{i,j=1}^k$ such that $m_{ij} = 1$ if and only if there is an arrow from v_i to v_j, and $m_{ij} = 0$ otherwise.

Let (T, Q, f) be a monotone model such that $Q \supset V(T)$. Note that, in this case, any connected component of $T \setminus Q$ is an open interval. An interval of T will be called Q-*basic* if it is the closure of a connected component of $T \setminus Q$. Observe that two different Q-basic intervals have pairwise disjoint interiors. Given $K, L \subset T$, we will say that K f-*covers* L if $f(K) \supset L$. Consider a labelling $I_1, I_2, \dots I_k$ of all Q-basic intervals. The *Markov graph of* (T, Q, f) associated to this labelling is a combinatorial directed graph whose vertices are the Q-basic intervals and there is an arrow from I_i to I_j if and only if I_i f-covers I_j. On the other hand, the *Markov matrix of* (T, Q, f) associated to this labelling is the transition matrix of the corresponding Markov graph of (T, Q, f). Given two different labellings of the set of Q-basic intervals and their associated Markov matrices M and N, there exists a permutation matrix A such that $M = A^T N A$ (where A^T denotes the transpose of A), and the corresponding Markov graphs are isomorphic.

For any square matrix M, we will denote its *spectral radius* by $\sigma(M)$. We recall that it is defined as the maximum of the moduli of the eigenvalues of M.

Remark 2.2. Let (T, Q, f) be a monotone model such that $Q \supset V(T)$. Let M be the Markov matrix of (T, Q, f). By standard arguments (see for instance [14] or [7, Theorem 4.4.5]), the topological entropy of f can be computed as

$$h(f) = \log \max\{\sigma(M), 1\}.$$

Recall that if (T, P, f) is the canonical model of a pattern $\mathcal{P}$ then the model $(T, P \cup V(T), f)$ is monotone. Thus, according to the previous paragraphs, we can consider the associated Markov graph and matrix. Since both objects depend only on the canonical model of $\mathcal{P}$, which is uniquely determined by the combinatorial data of the pattern $\mathcal{P}$, they will be respectively called *Markov graph of* $\mathcal{P}$ and *Markov matrix of* $\mathcal{P}$.

Remark 2.3. Let $\mathcal{P}$ be a pattern and let M be its Markov matrix. From Theorem 2.1(b) and Remark 2.2 we get that $h(\mathcal{P}) = \log \max\{\sigma(M), 1\}$.

Next we introduce our candidates for minimum entropy in the class Pos_n of positive entropy n-periodic patterns. Of course every periodic pattern of period 1 or 2 has entropy zero, so the problem makes sense only for $n \geq 3$.

Let $n \in \mathbb{N}$ with $n \geq 3$. Let $\mathcal{Q}_n$ be the n-periodic pattern $([T, P], [\theta])$ such that $P = \{x_1, x_2, \dots, x_n\}$ is time labeled and (T, P) has two discrete components, $\{x_n, x_1\}$ and $\{x_1, x_2, \dots, x_{n-1}\}$. It is straightforward to check that the canonical model (T, P, f) of $\mathcal{Q}_n$ satisfies:

(a) T is an $(n-1)$-star with endpoints $\{x_2, x_3, \dots, x_n\}$.
(b) The central point of T is fixed by f.
(c) x_1 belongs to the interior of the edge that has x_n as an endpoint.

In Figure 2 we show the canonical model of $\mathcal{Q}_n$. Observe that $\mathcal{Q}_3$ is the 3-periodic Štefan cycle of the interval [24].

Now we are ready to state the main result of this paper.

Theorem A. *Let n be a positive integer such that $n = m^k \geq 3$ with m prime and $k \geq 1$. Then, the pattern $\mathcal{Q}_n$ has minimum entropy in the set of all n-periodic patterns with positive entropy.*

As it has been explained in Section 1, there is an important subclass of patterns with positive entropy: the irreducible patterns. The characterization of such patterns relies on the idea of *block structure*. This notion appeared early in the literature for interval patterns: the *square root construction* [7, 23] or the notion of *extension*, first appeared in [15], are examples of block structures for interval periodic orbits. The generalization to periodic patterns on trees has been recently introduced in [5].

Let $\mathcal{P}$ be an n-periodic pattern and let (T, P, f) be a model of $\mathcal{P}$. For $n > p \geq 2$, we will say that $\mathcal{P}$ *has a p-block structure* (or simply a *block structure*) if there exists a partition $P = P_1 \cup P_2 \cup \ldots \cup P_p$ such that $f(P_i) = P_{i+1}$ for $1 \leq i < p$, $f(P_p) = P_1$, and $\langle P_i \rangle_T \cap P_j = \emptyset$ whenever $i \neq j$. In this case, p is a strict divisor of n and $|P_i| = n/p$ for $1 \leq i \leq p$. It is easy to see that this definition does not depend on the particular chosen model (T, P, f). The trees $\langle P_i \rangle_T$ (which do depend on the particular model (T, P, f) realizing the pattern) will be called *blocks*. We note that if a pattern has a p-block structure, this p-block structure is essentially unique up to relabelling of blocks. Observe also that a pattern can have several different block structures [5]. A periodic pattern $\mathcal{P}$ with no block structures is said to be *irreducible*.

Remark 2.4. Let (T, P, f) be the canonical model of the pattern $\mathcal{Q}_n$ and let P be labeled as in the definition of $\mathcal{Q}_n$ above. If (T, P, f) had a block structure, then the convex hull of the block P_i containing x_n would contain also x_1. Since $f(x_n) = x_1$, $f(P_i)$ would intersect P_i, a contradiction. In consequence, the patterns $\mathcal{Q}_n$ are irreducible.

From the characterization of the zero entropy patterns given in [6] it follows that every pattern with entropy zero has a block structure (we will recall and use such a characterization in Section 9). In consequence, the set Irr_n of all irreducible n-periodic patterns is contained in Pos_n. By Remark 2.4, the patterns $\mathcal{Q}_n$ are irreducible. Hence, we have the following corollary of Theorem A.

Corollary B. *Let n be a positive integer such that $n = m^k \geq 3$ with m prime and $k \geq 1$. Then, the pattern $\mathcal{Q}_n$ has minimum entropy in the set of all n-periodic irreducible patterns.*

This paper is organized as follows. In Section 3 we compute the entropy of the patterns $\mathcal{Q}_n$. In Section 4 we recall the notion of *division* for periodic tree patterns, a particular case of block structure that will play a central role in the proof of Theorem A. In Section 5 we introduce a partial ordering $\leq$ in the set of all n-periodic patterns and prove that $\mathcal{P} \leq \mathcal{Q}$ implies $h(\mathcal{P}) \leq h(\mathcal{Q})$. In Section 6 we introduce the notion of *strongly centered* pattern. This means essentially that all its discrete components are adjacent to a central discrete component which satisfies an additional rotational property. We also prove that for any n-periodic pattern $\mathcal{Q}$ with no division there exists a strongly centered n-periodic pattern $\mathcal{P}$ with no division such that $\mathcal{P} \leq \mathcal{Q}$. In Section 7 we prove that, when $n = m^k$ for some m prime, given any strongly centered n-periodic pattern $\mathcal{Q}$ with no division there exists an n-periodic pattern $\mathcal{P} \leq \mathcal{Q}$ with no division such that $\mathcal{P}$ has only two discrete components. In Section 8 we show that the entropy of such patterns is greater than or equal to the entropy of $\mathcal{Q}_n$. Collecting it all, we prove that Theorem A is true when we restrict ourselves to the family of patterns with no division. Finally, Section 9 is devoted to the proof of Theorem A.

3. Computation and properties of the topological entropy of $\mathcal{Q}_n$

We start this section by introducing a standing notation. Let n be a positive integer with $n \geq 3$. From now on, q_n will stand for the polynomial $q_n(x) = x^n - 2x - 1$. Observe that $q_n(1) = -2$ and that $q_n(x)$ is strictly increasing for $x \geq 1$. In consequence, $q_n(x)$ has a unique real root in $(1, \infty)$. From now on, such a real root will be denoted by λ_n. The following result summarizes some properties of the sequence $(\lambda_n)_{n \geq 3}$.

Proposition 3.1. *Let n be a positive integer with $n \geq 3$. Then:*

(a) $\lambda_{n+1} < \lambda_n$

(b) $(\lambda_n)^{1/k} > \lambda_{kn}$ *for every* $k \in \mathbb{N}$ *with* $k \geq 2$.

Proof. Statement (a) follows from $q_n(1) = q_{n+1}(1) = -2$ and $q_{n+1}(x) > q_n(x)$ for $x > 1$. Next we prove (b). By definition,

$$(\lambda_n)^n - 2\lambda_n - 1 = 0. \tag{1}$$

On the other hand, since $q_{kn}(x)$ is strictly increasing for $x \geq 1$ and $q_{kn}(\lambda_{kn}) = 0$, to prove (b) it is enough to show that $q_{kn}((\lambda_n)^{1/k}) > 0$. Observe that

$$q_{kn}((\lambda_n)^{1/k}) = ((\lambda_n)^{1/k})^{kn} - 2(\lambda_n)^{1/k} - 1 = (\lambda_n)^n - 2(\lambda_n)^{1/k} - 1 - 2\lambda_n + 2\lambda_n$$

which, by virtue of (1), is equal to $2(\lambda_n - (\lambda_n)^{1/k})$. This expression is positive, since $\lambda_n > 1$. □

Next we recall a powerful tool, first introduced in [14], to compute the spectral radius of a transition matrix in terms of the loops of its associated combinatorial directed graph. The corresponding notions are defined for nonnegative matrices, but here we adapt the definitions to the setting of binary matrices.

Let $\mathcal{G}$ be a combinatorial directed graph of m vertices and let M be its associated transition matrix. Let α be a path $I_0 \to I_1 \to \ldots \to I_k$ of length k in $\mathcal{G}$. The length of α will be denoted by $l(\alpha)$. A subset R of the set of vertices of $\mathcal{G}$ is called a *rome* if there is no loop outside R. That is, there is no loop $I_0 \to I_1 \to \ldots \to I_k \to I_0$ in $\mathcal{G}$ such that $\{I_i : \ 0 \leq i \leq k\} \cap R = \emptyset$. For a rome R, a path $I_0 \to I_1 \to \ldots \to I_k$ in $\mathcal{G}$ is called *simple* if $\{I_0, I_k\} \subset R$ and $I_i \notin R$ for $0 < i < k$. If $R = \{J_1, J_2, \ldots, J_s\}$ is a rome of M then we define an $s \times s$ matrix-valued real function M_R by setting $M_R = (a_{ij})$, where $a_{ij}(x) = \sum_\alpha x^{-l(\alpha)}$, where the summation is over all simple paths from J_i to J_j.

Theorem 3.2 (Theorem 1.7 of [14])**.** *If R is a rome of an $m \times m$ transition matrix M then the characteristic polynomial of M is* $(-1)^{m-s}x^m \det(M_R(x) - \mathrm{Id})$*, where* $s = |R|$ *and* Id *is the* $s \times s$ *identity matrix.*

By means of Theorem 3.2 we are able to compute the topological entropy of the patterns $\mathcal{Q}_n$.

Proposition 3.3. *Let n be a positive integer with $n \geq 3$. The topological entropy of the pattern $\mathcal{Q}_n$ is equal to* $\log(\lambda_n)$.

Proof. Let M be the Markov matrix of $\mathcal{Q}_n$. By Remark 2.3, the topological entropy of $\mathcal{Q}_n$ is equal to $\log \max\{\sigma(M), 1\}$. Since M is a nonnegative matrix, the Perron-Frobenius Theorem tells us that $\sigma(M)$ is an eigenvalue of M. Then, since λ_n is the largest real root of $q_n(x)$, it is enough to prove that the characteristic polynomial of M is $\pm q_n(x)$. Let (T, P, f) be the canonical model of $\mathcal{Q}_n$, with $P = \{x_i\}_{i=1}^n$ time labeled. Then, (T, P, f) satisfies properties (a–c) as stated in page 6.

Let y be the central point of T, which satisfies $f(y) = y$. Next we fix a labelling for the set of $(P \cup V(T))$-basic intervals. Set $I_i = [y, x_i]$ for $1 \leq i < n$ and $I_n = [x_1, x_n]$. Then, it is straightforward to check that the Markov graph of $\mathcal{Q}_n$ contains exactly the following loops: $I_1 \to I_2 \to \ldots \to I_{n-1} \to I_1$, $I_2 \to I_3 \to \ldots \to I_n \to I_2$ and $I_1 \to I_2 \to \ldots \to I_{n-1} \to I_1$. We call such loops α, β and γ respectively. Then, $\{I_{n-1}\}$ is a rome and α, β and γ are simple loops of respective lengths $n-1$, $n-1$ and n. In consequence, by Theorem 3.2, the characteristic polynomial of M is $(-1)^{n-1}x^n(x^{-n} + 2x^{-(n-1)} - 1) = (-1)^n q_n(x)$. □

4. Division for periodic patterns

In this section we introduce the notions of *rotational component* and *division* for a periodic pattern, that will play an important role in the proof of the main result of this paper. A *division* is a classical example of block structure, first introduced in [19] for interval periodic orbits and generalized in [9] in order to study the topological entropy and the set of periods for tree maps. For practical reasons, the definition of division given here is a slightly modified version of the definition introduced in [9]. As we will remark, both notions are completely equivalent.

Let us start with a simple topological remark on the structure of the trees induced by the discrete components of a pointed tree.

Remark 4.1. Let (T, P, f) be a model and let C be a discrete component of (T, P). Recall that, by definition, $\mathrm{En}(T) \subset P$. It easily follows that $\mathrm{Int}(\langle C \rangle)$ is a connected set and $\mathrm{Bd}(\langle C \rangle) = \mathrm{En}(\langle C \rangle) = C$.

Let (T, P, f) be a model of a periodic pattern $\mathcal{P}$. Let C be a discrete component of (T, P). We will say that a point $x \in C$ *escapes from* C if $f(x)$ does not belong to the connected component of $T \setminus \{x\}$ that intersects $\mathrm{Int}(\langle C \rangle)$. As an example, observe that the point 1 escapes from C_2 in the 16-periodic pattern shown in Figure 3. Any discrete component C of (T, P) without points escaping from it will be called a *rotational component* of $\mathcal{P}$. Clearly, this notion does not depend on the particular chosen model of $\mathcal{P}$. So, it makes sense to say that the pattern $\mathcal{P}$ *has a rotational component*. For instance, the component C_3 is a rotational component of the pattern shown in Figure 3.

The next result states that every periodic pattern has rotational components, and its proof provides an algorithm to find out rotational components of a given periodic pattern. See again Figure 3 for an example.

