PACIFIC JOURNAL OF MATHEMATICS
Vol. 184, No. 1, 1998

BADLY ORDERED CYCLES OF CIRCLE MAPS

Lluís Alsedà, Jaume Llibre and Michał Misiurewicz

A cycle of a circle map of degree one is badly ordered if it cannot be divided into blocks of consecutive points, such that the blocks are permuted by the map like points of a cycle of a rational rotation. We find the smallest possible rotation intervals that a map with a badly ordered cycle of a given rotation number and period can have. Moreover, we show that if one of those intervals is contained in the interior of the rotation interval of a map then the map has a corresponding badly ordered cycle.

1. Introduction.

It is always very useful to be able to derive many consequences from a few bits of information. This situation arises in one dimensional dynamics, when knowing ordering of points along a cycle (a periodic orbit) we can often say a lot about a system. Here we want to study what happens if our system consists of iterations of a continuous degree one circle map and the information we have is that there is a badly ordered cycle of a given period and rotation number. The information we want to get is how large the rotation interval of the map has to be. This is the most important information. Once we know the rotation interval, we can estimate the set of periods and topological entropy (see e.g. [1]).

To state our assumptions and results in a more rigorous way, we introduce some notation in spirit of [1]. We will consider the circle $\mathbb{S}^1$ as $\{z \in \mathbb{C} : |z| = 1\}$ where $\mathbb{C}$ denotes the complex plane. Then we will denote by $e : \mathbb{R} \longrightarrow \mathbb{S}^1$ the natural projection $e(X) = \exp(2\pi i X)$, where $\mathbb{R}$ denotes the real line. A continuous map $F : \mathbb{R} \longrightarrow \mathbb{R}$ is called a *lifting* of a continuous map $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ if $e \circ F = f \circ e$. It can be seen that if f is a continuous map of the circle into itself of degree one and F is a lifting of f then $F(X+1) = F(X)+1$ for all $X \in \mathbb{R}$ and that $F(X + k) = F(X) + k$ for all $X \in \mathbb{R}$ and $k \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers. In the sequel, the class of all liftings of continuous maps of the circle into itself of degree 1 will be denoted by $\mathcal{L}$.

We shall say that a point $X \in \mathbb{R}$ is *periodic* (mod. 1) *of period* s for a map $F \in \mathcal{L}$ if $F^s(X) - X \in \mathbb{Z}$ but $F^i(X) - X \notin \mathbb{Z}$ for $i = 1, \dots, s-1$. We note that if F is a lifting of f then X is periodic (mod. 1) for F if and