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MINIMAL DYNAMICS FOR TREE MAPS

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Abstract. We prove that, given a tree pattern $\mathcal{P}$, the set of periods of a minimal representative $f\colon T \longrightarrow T$ of $\mathcal{P}$ is contained in the set of periods of any other representative. This statement is an immediate corollary of the following stronger result: there is a period-preserving injection from the set of periodic points of f into that of any other representative of $\mathcal{P}$. We prove this result by extending the main theorem of [6] to negative cycles.

1. **Introduction.** This paper is devoted to the study of the one-dimensional version of a well known problem in combinatorial dynamics: the so-called *dynamical minimality* problem, or *forcing* problem. The main question can be posed as follows: given a topological space X and a continuous self-map $f\colon X \longrightarrow X$ which is known to exhibit a periodic orbit P, what can be said about the rest of the periodic orbits of f only in terms of the combinatorial data supplied by $f\big|_P$?

When the space X is a closed interval of the real line, the solution to the forcing problem is a well known result in the theory of combinatorial dynamics. In this case, one considers the *pattern* of P, defined as the permutation induced by $f\big|_P$ (see [9] or [14]). To each pattern π one associates a π-*monotone model* $f_\pi\colon X \longrightarrow X$ which has an invariant set A such that the permutation induced by $f_\pi\big|_A$ is π and f_π is monotone between consecutive points of A (a "connect–the–dots" map). This map has minimal dynamics in several senses:

(1) f_π minimizes the topological entropy (a well known quantitative measure of the dynamical complexity of a map, first introduced in [1]; see also [7]) within the class of interval maps having a periodic orbit whose pattern is π.
(2) f_π admits a Markov graph. This is a combinatorial object which gives a good "coding" allowing one to describe the dynamics of the map f_π. The topological entropy and the periodic orbits of f_π may be calculated from the loops of this graph.

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