Lemma 4.2. *Any periodic pattern $\mathcal{P}$ has rotational components.*

Proof. Let (T, P, f) be any representative of $\mathcal{P}$. Set $T_1 := T$. Let C_1 be a non-rotational discrete component of (T, P). Then, C_1 contains an escaping point x_1. Let T_2 be the closure of the connected component of $T_1 \setminus \{x_1\}$ containing $f(x_1)$. Observe that $T_2 \subsetneq T_1$. On the other hand, $T_2 \cap P$ is a union of discrete components of (T, P). Note that C_1 is not a discrete component of (T, P) in T_2. Let C_2 be the only discrete component of (T, P) in T_2 such that $x_1 \in C_2$. Then, x_1 does not escape from C_2. If C_2 is a rotational component, then we are done. Otherwise, C_2 contains an escaping point x_3 and we can clearly iterate the above argument obtaining a subtree $T_3 \subsetneq T_2$ such that $\{C_1, C_2\}$ are not discrete components of (T, P) in T_3 and a discrete component C_3 of (T, P) in T_3. Proceeding in this way we obtain sequences of subtrees $T_1 \supsetneq T_2 \supsetneq \dots$ and discrete components $C_1, C_2, \dots$ of (T, P) such that

(a) C_i is contained in T_i, and
(b) $C_1, C_2, \dots, C_i$ are not contained in T_{i+1}.

From (a–b) it follows that C_i and C_j are different discrete components whenever $i \neq j$. Since the number of discrete components is finite, this construction stops after k steps for some $k \in \mathbb{N}$ with a component C_k that has to be rotational. □

The next technical lemma tells us that the convex hull of every rotational component contains a fixed point of the map.

Lemma 4.3. *Let (T, P, f) be a model of a periodic pattern. Let C be a rotational component of (T, P, f). Then, there are fixed points of f in $\mathrm{Int}(\langle C \rangle)$.*

Proof. By Remark 4.1, $\mathrm{Int}(\langle C \rangle)$ is connected and $\mathrm{Bd}(\langle C \rangle) = \mathrm{En}(\langle C \rangle) = C$. Let $r \colon T \longrightarrow \langle C \rangle$ be the natural retraction. Since $\langle C \rangle$ is a tree and $g := r \circ f\big|_{\langle C \rangle}$ is continuous, there exists a fixed point y of g. The definition of a rotational component clearly implies that C cannot contain fixed points of g. Hence, $y \in \mathrm{Int}(\langle C \rangle)$. If $f(y) \notin \mathrm{Int}(\langle C \rangle)$ then, by definition of a retraction, we would have that $y = g(y) \in \mathrm{Bd}(\langle C \rangle) = C$, a contradiction. Thus, $f(y) \in \mathrm{Int}(\langle C \rangle)$ and the

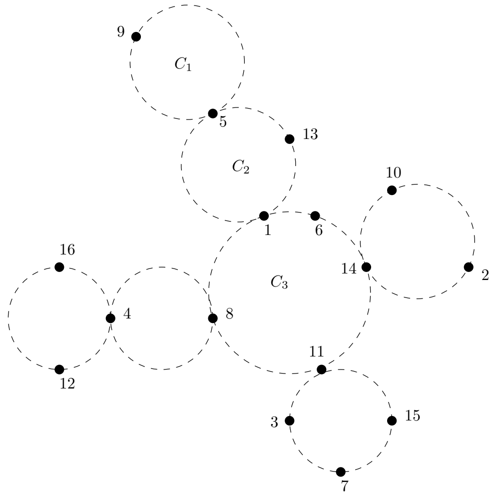

FIGURE 3. An example of application of Lemma 4.2 on a 16-periodic pattern $([T,P],[\theta])$. The points of P are time labeled with natural numbers. We start with $x_1 = 5$ and $C_1 = \{5,9\}$. The algorithm gives $x_2 = 1$, $C_2 = \{1,5,13\}$ and $C_3 = \{1,6,8,11,14\}$, which is a rotational component.

retraction r acts as the identity on $f(y)$. Summarizing, $y = r(f(y)) = f(y)$ and y is, in consequence, a fixed point of f. □

Next we introduce the notion of a *division*. Let (T,P,f) be a model of an n-periodic pattern $\mathcal{P}$. Let C be a discrete component of (T,P). By Remark 4.1, $\operatorname{Int}(\langle C\rangle)$ is connected. Let $Z_1, Z_2, \dots, Z_l$ be the connected components of $T \setminus \operatorname{Int}(\langle C\rangle)$. We will say that (T,P,f) has a *p-division with respect to C* (or simply a *p-division*) if there exists $\{M_1, M_2, \dots, M_p\}$ with $p \geq 2$, a partition of $T \setminus \operatorname{Int}(\langle C\rangle)$, such that each M_i is a union of some of the sets $Z_1, Z_2, \dots, Z_l$, $f(M_i \cap P) = M_{i+1} \cap P$ for $1 \leq i < p$ and $f(M_p \cap P) = M_1 \cap P$. In this case we will also say that the partition $\bigcup_{i=1}^{p} P_i$ *defines a p-division for* (T,P,f), where $P_i := M_i \cap P$. Again, this definition is independent of the particular chosen model of $\mathcal{P}$. In other words, if (S,Q,g) is another model of $\mathcal{P}$ and $\phi\colon P \longrightarrow Q$ is a bijection that preserves discrete components, then (S,Q,g) has a p-division with respect to $\phi(C)$. Therefore, it makes sense to say that the pattern $\mathcal{P}$ *has a p-division*.

The next result follows almost directly from the definition of a division. Its proof is left to the reader.

Lemma 4.4. *If a model (T,P,f) of a periodic pattern has a division with respect to a discrete component C, then C is rotational and (T,P) has no other rotational components.*

If a model (T,P,f) has a p-division with respect to a rotational component C then, by Lemma 4.3, $\operatorname{Int}(\langle C\rangle)$ contains a fixed point y of f. In this case, it is immediate to see that P has a p-division with respect to y according to the original definition (*division with respect to a fixed point*) given in [9]. Conversely, if a periodic orbit P of a tree map $f\colon T \longrightarrow T$ has a p-division with respect to a fixed point y of f according to that definition, then the model (T,P,f) has a p-division with respect to the unique discrete component C whose interior contains y.

Corollary C of [9] states that a tree map f has zero topological entropy if and only if, for every $k \in \mathbb{N}$, each periodic orbit of f^k has a division. Consequently, each periodic pattern with no division has positive entropy.

Finally, let us remark that Lemmas 4.2 and 4.4 can be used to quickly decide whether a given periodic pattern $\mathcal{P} = ([T, P], [\theta])$ has a division. To do it, use the procedure defined in the proof of Lemma 4.2 to detect a rotational component C. If $\mathcal{P}$ has a division then, by Lemma 4.4, C has to be the only rotational component of $\mathcal{P}$. Then, it is straightforward to check whether the points of P are separated with respect to the set $\mathrm{Int}(\langle C\rangle)$ in such a way that they are permuted by θ according to the definition of a division. If not, then $\mathcal{P}$ cannot have a division. For instance, consider again the 16-periodic pattern in Figure 3, for which $C_3 = \{1, 6, 8, 11, 14\}$ is a rotational component. Then, $P_1 := \{1, 5, 9, 13\}$, $P_2 := \{2, 6, 10, 14\}$, $P_3 := \{3, 7, 11, 15\}$ and $P_4 := \{4, 8, 12, 16\}$ are mapped cyclically by θ. Since every P_i is contained in a union of connected components of $T \setminus \mathrm{Int}(\langle C_3\rangle)$, $\mathcal{P}$ has a division.

5. Fine patterns

In this section we introduce a partial ordering in the set of all n-periodic patterns. It will be used as a tool to compare the entropy of two patterns in some particular cases.

Let (T, P, f) and (S, Q, g) be n-periodic models. A *time bijection* from Q to P is a bijective map $\varphi\colon Q \longrightarrow P$ such that $g\big|_Q = \varphi^{-1} \circ f \circ \varphi$. Observe that if we choose time labellings $P = \{x_i\}_{i=1}^n$ and $Q = \{z_i\}_{i=1}^n$ then there are n different time bijections $\varphi_i\colon Q \longrightarrow P$ (set $\varphi_i(z_1) := x_i$ and the remaining images are then decided by the condition $g\big|_Q = \varphi^{-1} \circ f \circ \varphi$). On the other hand, observe that given a time labelling $P = \{x_i\}_{i=1}^n$ and a time bijection $\varphi\colon Q \longrightarrow P$ there exists a unique time labelling $Q = \{y_i\}_{i=1}^n$ such that $\varphi(y_i) = x_i$ for $1 \le i \le n$. In this case we will say that P and Q are *consistently labeled* with respect to φ.

Let $\mathcal{P}$ and $\mathcal{Q}$ be n-periodic patterns. We will say that $\mathcal{Q}$ is *finer* than $\mathcal{P}$, denoted by $\mathcal{P} \le \mathcal{Q}$, if there exist respective n-periodic models (T, P, f) and (S, Q, g) and a time bijection $\varphi\colon Q \longrightarrow P$ such that for any discrete component C of (S, Q), $\varphi(C)$ is contained in a discrete component of (T, P). Is is easy to see that this definition is independent from the chosen models.

For an example, consider the 7-periodic patterns $\mathcal{P}$ and $\mathcal{Q}$ shown in Figure 4, with models labeled as in the upper part of the picture. To see that $\mathcal{P} \le \mathcal{Q}$, consider the time bijection $\varphi\colon Q \longrightarrow P$ such that $\varphi(z_1) = x_3$. Then, $\varphi(z_2) = x_4$, $\varphi(z_3) = x_5$, $\varphi(z_4) = x_6$, $\varphi(z_5) = x_7$, $\varphi(z_6) = x_1$ and $\varphi(z_7) = x_2$. The discrete components of (S, Q) are $C_1 = \{z_3, z_6, z_7\}$, $C_2 = \{z_1, z_4, z_5\}$ and $C_3 = \{z_2, z_5, z_6\}$. Then, $\varphi(C_1) = \{x_1, x_2, x_5\}$, which is a discrete component of (T, P). Analogously, $\varphi(C_2)$ and $\varphi(C_3)$ are contained in $\{x_1, x_3, x_4, x_6, x_7\}$, a discrete component of (T, P). Hence, $\mathcal{Q}$ is finer than $\mathcal{P}$. However, observe that P and Q are not consistently labeled with respect to φ. In order to have consistent time labellings for P and Q, we should consider a relabelling $Q = \{y_i\}_{i=1}^7$ such that $y_1 = z_6$, $y_2 = z_7$, $y_3 = z_1$, $y_4 = z_2$, $y_5 = z_3$, $y_6 = z_4$ and $y_7 = z_5$ (Figure 4, bottom).

Lemma 5.1. *The relation $\le$ is a partial ordering in the set of all n-periodic patterns.*

Proof. The reflexivity and transitivity of the relation $\le$ follow easily from its definition. Let us see that $\le$ is antisymmetric. Let $\mathcal{P}$ and $\mathcal{Q}$ be n-periodic patterns such that $\mathcal{P} \le \mathcal{Q} \le \mathcal{P}$. We have to show that $\mathcal{P} = \mathcal{Q}$. Assume that $n \ge 3$ to discard trivial cases.

Take models (T, P, f) and (S, Q, g) of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Since $\mathcal{P} \le \mathcal{Q} \le \mathcal{P}$, there are time bijections $\psi\colon P \longrightarrow Q$ and $\varphi\colon Q \longrightarrow P$ such that for any discrete component C of (T, P) (respectively, of (S, Q)), $\psi(C)$ (resp. $\varphi(C)$) is contained in a discrete component of (S, Q) (resp. (T, P)). We will prove that ψ preserves discrete components, that is, $\psi(C)$ is a discrete component of (S, Q) for every

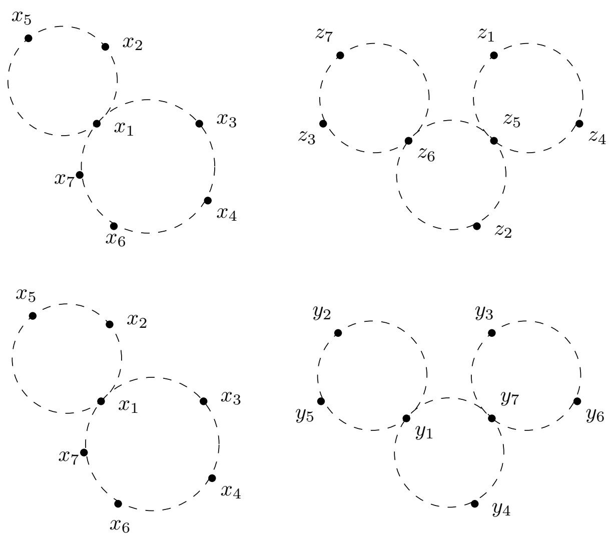

FIGURE 4. Top: two 7-periodic patterns $\mathcal{P}$ (left) and $\mathcal{Q}$ (right) of respective models (T, P, f) and (S, Q, g) with time labellings $P = \{x_i\}_{i=1}^7$ and $Q = \{z_i\}_{i=1}^7$. Here $\mathcal{Q}$ is finer than $\mathcal{P}$. Bottom: the same patterns consistently labeled with respect the time bijection φ such that $\varphi(z_1) = x_3$.

discrete component C of (T, P). Then, (T, P) and (S, Q) are equivalent pointed trees. Since, in addition, ψ is a time bijection, $f\big|_P$ and $g\big|_Q$ are equivalent maps. Summarizing, $([T, P], [f]) = ([S, Q], [g])$ and, in consequence, $\mathcal{P} = \mathcal{Q}$.

Note that $\varphi \circ \psi \colon P \longrightarrow P$ is a bijection. Moreover, $\varphi(\psi(C))$ is contained in a discrete component of (T, P) for any discrete component C of (T, P). We claim that in fact $\varphi(\psi(C))$ is a discrete component of (T, P). This is clearly true if C has maximum cardinality. Hence, $\varphi \circ \psi$ is a permutation in the set $\mathcal{X}$ of discrete components with maximum cardinality. Consequently, $\varphi(\psi(D))$ is not contained in a discrete component from $\mathcal{X}$, for every $D \notin \mathcal{X}$. So, $\varphi \circ \psi$ is a permutation in the set of discrete components not in $\mathcal{X}$ with maximum cardinality. Iterating this process, the claim follows.

