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Periodic behavior on trees

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Abstract. We characterize the set of periods for tree maps. More precisely, we prove that the set of periods of any tree map $f : T \to T$ is the union of finitely many initial segments of Baldwin's orderings ${}_p\!\geq$ and a finite set $\mathcal{F}$. The possible values of p and explicit upper bounds for the size of $\mathcal{F}$ are given in terms of the combinatorial properties of the tree T. Conversely, given any set $\mathcal{A}$ which is a union of finitely many initial segments of Baldwin's orderings ${}_p\!\geq$ with p of the above type and a finite set, we prove that there exists a tree map whose set of periods is $\mathcal{A}$.

1. *Introduction and statement of the main results*

In this paper we deal with the problem of determining the set of periods of all periodic orbits of a continuous map from a tree into itself. The widely known Sharkovskii's theorem (see [**24**]) which studies the set of periods of any continuous map defined on a closed interval was the first remarkable result in this setting. In order to state it, we introduce the Sharkovskii ordering $\trianglerighteq$ in the set $\mathbb{N} \cup \{2^\infty\}$:

$$3 \trianglerighteq 5 \trianglerighteq 7 \trianglerighteq \cdots \trianglerighteq 2 \cdot 3 \trianglerighteq 2 \cdot 5 \trianglerighteq 2 \cdot 7 \trianglerighteq \cdots \trianglerighteq 4 \cdot 3 \trianglerighteq 4 \cdot 5 \trianglerighteq 4 \cdot 7 \trianglerighteq \cdots \trianglerighteq \cdots \trianglerighteq 2^n \cdot 3 \trianglerighteq 2^n \cdot 5 \trianglerighteq 2^n \cdot 7 \trianglerighteq \cdots \trianglerighteq 2^\infty \trianglerighteq \cdots \trianglerighteq 2^n \trianglerighteq \cdots \trianglerighteq 16 \trianglerighteq 8 \trianglerighteq 4 \trianglerighteq 2 \trianglerighteq 1.$$

The Sharkovskii's theorem states that for each continuous map f from a closed interval into itself there exists some $n \in \mathbb{N} \cup \{2^\infty\}$ satisfying that the set of periods of f is the set of integers k such that $n \trianglerighteq k$. Conversely, given any $n \in \mathbb{N} \cup \{2^\infty\}$ there exists a continuous map defined on a closed interval whose set of periods is the set of all integers k such that $n \trianglerighteq k$.