ON THE MINIMUM POSITIVE ENTROPY FOR CYCLES ON TREES

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Abstract. Consider, for any $n \in \mathbb{N}$, the set Pos_n of all n-periodic tree patterns with positive topological entropy and the set $\mathrm{Irr}_n \subsetneq \mathrm{Pos}_n$ of all n-periodic irreducible tree patterns. The aim of this paper is to determine the elements of minimum entropy in the families Pos_n and Irr_n. Let λ_n be the unique real root of the polynomial $x^n - 2x - 1$ in $(1, +\infty)$. We explicitly construct an irreducible n-periodic tree pattern $\mathcal{Q}_n$ whose entropy is $\log(\lambda_n)$. For $n = m^k$, where m is a prime, we prove that this entropy is minimum in the set Pos_n. Since the pattern $\mathcal{Q}_n$ is irreducible, $\mathcal{Q}_n$ also minimizes the entropy in the family Irr_n.

1. Introduction

The notion of *pattern* plays a central role in the theory of topological and combinatorial dynamics. Consider a family $\mathcal{X}$ of topological spaces (such as the set of all trees, graphs, compact surfaces, closed intervals of the real line, etc) and the family $\mathcal{F}_{\mathcal{X}}$ of all maps $\{f\colon X \longrightarrow X : X \in \mathcal{X}\}$ satisfying a given restriction (continuous maps, homeomorphisms, etc). Given a map $f\colon X \longrightarrow X$ in $\mathcal{F}_{\mathcal{X}}$ which is known to exhibit a finite invariant set P, the *pattern of P in $\mathcal{F}_{\mathcal{X}}$* is the equivalence class $\mathcal{P}$ of all maps $g\colon Y \longrightarrow Y$ in $\mathcal{F}_{\mathcal{X}}$ having an invariant set $Q \subset Y$ that, at a combinatorial level, behaves like P. That is, the relative positions of the points of Q inside Y are the same as the relative positions of P inside X, and the way these positions are permuted under the action of g coincides with the way f acts on the points of P. In this case, it is said that every map g in the class *exhibits* the pattern $\mathcal{P}$. If in particular P is a periodic orbit of f, the pattern $\mathcal{P}$ is said to be *cyclic* or *periodic*.

When $\mathcal{F}_{\mathcal{X}}$ is the family of continuous self-maps of closed intervals, the points of an invariant set P of a map in $\mathcal{F}_{\mathcal{X}}$ are totally ordered and the pattern of P can be clearly identified with a permutation π in a natural way. The notion of pattern for interval maps has its roots in the well known Sharkovskii's Theorem [23], but it was formalized and developed in the early 1990s [10, 22].

As another important example, when $\mathcal{F}_{\mathcal{X}}$ is the family of surface homeomorphisms, the pattern (or *braid type*) of a cycle P of a map $f\colon M \longrightarrow M$ in $\mathcal{F}_{\mathcal{X}}$ is characterized by the isotopy class, up to conjugacy, of $f\big|_{M\setminus P}$ [16, 21].

Going back to one-dimensional spaces, recently there has been a growing interest in extending the notion of *pattern* from the interval case to more general spaces such as graphs [2, 8] or trees [6, 11, 12]. In this paper we deal with patterns of invariant sets of continuous maps defined on trees (simply connected graphs). From now on, such patterns will be called *tree patterns*.

Let us recall the notion of a tree pattern. If $f\colon T \longrightarrow T$ is a continuous map of a tree and $P \subset T$ is a finite invariant set of f, the triplet (T, P, f) will be called a *model*. Two points x, y of P will be called *consecutive* if the unique closed interval

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