Let C be a discrete component of (S, Q) and let D be the only discrete component of (T, P) such that $\psi(C) \subset D$. Since $\varphi(\psi(C))$ is a discrete component and $\varphi(D) \supset \varphi(\psi(C))$ is contained in a discrete component, it follows that $\varphi(D) = \varphi(\psi(C))$. Hence, $D = \psi(C)$. □

For practical purposes, it is convenient to give an alternative definition of *fine pattern*. Consider again the example in Figure 4 (bottom). One can think that the pattern $\mathcal{P}$ has been obtained from $\mathcal{Q}$ by "pulling out" the point y_7 in such a way that the two discrete components $\{y_1, y_4, y_7\}$ and $\{y_3, y_6, y_7\}$, adjacent at y_7, are joined together in a unique discrete component $\{x_1, x_3, x_4, x_6, x_7\}$. So, at least intuitively, a pattern $\mathcal{Q}$ is finer than a pattern $\mathcal{P}$ when one can iteratively join together several adjacent discrete components of $\mathcal{Q}$ to get finally the discrete components of $\mathcal{P}$. Let us introduce the precise notions.

Let (T, P, f) be a model of a pattern $\mathcal{P}$. We recall that two discrete components of (T, P) are either disjoint or intersect at a single point of P. Two discrete components C, D of (T, P) will be called *adjacent at* $x \in P$ (or simply *adjacent*) if $C \cap D = \{x\}$. A point $z \in P$ will be called *inner* if z belongs to $k \geq 2$ discrete components of (T, P), all being pairwise adjacent at z. In this case, $\mathrm{Val}_T(z) = k$.

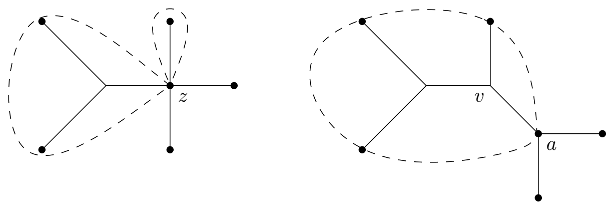

FIGURE 5. Replacing a point z by an interval $[a, v]$ in order to join together two discrete components.

Moreover, if (T', P', f') is another model of the same pattern and $\phi\colon P \longrightarrow P'$ is a bijection which preserves discrete components, then $\mathrm{Val}_{T'}(\phi(z)) = k$.

Let (T, P, f) and (S, Q, g) be n-periodic models. Let $\mathcal{J}$ be the set of discrete components of (S, Q) and let $z \in Q$ be an inner point. Let W be the set of $k \geq 2$ discrete components of (S, Q) which are adjacent at z. We will say that (T, P, f) is a *pull out* of (S, Q, g) if there exists a time bijection $\varphi\colon Q \longrightarrow P$ and a nonempty subset $Y \subset W$ of cardinality $m \leq k$ such that $\bigcup_{C \in Y} \varphi(C)$ is a discrete component of (T, P), and the rest of discrete components of (T, P) are $\{\varphi(C) : C \in \mathcal{J} \setminus Y\}$. This corresponds to joining together the m discrete components in Y and keeping intact the remaining discrete components of (S, Q). Note that in this case $\mathrm{Val}_T(\varphi(z)) = k - m + 1$. At a topological level, one can obtain T from S by expanding the point z to an interval $[a, v]$ in such a way that (see Figure 5 for an example):

(a) $\mathrm{Val}_T(a) = k - m + 1$
(b) $\mathrm{Val}_T(v) = m + 1$
(c) $S \setminus \{z\}$ and $T \setminus [a, v]$ are homeomorphic.

In the above definition, if $m = k$ (equivalently, $Y = W$) then $\varphi(z) \in \mathrm{En}(T)$ and we will say that (T, P, f) is a *complete pull out* of (S, Q, g). When we want to specify the time bijection φ, the inner point z and the set of discrete components joined together we will say that (T, P, f) is a *φ-pull out of* (S, Q, g) *with respect to* z *and* Y. Of course, when $Y = W$ we will also say that (T, P, f) is a *complete φ-pull out of* (S, Q, g) *with respect to* z. At the level of patterns, if $\mathcal{P} = ([T, P], [f\big|_P])$ and $\mathcal{Q} = ([S, Q], [g\big|_Q])$ we will say that $\mathcal{P}$ is a *pull out* of $\mathcal{Q}$. It is clear that this definition does not depend on the chosen models. Observe also that if in the above definitions we take $m = 1$ then $\mathcal{P} = \mathcal{Q}$. Of course, if $\mathcal{P}$ is a pull out of $\mathcal{Q}$ then $\mathcal{P} \leq \mathcal{Q}$.

The next result provides an alternative definition, in terms of pull outs, of the fact that a pattern is finer than another one. See Figure 6 for an example.

Theorem 5.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be n-periodic patterns. Then, $\mathcal{P} \leq \mathcal{Q}$ if and only if there exists a sequence of n-periodic patterns $\{\mathcal{P}_i\}_{i=0}^{l}$, with $l \geq 1$, such that $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_l = \mathcal{Q}$, and $\mathcal{P}_i$ is a pull out of $\mathcal{P}_{i+1}$ for $0 \leq i < l$.*

Proof. Observe that if $\mathcal{P}_i$ is a pull out of $\mathcal{P}_{i+1}$ then $\mathcal{P}_i \leq \mathcal{P}_{i+1}$ by definition. Then, the "if" part of the theorem follows from Lemma 5.1.

Let us prove the "only if" part. The result is trivial when $\mathcal{P} = \mathcal{Q}$. Assume that $\mathcal{P} \neq \mathcal{Q}$. Let (T, P, f) and (S, Q, g) be models of $\mathcal{P}$ and $\mathcal{Q}$ respectively, and let $\varphi\colon Q \longrightarrow P$ be a time bijection such that for any discrete component C of (S, Q), $\varphi(C)$ is contained in a discrete component of (T, P). In particular,

(2) if $\{a, b\}$ is a basic path of (T, P) then $\{\varphi(a), \varphi(b)\}$ is a basic path of (S, Q).

Since $\mathcal{P} \neq \mathcal{Q}$, there exists at least one discrete component C of (S, Q) such that $\varphi(C) \subsetneq D$ for some discrete component D of (T, P). Observe that there must exist

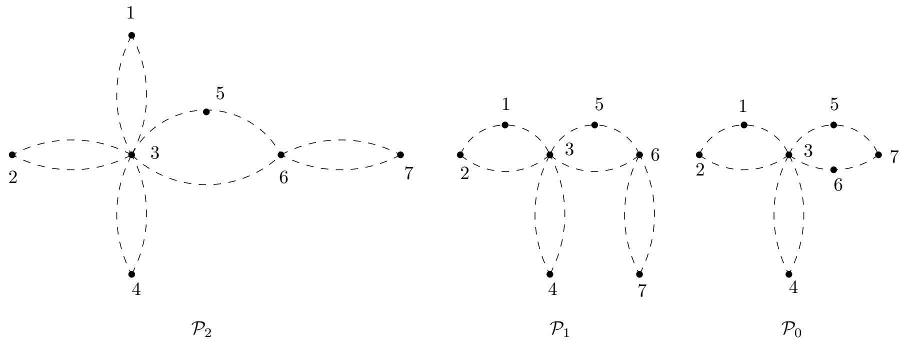

FIGURE 6. $\mathcal{P}_1$ is a pull out of $\mathcal{P}_2$ with respect the inner point 3 and the discrete components $\{2, 3\}$ and $\{1, 3\}$. $\mathcal{P}_0$ is a complete pull out of $\mathcal{P}_1$ with respect the inner point 6 and the discrete components $\{6, 7\}$ and $\{3, 5, 6\}$. Therefore, $\mathcal{P}_0 \leq \mathcal{P}_1 \leq \mathcal{P}_2$. The consistent time labellings have been specified with natural numbers.

at least one discrete component of (S, Q) adjacent to C. Otherwise, $\mathcal{Q}$ would be the trivial n-periodic pattern and, in consequence, $\mathcal{P} = \mathcal{Q}$.

Now we claim that D contains $\varphi(x)$ for some point x which belongs to $C' \setminus C$ for some discrete component C' adjacent to C. Let us prove the claim. Assume by way of contradiction that $\varphi(C' \setminus C) \cap D = \emptyset$ for all the discrete components C' adjacent to C. Take $z \in D \setminus \varphi(C)$. Since we are assuming that $\varphi^{-1}(z)$ cannot belong to any discrete component adjacent to C, there exists a sequence $\{a_i\}_{i=1}^k \subset Q$ with $k \geq 3$ such that $a_1 \in C$, $a_2 \notin C$, $a_k = \varphi^{-1}(z)$ and $(a_i, a_{i+1}) \cap Q = \emptyset$ for $1 \leq i < k$. Clearly a_2 belongs to a discrete component adjacent to C. Hence, by assumption, $\varphi(a_2) \notin D$. On the other hand, since $\varphi(a_k) = z \in D$, there is a minimum $m \in \{3, 4, \ldots, k\}$ such that $\varphi(a_m) \in D$. Let X be the connected component of $T \setminus \{\varphi(a_1)\}$ containing $\varphi(a_2)$. By (2), $(\varphi(a_1), \varphi(a_2)) \cap P = \emptyset$. Since $\varphi(a_1) \in D$ and $\varphi(a_2) \notin D$, it follows that

$$\langle D \rangle_S \subset T \setminus X. \tag{3}$$

Using (2) for each pair $\{a_i, a_{i+1}\}$ with $2 \leq i \leq m-2$ and the fact $\varphi(a_i) \notin D$ we easily get that $\varphi(a_i) \in X$ for $2 \leq i \leq m-1$. But, since $\varphi(a_m) \in D \setminus \{\varphi(a_1)\}$, from (3) we get that $\varphi(a_1) \in (\varphi(a_{m-1}), \varphi(a_m))$, in contradiction with (2). So the claim is proved.

By the previous claim, there exists a discrete component C' adjacent to C and a point $x \in C' \setminus C$ such that $\varphi(x) \in D$. Let z be the inner point in Q such that $\{z\} = C \cap C'$. Clearly, $\varphi(z) \in D$. Recall that, by hypothesis, $\varphi(C')$ is contained in a single discrete component of (T, P). Since $\{\varphi(x), \varphi(z)\} \subset D$, it follows that $\varphi(C') \subset D$. Thus, $\varphi(C) \cup \varphi(C') \subseteq D$.

Let $\mathcal{J}$ be the set of discrete components of (S, Q). Now consider a pull out $(\overline{S}, \overline{Q}, \overline{g})$ of (S, Q, g) with respect to z and $W := \{C, C'\}$. That is, there exists a time bijection $\phi \colon Q \longrightarrow \overline{Q}$ such that $\phi(C) \cup \phi(C')$ is a discrete component of $(\overline{S}, \overline{Q})$ and the remaining discrete components are $\{\phi(E) : E \in \mathcal{J} \setminus \{C, C'\}\}$. Set $\overline{\mathcal{Q}} := ([\overline{S}, \overline{Q}], [\overline{g}|_{\overline{Q}}])$. It is clear that $\varphi \circ \phi^{-1}$ is a time bijection from $\overline{Q}$ to P and that, for every discrete component E of $(\overline{S}, \overline{Q})$, $\varphi \circ \phi^{-1}(E)$ is contained in a discrete component of (T, P). Therefore, $\mathcal{P} \leq \overline{\mathcal{Q}}$. If $\overline{\mathcal{Q}} \neq \mathcal{P}$, then we can repeat the above construction replacing (S, Q, g) by $(\overline{S}, \overline{Q}, \overline{g})$ and φ by $\varphi \circ \phi^{-1}$. Since the number of discrete components of $(\overline{S}, \overline{Q})$ is one less than that of (S, Q), it is clear that this procedure can be iterated in a finite number of steps to get the desired result. $\square$

Going back again to Figure 4 (bottom), it is reasonable to expect that the entropy of $\mathcal{Q}$ is larger than that of $\mathcal{P}$, since the point y_7, which separates several points in $\mathcal{Q}$, has been pulled out in $\mathcal{P}$ to become the endpoint x_7. So, the coverings forced by the inner character of the point y_7 in $\mathcal{Q}$ are expected to be lost in $\mathcal{P}$. So, at least intuitively, if $\mathcal{Q}$ is finer than $\mathcal{P}$ then the entropy of $\mathcal{Q}$ should be greater than or equal to the entropy of $\mathcal{P}$. In other words, the topological entropy should respect the partial ordering $\leq$ in the set of all n-periodic patterns. This turns out to be true and is in fact the main result of this section.

Theorem 5.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be n-periodic patterns. If $\mathcal{P} \leq \mathcal{Q}$ then $h(\mathcal{P}) \leq h(\mathcal{Q})$.*

Proof. By Theorem 5.2, it is enough to prove the result when $\mathcal{P}$ is a pull out of $\mathcal{Q}$. Let (T, P, f) and (S, Q, g) be the canonical models of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Assume that, for a time bijection $\varphi\colon Q \longrightarrow P$, (T, P, f) is a φ-pull out of (S, Q, g) with respect to a point $z \in Q$ and a subset Y of the set W of all discrete components of (S, Q) adjacent at z.

If $|Y| = 1$ then $\mathcal{P} = \mathcal{Q}$ and the theorem follows trivially. So, we can assume that $|Y| \geq 2$.

To prove the theorem we will proceed as follows. From (S, Q, g) we will obtain a new model (S', Q', g') of the pattern $\mathcal{P}$ such that the map g' is $(Q' \cup V(S'))-$monotone. This model will be constructed in such a way that $h(g') = h(\mathcal{Q})$, as we will show at the end of the proof. Also, since $([S', Q'], [g']) = \mathcal{P}$, from the definition of $h(\mathcal{P})$ it follows that $h(\mathcal{P}) \leq h(g') = h(\mathcal{Q})$.

Next we start the construction of the model (S', Q', g').

Set $m := |Y| \geq 2$, $k := |W|$ and take a labelling $\{C_1, C_2, \dots, C_l\}$ of the discrete components of (S, Q) such that $W = \{C_1, C_2, \dots, C_k\}$ and $Y = \{C_1, C_2, \dots, C_m\}$.

Let $P = \{x_i\}_{i=1}^n$ and $Q = \{z_i\}_{i=1}^n$ be consistent time labellings with respect to φ. Recall that, by definition of a time bijection,

$$g\big|_Q = \varphi^{-1} \circ f \circ \varphi. \tag{4}$$

Consider a tree S' obtained from S by expanding the point z to an interval $[a, v]$ such that (see Figure 5):

(a) $\mathrm{Val}_{S'}(a) = k - m + 1$
(b) $\mathrm{Val}_{S'}(v) = m + 1$
(c) There exists a homeomorphism $\phi\colon S \setminus \{z\} \longrightarrow S' \setminus [a, v]$.

Set $Q' = \phi(Q \setminus \{z\}) \cup \{a\}$ and observe that $|Q'| = |Q| = |P| = n$. Consider the bijection $\mu\colon Q \longrightarrow Q'$ defined by

$$\mu(w) = \begin{cases} \phi(w) \text{ if } w \neq z \\ a \text{ if } w = z. \end{cases}$$

From (a–c) above it follows that $\mu(C_1 \cup C_2 \cup \ldots \cup C_m)$ is a discrete component of (S', Q'), and that the rest of discrete components of (S', Q') are $\mu(C_i)$, for $m < i \leq l$. From the definition of a pull out, this implies that $\varphi \circ \mu^{-1}$ is a bijection from Q' to P which preserves discrete components. In consequence,

$$(T, P) \text{ and } (S', Q') \text{ are equivalent pointed trees via } \varphi \circ \mu^{-1}. \tag{5}$$

Set $\theta' = \mu \circ g \circ \mu^{-1}\big|_{Q'}$. Since Q is an n-periodic orbit of g, Q' is an n-periodic orbit of θ'. Moreover, by (4), $\theta' = \mu \circ \varphi^{-1} \circ f \circ \varphi \circ \mu^{-1}\big|_{Q'}$. In other words, $[\theta'] = [f\big|_P]$. Together with (5), this yields

$$([S', Q'], [\theta']) = \mathcal{P}. \tag{6}$$

Set $Q'_V := Q' \cup V(S')$. Observe that $v \in V(S')$ because, by (b), $\mathrm{Val}_{S'}(v) \geq 3$. Next we define a map g' on Q'_V as follows:

(A) $g'\big|_{Q'} = \theta'\big|_{Q'}$
(B) $g'(v) = \theta'(a)$
(C) $g'(w) = \phi \circ g \circ \phi^{-1}(w)$ for any $w \in V(S') \setminus (Q' \cup \{v\})$ such that $g(\phi^{-1}(w)) \neq z$
(D) $g'(w) = a$ for any $w \in V(S') \setminus (Q' \cup \{v\})$ such that $g(\phi^{-1}(w)) = z$

Now we take an obvious piecewise monotone extension of g' to the whole S', obtaining a Q'_V-monotone map $g' \colon S' \longrightarrow S'$. From (A) and (6), we get that g' exhibits $\mathcal{P}$ over Q'. Hence, from the definition of $h(\mathcal{P})$,

$$h(\mathcal{P}) \leq h(g'). \tag{7}$$

Let M be the Markov matrix of $\mathcal{Q}$. By Remark 2.3,

$$h(\mathcal{Q}) = \log \max\{\sigma(M), 1\}. \tag{8}$$

Let $\{I_i\}_{i=1}^{s}$ be the set of $(Q \cup V(S))$-basic intervals (so that M is an $s \times s$ matrix). By construction (that is, by (a–c) above), there are $s+1$ Q'_V-basic intervals in S'. We take a labelling $\{J_i\}_{i=1}^{s+1}$ of the set of Q'_V-basic intervals in such a way that:

- $J_{s+1} = [v, a]$
- If $I_i = [x, y]$ and I_i is contained in a connected component of $S \setminus \{z\}$, then $J_i = [\phi(x), \phi(y)]$
- If $I_i = [x, z]$ and I_i is contained in $\langle C_j \rangle_S$ for some $1 \leq j \leq m$, then $J_i = [\phi(x), v]$
- If $I_i = [x, z]$ and I_i is contained in $\langle C_j \rangle_S$ for some $m < j \leq k$, then $J_i = [\phi(x), a]$.

Let M' be the Markov matrix of g' with respect to this labelling. Since g' is Q'_V-monotone, by Remark 2.2 we have that

$$h(g') = \log \max\{\sigma(M'), 1\}. \tag{9}$$

From the definitions it easily follows that, for any $1 \leq i \leq s$, if I_i g-covers I_j then J_i g'-covers J_j and, perhaps, J_{s+1}. Moreover, J_{s+1} does not cover any Q'_V-basic interval. This implies that all entries in the $(s+1)$-th row of M' are zero, and that after deleting the $(s+1)$-th row and column from M' we get the matrix M. In consequence, $\sigma(M') = \sigma(M)$. Then, the theorem follows from (7), (8) and (9). □

6. Centered patterns. Reduction to a strongly centered pattern

In this section we prove that for any n-periodic pattern $\mathcal{Q}$ with no division there exists another n-periodic pattern $\mathcal{P}$ with no division such that $\mathcal{Q}$ is finer than $\mathcal{P}$ (so, $h(\mathcal{P}) \leq h(\mathcal{Q})$ by Theorem 5.3) and the distribution of the discrete components of $\mathcal{P}$ is particularly simple.

Let (T, P, f) be a model of a periodic pattern $\mathcal{P}$. We will say that $\mathcal{P}$ is *centered* if there exists a rotational discrete component C of (T, P) such that all the inner points of P belong to C. If in addition each inner point belongs exactly to two discrete components then we will say that $\mathcal{P}$ is *strongly centered.* Of course, these definitions do not depend on the particular choice of the model. For an example, consider the patterns $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ in Figure 6. The discrete component $\{3, 4\}$ is rotational for the patterns $\mathcal{P}_1$ and $\mathcal{P}_2$. However, it does not contain the point 6, which is inner. Both patterns have also another rotational component, $\{6, 7\}$, which in both cases does not contain the inner point 3. Hence, $\mathcal{P}_1$ and $\mathcal{P}_2$ are not centered patterns. On the other hand, $\{3, 4\}$ is the only rotational component of $\mathcal{P}_0$. This component contains 3, which is the only inner point. In consequence, $\mathcal{P}_0$ is a centered pattern. However, it is not strongly centered, since 3 belongs to three discrete components. The 6-periodic pattern $\mathcal{Q}$ in Figure 7 is an example of a strongly centered pattern.

Now we are ready to state the main result of this section.

Theorem 6.1. *Let $\mathcal{Q}$ be an n-periodic pattern with no division. Then, there exists a strongly centered n-periodic pattern $\mathcal{P}$ with no division such that $\mathcal{P} \le \mathcal{Q}$.*

Proof. We will proceed in two steps: first we will construct a centered pattern $\mathcal{R}$ with no division such that $\mathcal{R} \le \mathcal{Q}$, and then we will construct a strongly centered pattern $\mathcal{P}$ with no division such that $\mathcal{P} \le \mathcal{R}$.

First step: construction of $\mathcal{R}$. Let (S, Q, g) be a model of $\mathcal{Q}$. By Lemma 4.2, there exists a rotational discrete component D of $\mathcal{Q}$. If all inner points are in D, then $\mathcal{Q}$ is centered and we are done simply by taking $\mathcal{R} = \mathcal{Q}$.

Let $z \in Q$ be an inner point of $\mathcal{Q}$ such that $z \notin D$. Let $\mathcal{J}$ be the set of discrete components of (S, Q) and let $W \subset \mathcal{J}$ be the set of all discrete components adjacent at z. In particular, $D \notin W$. Let $\mathcal{Q}'$ be a complete pull out of $\mathcal{Q}$ with respect to z. Take the canonical model $(S'Q', g')$ of $\mathcal{Q}'$ and a time bijection $\varphi\colon Q \longrightarrow Q'$ such that (S', Q', g') is a complete φ-pull out of (S, Q, g) with respect to z. By definition, $\bigcup_{C \in W} \varphi(C)$ is a discrete component of $\mathcal{Q}'$, and the rest of discrete components are $\{\varphi(C) : C \in \mathcal{D} \setminus W\}$. In particular, $\varphi(D)$ is a discrete component of $\mathcal{Q}'$. Moreover, since we have joined together all the components adjacent to z, it follows that $\varphi(z)$ is an endpoint of S'. Thus, $\varphi(z)$ is not an inner point. On the other hand, the φ-image of every inner point of $\mathcal{Q}$ different from z is an inner point of $\mathcal{Q}'$.

Let us see that $\varphi(D)$ is a rotational component of $\mathcal{Q}'$. Let $D = \{a_1, a_2, \dots, a_l\}$. For every $1 \le i \le l$, let Z_i be the connected component of $S \setminus \operatorname{Int}(\langle D \rangle_S)$ containing a_i and let Z'_i be the connected component of $S' \setminus \operatorname{Int}(\langle \varphi(D) \rangle_{S'})$ containing $\varphi(a_i)$. From the definition of a pull out if follows that

$$\varphi(Z_i \cap Q) = Z'_i \cap Q' \text{ for every } 1 \le i \le l. \tag{10}$$

Since a_i does not escape from D, $g(a_i) \in Z_j$ from some $j \neq i$. On the other hand, $g'(\varphi(a_i)) = \varphi(g(a_i))$. Then, by (10), $g'(\varphi(a_i)) \in Z'_j$. Therefore, $\varphi(D)$ does not contain escaping points and is, in consequence, a rotational component.

Next let us show that $\mathcal{Q}'$ has no division. Since $\varphi(D)$ is a rotational component, by Lemma 4.4 it is enough to see that (S', Q', g') has not a division with respect to $\varphi(D)$. Assume the contrary. That is, for some $p \ge 2$ there is a partition $\{M'_1, M'_2, \dots, M'_p\}$ of $S' \setminus \operatorname{Int}(\langle \varphi(D) \rangle_{S'})$ such that each M'_i is a union of some of the sets $Z'_1, Z'_2, \dots, Z'_l$, $g'(M'_i \cap Q') = M'_{i+1} \cap Q'$ for $1 \le i < p$ and $g'(M'_p \cap Q') = M'_1 \cap Q'$. Then, (10) and the fact that $\varphi \circ g\big|_Q = g'\big|_{Q'} \circ \varphi$ imply that (S, Q, g) has a p-division with respect to D, a contradiction.

Summarizing, we have proved that there is an n-periodic pattern $\mathcal{Q}'$ such that:

- $\mathcal{Q}'$ is a pull out of $\mathcal{Q}$ (in particular, $\mathcal{Q}' \le \mathcal{Q}$)
- $\varphi(D)$ is a rotational component of $\mathcal{Q}'$
- The number of inner points of $\mathcal{Q}'$ which are not in D is exactly that of $\mathcal{Q}$ minus one
- $\mathcal{Q}'$ has no division.

It is clear that this procedure can be iterated as many times as the number of inner points of $\mathcal{Q}$ which are not in D, in order to finally obtain the prescribed pattern $\mathcal{R}$.

Second step: construction of $\mathcal{P}$. Let (S, Q, g) be a model of $\mathcal{R}$. Since $\mathcal{R}$ is centered, there exists a rotational discrete component D of $\mathcal{R}$ containing all inner points. If all such points belong exactly to two discrete components, then $\mathcal{R}$ is strongly centered and we are done simply by taking $\mathcal{P} = \mathcal{R}$.

Assume that there exists an inner point $z \in D$ and $m \ge 2$ such that there are $m + 1$ discrete components of (S, Q) adjacent to z. Let $\{D, C_1, C_2, \dots, C_m\}$ be the set of such discrete components. As in the first step of the proof, we can construct a pull out $\mathcal{Q}'$ of $\mathcal{Q}$ with respect to z and $\{C_1, C_2, \dots, C_m\}$. If (S', Q', g') is any model of $\mathcal{Q}'$ and $\varphi\colon Q \longrightarrow Q'$ is a time bijection such that (S', Q', g') is a φ-pull out of (S, Q, g), then $\varphi(z')$ is an inner point of $\varphi(D)$ which belongs to the two discrete components $\varphi(D)$ and $\bigcup_{i=1}^{m} \varphi(C_i)$. Moreover, the same arguments used in the first step of the proof show that $\varphi(D)$ is a rotational discrete component of $\mathcal{Q}'$ and that $\mathcal{Q}'$ has no division.

It is clear that this procedure can be iterated finitely many times in order to finally obtain the prescribed pattern $\mathcal{P}$. □

7. Reduction to a pattern with two discrete components

In this section we will prove that, when $n = m^k$ for some m prime, given any n-periodic strongly centered pattern $\mathcal{Q}$ with no division there exists an n-periodic pattern $\mathcal{P}$ with two discrete components such that $\mathcal{Q}$ is finer than $\mathcal{P}$ (so, $h(\mathcal{P}) \leq h(\mathcal{Q})$ by Theorem 5.3) and $\mathcal{P}$ has no division. For a counterexample of this result when n is not a power of a prime, consider the 6-periodic pattern $\mathcal{Q}$ shown in Figure 7. There are only three 6-periodic patterns smaller than $\mathcal{Q}$ with respect to the partial ordering $\leq$: the trivial pattern with only one discrete component, which has obviously a division, and the patterns $\mathcal{Q}'$ and $\mathcal{Q}''$ shown in Figure 7, having also a division in each case.

We start this section with an intuitively clear lemma.

Lemma 7.1. *Let $\mathcal{Q}$ be a strongly centered n-periodic pattern and let $\mathcal{Q}'$ be a complete pull out of $\mathcal{Q}$ with respect to an inner point. Then, $\mathcal{Q}'$ is strongly centered.*

Proof. Let (S, Q, g) and (S', Q', g') be models of $\mathcal{Q}$ and $\mathcal{Q}'$ respectively. Let $x \in Q$ be an inner point such that (S', Q', g') is a complete φ-pull out of (S, Q, g) with respect to x for some time bijection $\varphi\colon Q \longrightarrow Q'$. Since $\mathcal{Q}$ is strongly centered, x belongs exactly to two adjacent discrete components C, D of (S, Q) such that D is rotational. By definition of a complete pull out, $\varphi(C \cup D)$ is a discrete component of (S', Q'). Moreover, from the definition of a pull out and since x is the only inner point in C, any $y \in \varphi(C \cup D)$ verifies one of the following statements:

(a) $y = \varphi(x)$ and $\mathrm{Val}_{S'}(y) = 1$
(b) $y \in \varphi(C \setminus \{x\})$ and $\mathrm{Val}_{S'}(y) = 1$
(c) $y \in \varphi(D \setminus \{x\})$ and $\mathrm{Val}_{S'}(y) = \mathrm{Val}_S(\varphi^{-1}(y))$.

Since $\mathcal{Q}$ is strongly centered, all points in D have valence 1 or 2. Then, from (a–c) it follows that all points in $\varphi(C \cup D)$ have valence 1 or 2. Also, since a pull out does not increase the valence of any point, each point in $Q' \setminus \varphi(C \cup D)$ has valence 1. Hence, to prove the lemma it is enough to show that $\varphi(C \cup D)$ is a rotational component. Take any $y \in \varphi(C \cup D)$. We have to see that y is not an escaping point.

Since an endpoint cannot be an escaping point, according to (a–c) we are left with the case $y = \varphi(z)$ with $z \in D \setminus \{x\}$ and $\mathrm{Val}_S(z) = 2$. Let E be the only discrete component of (S, Q) different from D containing z. Observe that $\varphi(E)$ and $\varphi(C \cup D)$ are the two discrete components of (S', Q') containing y. Since D is rotational, z is not an escaping point and $g(z) \notin E$. Then, $g'(y) = g'(\varphi(z)) = \varphi(g(z)) \notin \varphi(E)$. In consequence, y is not an escaping point. □

The following technical lemma will be essential to obtain the main result of this section.

Lemma 7.2. *Let (S, Q, g) be a model of a strongly centered n-periodic pattern with no division and at least two inner points x, y. Let (S', Q', g') and (S'', Q'', g'') be complete pull outs of (S, Q, g) with respect to x and y respectively. Assume that*

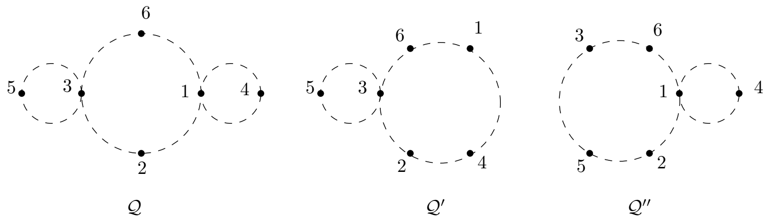

FIGURE 7. The time labellings have been specified with natural numbers. $\mathcal{Q}$ is strongly centered with no division. The only rotational component is $\{1,2,3,6\}$. The patterns $\mathcal{Q}'$ and $\mathcal{Q}''$ have been obtained after a complete pull out with respect the inner points 1 and 3, respectively. $\mathcal{Q}'$ has a 2-division defined by $\{1,3,5\}\cup\{2,4,6\}$ and $\mathcal{Q}''$ has a 3-division defined by $\{1,4\}\cup\{2,5\}\cup\{3,6\}$.

both models have divisions defined respectively by partitions $Q' = Q'_1 \cup Q'_2 \cup \ldots \cup Q'_p$ and $Q'' = Q''_1 \cup Q''_2 \cup \ldots \cup Q''_q$ with $p \le q$. Then, q is not a multiple of p.

Proof. Let $\varphi'\colon Q \longrightarrow Q'$, $\varphi''\colon Q \longrightarrow Q''$ be the time bijections such that (S',Q',g') is a φ'-pull out of (S,Q,g) and (S'',Q'',g'') is a φ''-pull out of (S,Q,g). Let $Q = \{x_i\}_{i=1}^n$, $Q' = \{x'_i\}_{i=1}^n$ and $Q'' = \{x''_i\}_{i=1}^n$ be consistent time labellings with respect to φ' and φ''. We can assume without loss of generality that $x = x_1$, $x'_1 \in Q'_1$ and $x''_1 \in Q''_1$.

Assume by way of contradiction that $q = lp$ for some $l \ge 1$. By definition of a division, we have that $g'(Q'_i) = Q'_{i+1 \bmod p}$ and $g''(Q''_i) = Q''_{i+1 \bmod q}$. Since $x'_1 \in Q'_1$ and $x''_1 \in Q''_1$, the fact that $q = lp$ implies that $Q'_i = \{x'_{i+jp}\}_{j=0}^{\frac{n}{p}-1}$ for $1 \le i \le p$ and $Q''_i = \{x''_{i+jlp}\}_{j=0}^{\frac{n}{lp}-1}$ for $1 \le i \le lp$. In consequence,

$$(\varphi')^{-1}(Q'_i) = \bigcup_{j=0}^{l-1} (\varphi'')^{-1}(Q''_{i+jp}) \text{ for } 1 \le i \le p.$$

In particular,

$$(\varphi')^{-1}(Q'_1) \supset (\varphi'')^{-1}(Q''_1). \tag{11}$$

On the other hand, since $g'\big|_{Q'} = \varphi' \circ g\big|_Q \circ (\varphi')^{-1}$, then

$$\begin{array}{rcl} g((\varphi')^{-1}(Q'_i)) &=& (\varphi')^{-1}(Q'_{i+1}) \text{ for } 1 \le i < p, \\ g((\varphi')^{-1}(Q'_p)) &=& (\varphi')^{-1}(Q'_1). \end{array} \tag{12}$$

Since (S,Q,g) is strongly centered, there exists a rotational discrete component D of (S,Q) containing x and y. Moreover, there exists exactly two discrete components C_x and C_y, different from D, such that $x \in C_x$ and $y \in C_y$. Set $Z = \operatorname{Int}(\langle D\rangle_S)$, $Z' = \operatorname{Int}(\langle \varphi'(C_x \cup D)\rangle_{S'})$ and $Z'' = \operatorname{Int}(\langle \varphi''(C_y \cup D)\rangle_{S''})$. For each inner point w in Q (respectively, in Q' and Q''), let Z_w (respectively, Z'_w and Z''_w) be the connected component of $S \setminus Z$ (respectively, of $S' \setminus Z'$ and of $S'' \setminus Z''$) containing w. Observe that $Z_x \cap Q = C_x$ and $Z_y \cap Q = C_y$. Moreover, since (S'',Q'',g'') is a complete pull out of (S,Q,g) with respect to y, it follows that

$$(\varphi'')^{-1}(Z''_{x''_1} \cap Q'') = Z_{x_1} \cap Q. \tag{13}$$

Note that Q''_1 is a union of sets of the form $Z''_w \cap Q''$, because the partition $\{Q''_i\}$ defines a division for (S'',Q'',g''). Since $x''_1 \in Q''_1$, it follows that $Q''_1 \supset Z''_{x''_1} \cap Q''$. Therefore, $(\varphi'')^{-1}(Q''_1) \supset (\varphi'')^{-1}(Z''_{x''_1} \cap Q'')$, which is equal to $Z_{x_1} \cap Q$ by (13). Together with (11), this yields

$$(\varphi')^{-1}(Q'_1) \supset Z_{x_1} \cap Q. \tag{14}$$

Now we claim that the partition $\{(\varphi')^{-1}(Q'_i)\}_{i=1}^{p}$ defines a p-division for (S, Q, g). This claim is in contradiction with the fact that (S, Q, g) has no division, thus proving the proposition.

In view of (12), to prove the claim it is enough to show that each set $(\varphi')^{-1}(Q'_i)$ is a union of sets of the form $Z_t \cap Q$ with $t \in D$.

Since (S', Q', g') is a complete pull out with respect to x, it follows that

$$(\varphi')^{-1}(Z'_w \cap Q') = Z_{(\varphi')^{-1}(w)} \cap Q \text{ whenever } w \notin \varphi'(Z_{x_1} \cap Q). \tag{15}$$

On the other hand, since the partition $\{Q'_i\}$ defines a division for (S', Q', g'), every Q'_i is a union of sets of the form $Z'_w \cap Q'$. From (14) we get that $Q'_1 = \varphi'(Z_{x_1} \cap Q) \cup \mathcal{K}$, where $\mathcal{K}$ is either empty or a union of sets of the form $Z'_w \cap Q'$ with $w \notin \varphi'(Z_{x_1} \cap Q)$. Moreover, for every $2 \le i \le p$, Q'_i is a union of sets of the form $Z'_w \cap Q'$ with $w \notin \varphi'(Z_{x_1} \cap Q)$. In any case, from (15) we get that $(\varphi')^{-1}(Q'_i)$ is a union of sets of the form $Z_t \cap Q$ for $1 \le i \le p$ and the claim is proved. $\square$

As a corollary of Lemma 7.2, we obtain the main result of this section.

Proposition 7.3. *Let $n = m^k$ where m is a prime number and $k \in \mathbb{N}$. Let $\mathcal{Q}$ be a strongly centered n-periodic pattern with no division. Then, there exists an n-periodic pattern $\mathcal{P}$ with two discrete components such that $\mathcal{P} \le \mathcal{Q}$ and $\mathcal{P}$ has no division.*

Proof. Let (S, Q, g) be a model of $\mathcal{Q}$. Since $\mathcal{Q}$ is strongly centered, there exists a rotational component D containing all the inner points, and each inner point belongs exactly to two discrete components. If $\mathcal{Q}$ has two discrete components, we are done simply by setting $\mathcal{P} = \mathcal{Q}$. Assume that $\mathcal{Q}$ has $l \ge 3$ discrete components. This implies that there are at least two inner points in D. Recall that if an n-periodic pattern has a p-division then p is a strict divisor of n. Since $n = m^k$, from Lemma 7.2 it follows that we can choose an inner point $x \in D$ and a complete pull out (S', Q', g') of (S, Q, g) with respect to x such that (S', Q', g') has no division. Set $\mathcal{Q}' = ([S', Q'], [g'\big|_{Q'}])$. By Lemma 7.1, $\mathcal{Q}'$ is strongly centered. Moreover, from the definition of a pull out we get that the number of discrete components of $\mathcal{Q}'$ is $l - 1$. It is clear that this argument can be iterated as many times as necessary in order to obtain the prescribed pattern $\mathcal{P}$. $\square$

8. A lower bound for the entropy of no division patterns with two discrete components

In this section we prove that the entropy of any n-periodic pattern with two discrete components and no division is greater than or equal to the entropy of the pattern $\mathcal{Q}_n$. We recall that the patterns $\mathcal{Q}_n$ are our candidates for minimum (positive) entropy in the class of n-periodic patterns. They were defined in page 6. Recall also that that the entropy of $\mathcal{Q}_n$ is $\log(\lambda_n)$, where λ_n is the unique real root of the polynomial $q_n(x) = x^n - 2x - 1$ in $(1, \infty)$ (see Section 3).

So far, the entropy of any pattern $\mathcal{P}$ has been obtained by constructing the canonical model (T, P, f) and computing the logarithm of the spectral radius of the Markov matrix associated to the monotone model $(T, P \cup V(T), f)$. However, there is an alternative way which depends only on the combinatorial data of $\mathcal{P}$ and does not require the construction of the canonical model. Indeed, $h(\mathcal{P})$ can be obtained from the transition matrix of a combinatorial directed graph that can be constructed independently of the images of the vertices in any particular monotone model of the pattern. To prove the main theorem of this section it is convenient to use this alternative approach. Let us introduce the necessary notions. Let $\{\pi_1, \pi_2, \dots, \pi_k\}$ be the set of basic paths of the pointed tree (T, P). We will say that π_i *f-covers* π_j, denoted by $\pi_i \to \pi_j$, whenever $\pi_j \subset \langle f(\pi_i) \rangle_T$. The *$\mathcal{P}$-path graph* is the combinatorial directed graph whose vertices are in one-to-one correspondence with the basic paths of (T, P), and there is an arrow from the vertex i to the vertex j if and only if π_i f-covers π_j. The associated transition matrix, denoted by $M_{\mathcal{P}}$, will be called the *path transition matrix of* $\mathcal{P}$. It can be seen that the definitions of the $\mathcal{P}$-path graph and the matrix $M_{\mathcal{P}}$ are independent of the particular choice of the model (T, P, f). Thus, they are well-defined pattern invariants. The crucial fact about the path transition matrix $M_{\mathcal{P}}$ is the following (see [6]):

$$h(\mathcal{P}) = \max\{0, \log \sigma(M_{\mathcal{P}})\}. \tag{16}$$

Now we are ready to prove the main theorem of this section. We suggest the reader to accompany the reading of the proof of Theorem 8.1 with an example, as the one shown in Figure 8.

Theorem 8.1. *Let $\mathcal{P}$ be an n-periodic pattern with two discrete components. If $\mathcal{P}$ has no division, then $h(\mathcal{P}) \geq \log(\lambda_n)$.*

Proof. Until the end of this proof, for any $k \geq 1$ we will take $\{1, 2, \ldots, k\}$ as the representatives of the classes of $\mathbb{Z}/k\mathbb{Z}$.

Let (T, P, f) be the canonical model of $\mathcal{P}$ and let $P = \{x_i\}_{i=1}^n$ be a time labelling such that x_1 is the only inner point of P. Let C_1 and C_2 be the discrete components of (T, P), labeled in such a way that $x_2 \in C_2$. Set $L = \langle C_1 \rangle$ and $R = \langle C_2 \rangle$. Observe that $C_1 \cap C_2 = L \cap R = \{x_1\}$. Note also that C_2 is a rotational component since x_1 does not escape from it and the remaining points of C_2 are endpoints.

By Lemma 4.3, there exists a fixed point y of f in $\mathrm{Int}(R)$. This fixed point is unique, since if there were another fixed point y' of f in $\mathrm{Int}(R)$ then $[y, y']$ would be an invariant forest, in contradiction with the fact that (T, P, f) is a canonical model. Observe that f is $(P \cup \{y\})$-monotone.

Let p be the minimum positive integer such that $x_i \in R$ for $1 \leq i \leq p$ and $x_{p+1} \in L$. Since $\{x_1, x_2\} \subset R$ and $L \neq \emptyset$, it follows that $2 \leq p < n$. Assume that $p = n - 1$. Then, $C_1 = \{x_1, x_n\}$ and $C_2 = \{x_1, x_2, \ldots, x_{n-1}\}$. In this case, the pattern $\mathcal{P}$ coincides with $\mathcal{Q}_n$. In consequence, $h(\mathcal{P}) = \log(\lambda_n)$ by Proposition 3.3 and the theorem holds. So, from now on we assume that

$$2 \leq p \leq n - 2. \tag{17}$$

Let $m = \mathrm{Val}(y) \geq 2$ and let $\{a_i\}_{i=1}^m$ be the set of m points of $P \cup V(T)$ closest to y. That is, $(y, a_i) \cap (P \cup V(T)) = \emptyset$. Assume without loss of generality that they have been labeled in such a way that $a_1 \in (y, x_1]$. Since f is P-monotone, it follows that

$$f \text{ is monotone on each interval of the form } [a_i, a_j] \tag{18}$$

and

$$f \text{ is monotone on each interval of the form } [y, a_i]. \tag{19}$$

Note that $f(a_i) \neq y$ for all i: this is obvious when $a_i \in P$ and, if $a_i \in V(T) \setminus P$, then $f(a_i) \neq y$ because otherwise $[y, a_i]$ would be an invariant forest. Therefore, each interval of the form $[y, a_i]$ f-covers one of such intervals, which is unique by (19). Hence, we have a well defined map

$$\phi\colon \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, m\}$$

such that $\phi(i) = j$ if and only if $[y, a_j]$ is the only interval adjacent to y f-covered by $[y, a_i]$. Observe that if $i \neq j$ then $\phi(i) \neq \phi(j)$ since, otherwise, using (18) and the fact that $f(y) = y$ would lead us to a contradiction. In consequence, ϕ is injective. It follows that $\{1, 2, \ldots, m\}$ is a union of periodic orbits of ϕ.

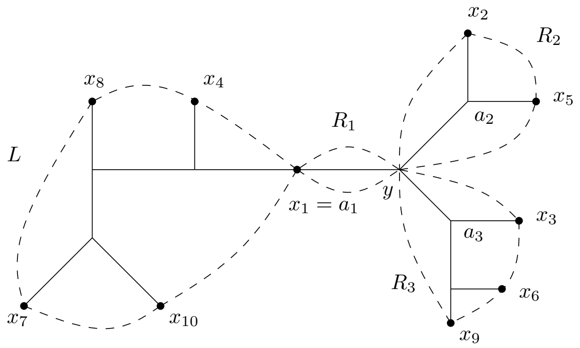

FIGURE 8. The canonical model of a 10-periodic pattern with two discrete components. In the notation of the proof of Theorem 8.1, $p = 3$. The subtrees L, R_1, R_2, R_3 and the fixed point y are indicated. For this example $a_1 = x_1$.

For $1 \leq i \leq m$, let R_i be the closure of the connected component of $R \setminus \{y\}$ containing a_i. Note that $R = \bigcup_1^m R_i$. The $(P \cup \{y\})$-monotonicity of f easily yields that

$$\text{If } \phi(i) \neq 1 \text{ then } f(R_i) \subset R_{\phi(i)}, \text{ while if } \phi(i) = 1 \text{ then } f(R_i) \subset R_1 \cup L. \tag{20}$$

Now we claim that all the elements in $\{1, 2, \ldots, m\}$ form in fact a unique m-periodic orbit of ϕ. Otherwise, there exists an l-periodic orbit $Q = \{i_1, i_2, \ldots, i_l\} \subsetneq \{1, 2, \ldots, m\}$ of ϕ such that $1 \notin Q$. In particular, $x_1 \notin \cup_{j=1}^l R_{i_j}$. By (20), $P \cap \left(\cup_{j=1}^l R_{i_j}\right)$ contains an invariant subset of P disjoint from $\{x_1\}$; a contradiction. From now on we assume that the points a_i (and, consequently, the trees R_i) are labeled in such a way that $f([y, a_i]) \supset [y, a_{i+1 \bmod m}]$ for $1 \leq i \leq m$. Then,

$$f(R_i) \subset R_{i+1} \text{ for } 1 \leq i < m \text{ and } f(R_m) \subset R_1 \cup L. \tag{21}$$

We claim that $m = p$. To prove this claim, recall that $x_1 \in R_1$ and that $f(x_i) = x_{i+1}$ for $1 \leq i < p$ and $f(x_p) \in L$. Then, from (21) it follows that $p = tm$ for some $t \geq 1$ and $x_i \in R_{i \bmod m}$ for $1 \leq i \leq p$. Since $p, m \geq 2$, the claim is trivially true when $p = 2$ or $p = 3$. So, assume that $p \geq 4$. Set $X := \{x_i\}_{i=1}^p \subset P$. Then, $X \subset \operatorname{En}(R)$ and $X \cap R_i = \{x_i, x_{i+m}, \ldots, x_{i+(t-1)m}\}$. Consider the set V of vertices v satisfying the following property: there exists a pair of points $\{x_i, x_j\} \subset X$ such that $i \equiv j \bmod m$ (that is, $\{x_i, x_j\} \subset R_{i \bmod m}$) and v is the central point of the 3-star $\langle\{y, x_i, x_j\}\rangle$. If $t > 1$, then $V \neq \emptyset$ since each tree R_i contains $t \geq 2$ points of X. In this case, take any $v \in V$. By the $(P \cup \{y\})$-monotonicity of f we have that either $x_p \notin \{x_i, x_j\}$ and $f(v)$ is the central point of the 3-star $\langle\{y, x_{i+1}, x_{j+1}\}\rangle$ or $\{x_i, x_j\} = \{x_i, x_p\}$ and $f(v)$ is the central point of the 3-star $\langle\{y, x_{i+1}, x_1\}\rangle$. In any case, we have that $f(v) \in V$ for any $v \in V$. Since V is finite, it follows that there exists a periodic orbit Q of vertices in $V \subset \operatorname{Int}(R)$. Then, $\langle Q \cup \{y\}\rangle$ is an invariant forest; a contradiction. In consequence, $t = 1$ and the claim is proved.

Summarizing, we have proved that the fixed point y defines a partition of R into p subtrees $\{R_i\}_{i=1}^p$ such that

$$f(R_i) \subset R_{i+1} \text{ for } 1 \leq i < p \text{ and } f(R_p) \subset R_1 \cup L. \tag{22}$$

Set $S_1 := R_1 \cup L$, $S_i := R_i$ for $2 \leq i \leq p$. Then, $x_i \in S_{i \bmod p}$ for $1 \leq i \leq p + 1$. Now observe that if $x_i \in S_{i \bmod p}$ for $1 \leq i \leq n$ then $\mathcal{P}$ would have a p-division with respect to the discrete component C_2. Since $\mathcal{P}$ has no division by hypothesis, it follows that there exists an integer k such that

$$p + 1 \leq k \leq n \tag{23}$$

with $x_i \in S_{i \bmod p}$ for $1 \le i \le k$ and $x_{k+1} \notin S_{i+1 \bmod p}$. From (22) it follows that

$$x_k \in L \setminus \{x_1\} \text{ and } k \equiv 1 \bmod p. \tag{24}$$

Since f is $(P \cup \{y\})$-monotone, from Theorem 2.1(b) we get that $h(\mathcal{P}) = h(f) = h(\mathcal{P}_y)$, where $\mathcal{P}_y$ is the pattern $([T, P \cup \{y\}], [f\big|_{P\cup\{y\}}])$. So, to prove the theorem it is enough to show that $h(\mathcal{P}_y) \ge \log(\lambda_n)$. To do it, we will find some loops of the path transition matrix of $\mathcal{P}_y$, thus obtaining a subgraph $\mathcal{G}$ of the $\mathcal{P}_y$-path graph. Then we will prove that the spectral radius of the transition matrix $M_{\mathcal{G}}$ associated to $\mathcal{G}$ is greater than or equal to λ_n. Proceeding in this way, we get that $\lambda_n \le \sigma(M_{\mathcal{G}}) \le \sigma(M_{\mathcal{P}_y})$ and then the theorem follows from (16), since $h(\mathcal{P}) = h(\mathcal{P}_y)$.

By construction, each basic path of $\mathcal{P}_y$ has one of the following forms:

- $\{y, x\}$ with $x \in P \cap R_i$ for some $1 \le i \le p$
- $\{x, z\}$ with $x, z \in P \cap R_i$ for some $1 \le i \le p$
- $\{x, z\}$ with $x, z \in P \cap L$.

For the sake of brevity, from now on we will write $\{a, b\} \to \{c, d\}$ to indicate that $\{a, b\}, \{c, d\}$ are basic paths of $\mathcal{P}_y$ and that $\{a, b\}$ f-covers $\{c, d\}$.

Set $\pi_i := \{y, x_i\}$ for $1 \le i \le p$.

From (22) and the fact that $f(y) = y$ we get that

$$\text{If } x \in P \cap R_i \text{ for some } 1 \le i < p \text{ then } \{y, x\} \to \{y, f(x)\} \subset R_{i+1}. \tag{25}$$

Moreover,

$$\text{If } x \in P \cap R_p \text{ then } \{y, x\} \to \begin{cases} \{y, f(x)\} \text{ if } f(x) \in R_1 \\ \pi_1 \text{ and } \{x_1, f(x)\} \text{ if } f(x) \in L \setminus \{x_1\}. \end{cases} \tag{26}$$

Since $f(x_p) \in L \setminus \{x_1\}$, from (25) and (26) it follows that the $\mathcal{P}_y$-path graph contains the following subgraph:

$$\begin{tikzcd} \pi_1 \arrow[r] & \pi_2 \arrow[r] & \dots \arrow[r] & \pi_p \arrow[r] \arrow[lll, bend right] & \{x_1, x_{p+1}\} \end{tikzcd}. \tag{27}$$

On the other hand, from (22) and the definition of x_k we get that

$$\begin{aligned} &\text{If } \{x_i, x_j\} \text{ is a basic path in } S_m \text{ with } 1 \le m < p \text{ and } i < j < k, \\ &\text{then } \{x_i, x_j\} \to \{x_{i+1 \bmod n}, x_{j+1 \bmod n}\} \subset R_{m+1}. \end{aligned} \tag{28}$$

When $\{x_i, x_j\} \subset R_p$, then either $\{x_{i+1 \bmod n}, x_{j+1 \bmod n}\}$ is a basic path (contained in either R_1 or L) f-covered by $\{x_i, x_j\}$, or $x_{i+1 \bmod n}$ and $x_{j+1 \bmod}$ are separated by x_1 and, in this case, $\{x_i, x_j\}$ f-covers the two basic paths $\{x_1, x_{i+1 \bmod n}\}$ and $\{x_1, x_{j+1 \bmod n}\}$. One of these two basic paths is contained in L and the other one in R_1. Hence,

$$\begin{aligned} &\text{If } \{x_i, x_j\} \subset R_p, \text{ then} \\ &\{x_i, x_j\} \to \begin{cases} \text{either } \{x_{i+1 \bmod n}, x_{j+1 \bmod n}\} \subset S_1 \\ \text{or } \{x_1, x_{i+1 \bmod n}\} \subset S_1 \text{ and } \{x_1, x_{j+1 \bmod n}\} \subset S_1. \end{cases} \end{aligned} \tag{29}$$

We consider two cases.

Case 1. $k = p + 1$.

In this case, $x_k = x_{p+1} \in L$ and $x_{k+1} = x_{p+2} \notin R_2$. If $x_{p+2} \in L$, then $\{x_1, x_{p+1}\}$ f-covers both π_1 and π_2. Together with (27), this amounts to the subgraph of the $\mathcal{P}_y$-path graph shown at the top of Figure 9. When $x_{p+2} \notin L$, then $x_{p+2} \in R_i$ for some $i \ne 2$. In consequence, $\{x_1, x_{p+1}\}$ f-covers both π_2 and $\{y, x_{p+2}\}$. Now starting with $\{y, x_{p+2}\}$ and iteratively using (25) and/or (26) we easily get that there exist $p + 2 \le l \le n$ and a sequence of coverings of the form

$$\{y, x_{p+2}\} \to \{y, x_{p+3}\} \to \dots \to \{y, x_l\} \to \pi_1,$$

$$\pi_1 \to \pi_2 \to \ldots \to \pi_p \to \{x_1, x_{p+1}\}$$

$$\pi_1 \to \pi_2 \to \ldots \to \pi_p \to \{x_1, x_{p+1}\} \to \{y, x_{p+2}\} \to \ldots \to \{y, x_l\}$$

FIGURE 9. The two possibilities for a subgraph $\mathcal{G}$ of the $\mathcal{P}_y$-path graph in Case 1 of the proof of Theorem 8.1.

which amounts to the the subgraph shown at the bottom of Figure 9. In any case, the $\mathcal{P}_y$-path graph contains a subgraph $\mathcal{G}$ for which $\{\pi_p\}$ is a rome and there are 3 simple loops starting and ending at π_p, of lengths p, p and l, where $p+1 \leq l \leq n$. Let us prove that $\lambda_n \leq \sigma(M_{\mathcal{G}})$. By Theorem 3.2, the characteristic polynomial of $M_{\mathcal{G}}$ is $\pm\, x^l(2x^{-p} + x^{-l} - 1) = \pm(x^l - 2x^{l-p} - 1)$. Let us see that the polynomial $F(x) := x^l - 2x^{l-p} - 1$ has a real root greater than or equal to λ_n. Since $F(x) \to +\infty$ when $x \to +\infty$, it is enough to prove that $F(\lambda_n) \leq 0$. Recall that, by definition,

$$(\lambda_n)^n - 2\lambda_n - 1 = 0. \tag{30}$$

Then, $F(\lambda_n) = (\lambda_n)^l - 2(\lambda_n)^{l-p} - 1 = (\lambda_n)^l - 2(\lambda_n)^{l-p} - (\lambda_n)^n + 2\lambda_n$. Since $\lambda_n > 1$ and $p+1 \leq l \leq n$, we get that $(\lambda_n)^l - (\lambda_n)^n \leq 0$ and $2\lambda_n - 2(\lambda_n)^{l-p} \leq 0$. In consequence, $F(\lambda_n) \leq 0$ and we are done in this case.

Case 2. $k > p+1$.

From (24), $k \geq 2p+1$. Here we have to consider two different situations.

Subcase 2.1. $x_{jp+1} \in L$ *for each* $j \geq 1$ *such that* $jp+1 \leq k$.

From (24) and the definition of k we get that $x_k \in L$ and $x_{k-p+1} \in R_2$. Moreover, in this subcase we also get that $x_{k-p} \in L$.

If $x_{k+1} \in L$ it follows that $\{x_{k-p}, x_k\}$ f-covers π_1 and $\{y, x_{k-p+1}\}$, and $\{x_1, x_k\}$ f-covers π_1 and π_2, by (22). Consequently, if $x_{k+1} \in L$, from (25)–(28) we get that the $\mathcal{P}_y$-path graph contains the subgraph shown at the top of Figure 10.

If $x_{k+1} \notin L$ then $\{x_1, x_k\}$ f-covers $\{y, x_{k+1}\}$ and π_2, and $\{x_{k-p}, x_k\}$ f-covers $\{y, x_{k+1}\}$ and $\{y, x_{k-p+1}\}$. So, as above, it follows from (25)–(28) that the $\mathcal{P}_y$-path graph contains the subgraph shown at the bottom of Figure 10.

Both graphs from Figure 10 have $\{\pi_p\}$ as a rome. In the graph at the top of Figure 10 there are unique simple loops of lengths p, k, $k+p$ and two of length $k+p-1$. Consequently, Theorem 3.2 yields that the characteristic polynomial of the Markov matrix associated to this graph is

$$\pm\, x^{k+p}(x^{-p} + x^{-k} + x^{-k-p} + 2x^{-k-p+1} - 1) = \pm\, G(x)$$

with $G(x) := x^{k+p} - x^k - 2x - x^p - 1$.

In the graph at the bottom of Figure 10 there are unique simple loops of lengths p, l, $l+p$ and two of length $k+p-1$. In this case, Theorem 3.2 gives

$$\pm\, x^{l+p}(x^{-p} + x^{-l} + x^{-l-p} + 2x^{-k-p+1} - 1) = \pm\, F(x)$$

with $F(x) := x^{l+p} - x^l - 2x^{l-k+1} - x^p - 1$ and $l > k$ as the characteristic polynomial of the Markov matrix associated to this graph.

As above, to end the proof of the theorem in this subcase, we have to see that $F(\lambda_n) \leq 0$ and $G(\lambda_n) \leq 0$.

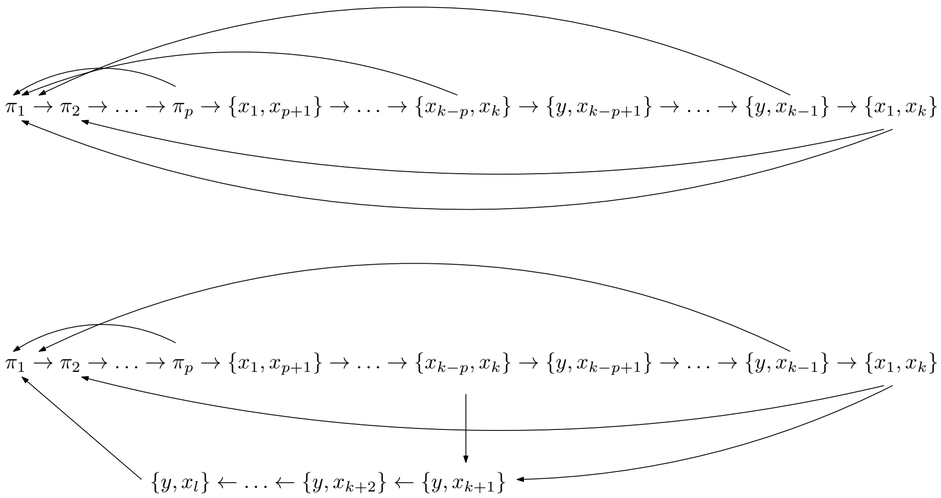

FIGURE 10. The two possibilities for a subgraph $\mathcal{G}$ of the $\mathcal{P}_y$-path graph in Subcase 2.1 of the proof of Theorem 8.1.

Since $k < l$ and $x^{l+p} - x^l - 2x^{l-k+1} = x^{l-k}(x^{k+p} - x^k - 2x)$, it follows that $G(x) \leq F(x)$ for $x \geq 1$. Therefore, it is enough to show that $F(\lambda_n) \leq 0$.

Observe that, fixing $x > 1$ and $p \geq 1$, the expression $x^{l+p} - x^l$ is increasing in l. Since $\lambda_n > 1$ and $k < l \leq n$, it follows that

$$F(\lambda_n) \leq (\lambda_n)^{n+p} - (\lambda_n)^n - 2(\lambda_n)^{l-k+1} - (\lambda_n)^p - 1 < (\lambda_n)^{n+p} - (\lambda_n)^n - 2\lambda_n - (\lambda_n)^p - 1.$$

From (30) we get

$$(\lambda_n)^{n+p} - 2(\lambda_n)^{p+1} - (\lambda_n)^p = 0 \tag{31}$$

and

$$\begin{aligned}(\lambda_n)^{n+p} - (\lambda_n)^n - 2\lambda_n - (\lambda_n)^p - 1 &= -(\lambda_n)^n - 2\lambda_n - 1 + 2(\lambda_n)^{p+1} \\ &= -2(\lambda_n)^n + 2(\lambda_n)^{p+1}.\end{aligned}$$

This expression is negative since $2p + 1 \leq k \leq n$. So we are done in this case.

Subcase 2.2. $x_{jp+1} \notin L$ for some j such that $jp + 1 \leq k$.

Let s be the minimum of such j. Hence, $2 \leq s$ because $x_{p+1} \in L$. Moreover, by (24), $k \equiv 1 \mod p$ and $x_k \in L$. Therefore, $sp + 1 \leq k - p$. In consequence,

$$3p + 1 \leq (s + 1)p + 1 \leq k. \tag{32}$$

From (22) and the definition of s we get that $\{x_i, x_{i+p}\}$ is a $\mathcal{P}_y-$basic path for $1 \leq i \leq (s-1)p$. By (29), $\{x_{(s-1)p}, x_{sp}\}$ f-covers both $\{x_1, x_{(s-1)p+1}\}$ and $\{x_1, x_{sp+1}\}$. In addition, since $x_{p+1} \in L$ and $x_{sp+1} \notin L$, $\{x_p, x_{sp}\}$ f-covers both $\{x_1, x_{p+1}\}$ and $\{x_1, x_{sp+1}\}$.

From (29) we also get the following sequence of coverings in the $\mathcal{P}_y$-path graph:

$$\{x_1, x_{sp+1}\} \to \tau_{sp+2} \to \tau_{sp+3} \to \ldots \to \tau_k,$$

where every path τ_i has the form $\{x_{t(i)}, x_i\}$ for some $t(i) < i$ such that $t(i) \equiv i \mod p$. Since $t(k) \equiv k \mod p$ and $t(k) < k$, it follows that $x_{t(k)+1} \in R_2$. On the other hand, by the definition of k, $x_{k+1} \notin R_2$. In consequence, τ_k f-covers either $\{y, x_{k+1}\}$ when $x_{k+1} \notin L$ or $\{y, x_1\}$ otherwise. So, by (25)–(28) we get that the $\mathcal{P}_y$-path graph contains one of the two subgraphs $\mathcal{G}$ shown in Figure 11: the one at the top when $x_{k+1} \in L$ and the one at the bottom when $x_{k+1} \notin L$, where l satisfies $k \leq l \leq n$.

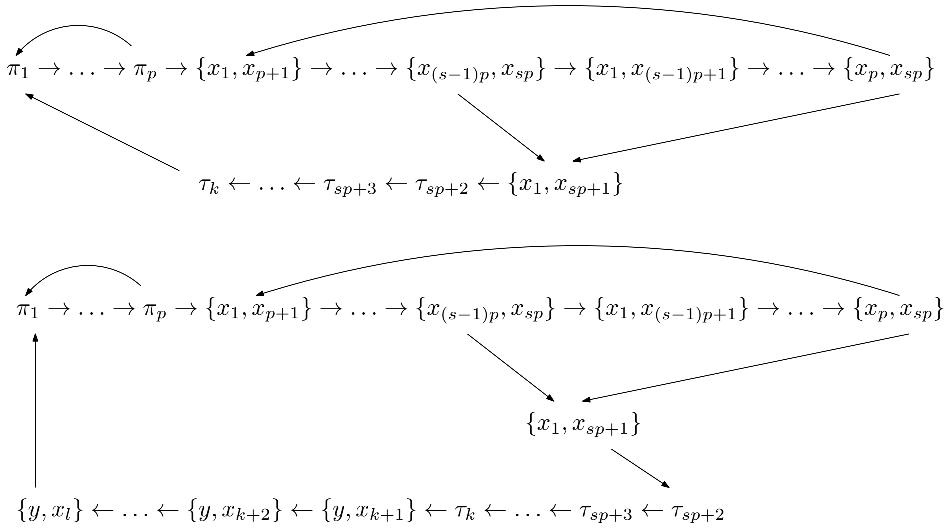

FIGURE 11. The two possibilities for a subgraph $\mathcal{G}$ of the $\mathcal{P}_y$-path graph in Subcase 2.2 of the proof of Theorem 8.1.

In any case, the set $\{\pi_p, \{x_1, x_{p+1}\}\}$ is a rome for $\mathcal{G}$. In both graphs, there is a simple loop of length p starting and ending at π_p. There is also a simple loop of length sp starting and ending at $\{x_1, x_{p+1}\}$. On the other hand, $\pi_p \to \{x_1, x_{p+1}\}$ is a simple path of length 1. Finally, there are two simple paths of lengths $l-1$ and $l+p-1$ from $\{x_1, x_{p+1}\}$ to π_p. Moreover, these are the only simple paths in the two graphs of Figure 11. In consequence, from Theorem 3.2 we get that the characteristic polynomial of $M_{\mathcal{G}}$ is $\pm F(x) := \pm x^{l+p}\det(M)$, where

$$M = \begin{pmatrix} x^{-p} - 1 & x^{-1} \\ x^{-l+1} + x^{-l-p+1} & x^{-sp} - 1 \end{pmatrix}.$$

Then, $F(x) = x^{l-sp} - x^l - x^{l+p-sp} + x^{l+p} - x^p - 1$. As above, it is enough to see that $F(\lambda_n) \le 0$. Since $x^{l-sp} - x^{l-sp+p} < 0$ for $x > 1$, it suffices to show that $G(\lambda_n) := -(\lambda_n)^l + (\lambda_n)^{l+p} - (\lambda_n)^p - 1 \le 0$. Recall that $x^{l+p} - x^l$ is increasing in l for $x > 1$ and $p \ge 1$ fixed. Since $\lambda_n > 1$ and $l \le n$, it follows that $G(\lambda_n) < -(\lambda_n)^n + (\lambda_n)^{n+p} - (\lambda_n)^p - 1$ which, by (31), is equal to $-(\lambda_n)^n + 2(\lambda_n)^{p+1} - 1$. Since $k \le n$, from (32) we get that $n > 3p$. Therefore, the previous expression is smaller than $H(\lambda_n)$, where $H(x) := -x^n + 2x^{\frac{n}{3}+1} - 1$. Then, $H'(x) = -nx^{n-1} + 2(\frac{n}{3}+1)x^{n/3}$. Since $k \le n$ and $p \ge 2$, (32) yields $n \ge 7$. Thus, $x^{n-1} > x^{n/3}$ for $x > 1$ and $n > 2(\frac{n}{3}+1)$, what implies that $H'(x) < 0$ for $x > 1$. Since $H(1) = 0$, it follows that $H(x) < 0$ for $x > 1$. In particular, $H(\lambda_n) < 0$. This completes the proof of the theorem. □

Now we are ready to prove that $\mathcal{Q}_n$ minimizes the entropy in the set of all n-periodic patterns with no division, when n is a power of a prime. This result will play a central role in the proof of Theorem A.

Corollary 8.2. *Let $n = m^k$ where m is a prime number and $k \in \mathbb{N}$. Then, the pattern $\mathcal{Q}_n$ has minimum entropy in the set of all n-periodic patterns with no division.*

Proof. Let $\mathcal{Q}$ be an n-periodic pattern with no division. By Theorem 6.1, there exists a strongly centered n-periodic pattern $\mathcal{Q}'$ with no division such that $\mathcal{Q}' \le \mathcal{Q}$. By Theorem 5.3, $h(\mathcal{Q}') \le h(\mathcal{Q})$. On the other hand, by Proposition 7.3 there exists an n-periodic pattern $\mathcal{Q}''$ with two discrete components such that $\mathcal{Q}'' \le \mathcal{Q}'$ and

$\mathcal{Q}''$ has no division. Using again Theorem 5.3, we get that $h(\mathcal{Q}'') \leq h(\mathcal{Q}')$. Finally, by Theorem 8.1 we have that $h(\mathcal{Q}'') \geq \log(\lambda_n)$, which equals $h(\mathcal{Q}_n)$ by virtue of Proposition 3.3. □

9. Proof of Theorem A

We start by giving a sketch of the proof of Theorem A. Let $\mathcal{P} = ([T, P], [f])$ be an n-periodic pattern. A pattern $\mathcal{P}'$ will be called *subordinated to* $\mathcal{P}$ if for some divisor $n > p > 1$ of n there is an (n/p)-periodic orbit $P' \subset P$ of f^p such that $\mathcal{P}' = ([\langle P' \rangle_T, P'], [f^p\big|_{P'}])$. Clearly, this definition is independent of the particular model (T, P, f) representing $\mathcal{P}$. The following result states that when an n-periodic pattern $\mathcal{P}$ has a subordinated pattern $\mathcal{P}'$ with positive entropy, then we can reduce the problem of determining whether $h(\mathcal{P}) \geq h(\mathcal{Q}_n)$ to determining whether $h(\mathcal{P}') \geq h(\mathcal{Q}_{n/p})$.

Lemma 9.1. *Let $\mathcal{P}$ be an n-periodic pattern. Let $\mathcal{P}'$ be an n'-periodic pattern subordinated to $\mathcal{P}$. If $n' \geq 3$ and $h(\mathcal{P}') \geq h(\mathcal{Q}_{n'})$ then $h(\mathcal{P}) \geq h(\mathcal{Q}_n)$.*

Proof. Let (T, P, f) be the canonical model of $\mathcal{P}$. By definition of a subordinated pattern, there exists a strict divisor p of n such that $n' = n/p$ and $\mathcal{P}' = ([\langle P' \rangle_T, P'], [f^p\big|_{P'}])$ for some n'-periodic orbit $P' \subset P$ of f^p. Since $h(\mathcal{P}')$ is smaller than or equal to the entropy of any map exhibiting $\mathcal{P}'$ and f^p exhibits $\mathcal{P}'$,

$$h(\mathcal{P}') \leq h(f^p) = p \cdot h(f) = p \cdot h(\mathcal{P}). \tag{33}$$

We are assuming that $h(\mathcal{P}') \geq h(\mathcal{Q}_{n/p})$, which, by Proposition 3.3, is equal to $\log(\lambda_{n/p})$. Then, from (33) it follows that

$$h(\mathcal{P}) \geq \frac{1}{p} \log(\lambda_{n/p}) = \log((\lambda_{n/p})^{1/p}),$$

which is greater than or equal to $\log(\lambda_n)$ by Proposition 3.1(b). Since $\log(\lambda_n) = h(\mathcal{Q}_n)$ by Proposition 3.3, the lemma follows. □

Remark 9.2. Surprisingly, it is not always true that a positive entropy pattern has subordinated patterns with positive entropy. See Figure 12 for a counterexample. The pattern $\mathcal{P}$ has a 2-division, but a direct computation using the Markov matrix of $\mathcal{P}$ shows that $h(\mathcal{P}) \approx \log(1.272) > 0$. It turns out that all the subordinated patterns induced by f^p for $p = 2, 3, 4, 6$ have entropy zero. Indeed: all patterns of f^6 and f^4 are trivial 2-periodic and 3-periodic patterns respectively. On the other hand, f^3 induces three 4-periodic patterns on the sets $P'_1 := \{1, 4, 7, 10\}$, $P'_2 := \{2, 5, 8, 11\}$ and $P'_3 := \{3, 6, 9, 12\}$ and f^2 induces two 6-periodic patterns on $P'_4 := \{1, 3, 5, 7, 9, 11\}$ and $P'_5 := \{2, 4, 6, 8, 10, 12\}$. The patterns $([\langle P'_1 \rangle_T, P'_1], [f^3])$ and $([\langle P'_2 \rangle_T, P'_2], [f^3])$ are trivial. The pattern $([\langle P'_3 \rangle_T, P'_3], [f^3])$ is not trivial since it has two discrete components: $\{3, 6, 9\}$ and $\{6, 12\}$. The same happens for $([\langle P'_4 \rangle_T, P'_4], [f^2])$ and $([\langle P'_5 \rangle_T, P'_5], [f^2])$. However, it is not difficult to show that all these patterns have entropy zero (use the characterization of zero entropy patterns that we explain below). The existence of such counterexamples explains why we cannot use these techniques to prove Theorem A for any $n \in \mathbb{N}$.

The core idea of the proof of Theorem A is that a counterexample like the one shown in Figure 12 cannot be found in the family of n-periodic patterns when n is a power of a prime. More precisely, we will see that, in this case, any n-periodic pattern with a division and positive entropy has subordinated patterns of positive entropy (Proposition 9.7).

Let us recall now the description of zero entropy patterns first given in [6]. This characterization applies to general (not specifically periodic) patterns. However, as

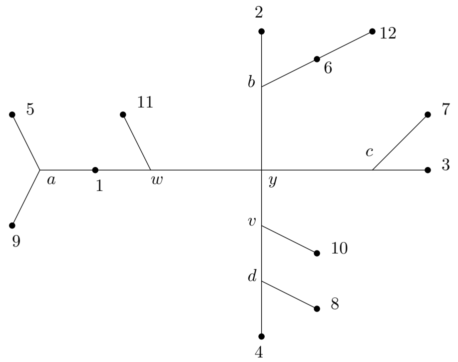

FIGURE 12. The canonical model of a 12-periodic pattern $\mathcal{P}$. The point y is fixed by f and the images of the vertices are: $f(a) = f(w) = b$, $f(b) = c$, $f(c) = d$, $f(d) = a$, $f(v) = w$. All subordinated patterns have entropy zero, while $h(\mathcal{P}) > 0$.

we will see, the zero entropy periodic patterns exhibit some additional properties that will be used in the proof of Theorem A.

Let (T, P, f) be a monotone model of a pattern $\mathcal{P}$. Let π be a basic path of (T, P). We say that $\mathcal{P}$ is *π-reducible* if $f^n(\pi)$ is contained in a single discrete component of (T, P) for every $n \geq 0$. In this case, let $X = \bigcup_{i\geq 0}\langle f^i(\pi)\rangle$ and let $C_1, C_2, \ldots, C_p$ be the connected components of X. Note that $P \subset X$. It is easy to see that for each $1 \leq i \leq p$ there exists j_i such that $f(C_i) \subset C_{j_i}$. Then we take the tree T' obtained from T by collapsing each C_i to a point c_i. Let $\kappa\colon T \longrightarrow T'$ be the standard projection. We set $P' = \kappa(P)$ and define $f'\colon P' \longrightarrow P'$ as $f' = \kappa \circ f \circ \kappa^{-1}$. It is easy to see that $([T', P'], [f'])$ is a well defined pattern, which we call a *π-reduced* (or simply *reduced*) *pattern of* $\mathcal{P}$. The process of obtaining this pattern from $\mathcal{P}$ is called a *reduction*. The entropies of $\mathcal{P}$ and the reduced pattern coincide, as the following result (Proposition 8.1 of [6]) states.

Proposition 9.3. *Let $\mathcal{P}$ be a pattern. Let $\mathcal{P}'$ be a reduced pattern of $\mathcal{P}$. then, $h(\mathcal{P}') = h(\mathcal{P})$.*

A pattern will be called *strongly reducible* if there is a finite sequence of reductions leading to a pattern consisting of a single point. The notion of a strongly reducible pattern depends apparently on the chosen sequence of basic paths and monotone models. From the next theorem, which is the characterization of zero entropy patterns given in [6], it follows that this notion is well defined.

Theorem 9.4 (Theorem E of [6])**.** *A pattern has zero entropy if and only if it is strongly reducible.*

The following result summarizes some specific properties of the reduced patterns of a periodic pattern. Statements (a) and (b) are immediate and statement (c) is Proposition 5.2 of [5].

Proposition 9.5. *Let $\mathcal{P} = ([T, P], [f])$ be an n-periodic pattern that is π-reducible for a basic path π. Let $C_1, C_2, \ldots, C_s$ be the connected components of $\bigcup_{i\geq 0}\langle f^i(\pi)\rangle$. Then, $n > s \geq 1$ and the following statements hold:*

(a) $\mathrm{En}(C_i) \subset P$ *for* $1 \leq i \leq s$.
(b) *The sets C_i can be labeled in such a way that $f(C_i) = C_{i+1}$ for $1 \leq i < s$ and $f(C_s) = C_1$. Thus, s divides n, $C_i \cap P$ is an (n/s)-periodic orbit of f^s for each $1 \leq i \leq s$ and the π-reduced pattern is s-periodic.*
(c) *The pointed tree $(C_i, C_i \cap P)$ has a unique discrete component for $1 \leq i \leq s$.*

Proposition 9.5(b) tells us that a reduced pattern of a periodic pattern is also periodic, with a period strictly smaller than that of the original pattern. The following result is the analog of Lemma 9.1 for reduced patterns.

Lemma 9.6. *Let $\mathcal{P}$ be an n-periodic pattern. Let $\mathcal{P}'$ be an n'-periodic reduced pattern of $\mathcal{P}$. If $n' \geq 3$ and $h(\mathcal{P}') \geq h(\mathcal{Q}_{n'})$ then $h(\mathcal{P}) \geq h(\mathcal{Q}_n)$.*

Proof. Using Propositions 9.3 and 3.3 we obtain that

$$h(\mathcal{P}) = h(\mathcal{P}') \geq h(\mathcal{Q}_{n'}) = \log(\lambda_{n'}).$$

Since $n > n'$, the lemma follows from Proposition 3.1(a). □

The following proposition is the central result in the proof of Theorem A.

Proposition 9.7. *Let $\mathcal{P}$ be an n-periodic pattern such that $n = m^k$ for some prime m. Assume that $\mathcal{P}$ is not π-reducible for any basic path π and that $\mathcal{P}$ has a p-division. Then, $\mathcal{P}$ has subordinated patterns of positive entropy.*

Proof. From Theorem 9.4 we get that $h(\mathcal{P}) > 0$. By definition of a p-division, p is a divisor of n and $p > 1$. Moreover, if $p = n$ then $\mathcal{P}$ would be a trivial pattern, a contradiction. In consequence, $n > p > 1$.

Let (T, P, f) be the canonical model of $\mathcal{P}$ and let C be a discrete component of (T, P) such that (T, P, f) has a p-division with respect to C. Set $Z = \operatorname{Int}(\langle C \rangle)$ and let $Z_1, Z_2, \ldots, Z_l$ be the connected components of $T \setminus Z$. By definition of a p-division, there exists $\{M_1, M_2, \ldots, M_p\}$, a partition of $T \setminus Z$, such that each M_i is a union of some of the sets $Z_1, Z_2, \ldots, Z_l$, $f(M_i \cap P) = M_{i+1} \cap P$ for $1 \leq i < p$ and $f(M_p \cap P) = M_1 \cap P$. Observe that, for any $1 \leq i \leq p$, the set $P_i := M_i \cap P$ is an (n/p)-periodic orbit of f^p.

Set $g := f^p$ and consider the (n/p)-periodic patterns $\mathcal{P}_i := ([\langle P_i \rangle_T, P_i], [g\big|_{P_i}])$ for $1 \leq i \leq p$. Since $n > p > 1$, all these patterns are subordinated to $\mathcal{P}$. Now we claim that

$$\text{If } \{a, b\} \text{ is a basic path of } \mathcal{P}_i, \text{ then } \{a, b\} \text{ is a basic path of } \mathcal{P}. \tag{34}$$

Indeed, this is clear when $\{a, b\} \subset Z_j$ for some j or when $\{a, b\} \subset \operatorname{Cl}(Z)$. These are the only two possibilities, because if $a \in Z_r \setminus \operatorname{Cl}(Z)$ and $b \in Z_s \setminus \operatorname{Cl}(Z)$ with $r \neq s$, then, since M_i is by definition a union of sets Z_j, the interval $[a, b]$ would contain the only point in $\operatorname{Cl}(Z) \cap Z_r$ and the only point in $\operatorname{Cl}(Z) \cap Z_s$, a contradiction with the fact that $\{a, b\}$ is a basic path of $\mathcal{P}_i$. So the claim is proved.

To prove the proposition, assume by way of contradiction that $h(\mathcal{P}_i) = 0$ for each $1 \leq i \leq p$. In particular, by Theorem 9.4, every pattern $\mathcal{P}_i$ is π_i-reducible for a basic path π_i of $\mathcal{P}_i$. For $1 \leq i \leq p$, let $\{C^i_1, C^i_2, \ldots, C^i_{j_i}\}$ be the connected components of $\bigcup_{r \geq 0} \langle g^r(\pi_i) \rangle$. By Proposition 9.5(b), j_i divides n/p and $C^i_r \cap P_i$ is an (n/pj_i)-periodic orbit of g^{j_i}. So, for any $1 \leq r \leq j_i$,

$$\text{If } x \in P \cap C^i_r \text{ then } \{x, g^{j_i}(x), g^{2j_i}(x), \ldots, g^{(\frac{n}{p}-1)j_i}(x)\} \subset C^i_r. \tag{35}$$

Moreover, Proposition 9.5(c) tells us that

$$(C^i_r, C^i_r \cap P_i) \text{ has a unique discrete component for } 1 \leq r \leq j_i. \tag{36}$$

Now observe that, since $n = m^k$ for some prime m, then $p = m^{k'}$ for some $k' < k$. Then, each pattern $\mathcal{P}_i$ is $(m^{k-k'})$-periodic and $j_i = m^{k_i}$ for some k_i such that $k - k' \geq k_i \geq 0$. Let $s \in \{1, 2, \ldots, p\}$ be such that

$$j_s = \max\{j_i : 1 \leq i \leq p\}.$$

Observe that j_i divides j_s for all i. Now take a point $x \in P \cap C^s_1$. By (35), the point $y := g^{j_s}(x)$ belongs to $P \cap C^s_1$. In consequence, (36) implies that $\{x, y\}$ is a basic path of $\mathcal{P}_s$. By (34), $\{x, y\}$ is also a basic path of $\mathcal{P}$. Now consider the pair $\{f(x), f(y)\}$. Then, $f(x) \in C_r^{s+1}$ for some $1 \leq r \leq j_{s+1}$ (here and in the rest of this paragraph, $s+1$ stands for 1 when $s = p$). On the other hand, since j_{s+1} divides j_s, there exists $1 \leq l \leq (n/p) - 1$ such that $j_s = l j_{s+1}$. Then, $f(y) = f(g^{j_s}(x)) = g^{j_s}(f(x)) = g^{l j_{s+1}}(f(x))$ which, by (35), belongs to C_r^{s+1}. In consequence, (36) implies that $\{f(x), f(y)\}$ is a basic path of $\mathcal{P}_{s+1}$ and, by (34), also of $\mathcal{P}$. Clearly one can iterate this argument n times to obtain that $\{f^i(x), f^i(y)\}$ is a basic path of $\mathcal{P}$ for $0 \leq i \leq n$. This means that $\mathcal{P}$ is $\{x, y\}$-reducible; a contradiction. □

Now we are ready to prove Theorem A.

Proof of Theorem A. Let $\mathcal{P}$ be an n-periodic pattern with positive entropy. Recall that, by assumption, $n = m^k \geq 3$ for some prime m. We have to show that $h(\mathcal{P}) \geq h(\mathcal{Q}_n)$. Next we construct sequences $\{n_i\}_{i=1}^l$ and $\{\mathcal{P}_i\}_{i=1}^l$ such that:

- $n_i = m^{k_i}$ and $\{k_i\}_{i=1}^l$ is strictly decreasing
- $\mathcal{P}_i$ is an n_i-periodic pattern with positive entropy and $n_i \geq 3$ for $1 \leq i \leq l$
- $\mathcal{P}_1 = \mathcal{P}$
- $\mathcal{P}_{i+1}$ is either subordinated to $\mathcal{P}_i$ or a reduced pattern of $\mathcal{P}_i$ for $1 \leq i < l$
- $\mathcal{P}_l$ has no division.

Set $k_1 := k$, $n_1 := n$ and $\mathcal{P}_1 := \mathcal{P}$. Then, $n_1 = m^{k_1}$. If $\mathcal{P}_1$ has no division, we simply set $l := 1$. Otherwise, to construct inductively the above sequences it is enough to explain how to construct $\mathcal{P}_{i+1}$ when the pattern $\mathcal{P}_i$ has positive entropy and a division. We consider two cases.

Case 1. $\mathcal{P}_i$ is π-reducible for some basic path π.

In this case, let $\mathcal{P}_{i+1}$ be the π-reduced pattern. By Proposition 9.5(b), $\mathcal{P}_{i+1}$ is n_{i+1}-periodic for some $n_{i+1} < n_i$. Moreover, n_{i+1} divides n_i and, in consequence, n_{i+1} has the form $m^{k_{i+1}}$ for some $k_{i+1} < k_i$. Finally, $n_{i+1} \geq 3$ since, otherwise, $\mathcal{P}_{i+1}$ would have entropy 0, in contradiction with the fact that, by Proposition 9.3, $h(\mathcal{P}_{i+1}) = h(\mathcal{P}_i)$.

Case 2. $\mathcal{P}_i$ is not π-reducible for any basic path π.

Since $\mathcal{P}_i$ has positive entropy, a division, and its period is a power of a prime then, by Proposition 9.7, $\mathcal{P}_i$ has a subordinated pattern $\mathcal{P}_{i+1}$ with positive entropy. By the definition of a subordinated pattern, $\mathcal{P}_{i+1}$ is n_{i+1}-periodic for some $1 < n_{i+1} < n_i$ which divides n_i. Hence, n_{i+1} has the form $m^{k_{i+1}}$ for some $k_{i+1} < k_i$. Finally, $n_{i+1} \geq 3$ since otherwise $\mathcal{P}_{i+1}$ would have entropy 0. This completes the induction step.

Now observe that, since $\mathcal{P}_l$ has no division, Corollary 8.2 yields that $h(\mathcal{P}_l) \geq h(\mathcal{Q}_{n_l})$. Then, using iteratively l times either Lemma 9.1 or Lemma 9.6 we obtain that $h(\mathcal{P}_1) = h(\mathcal{P}) \geq h(\mathcal{Q}_n)$. □

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