Minimizing topological entropy for continuous maps on trees

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Abstract

By an *interval* we mean the closed interval $[0,1]$ and any space homeomorphic to it. A (finite) *tree* is a uniquely arcwise-connected space that is a point or a union of a finite number of intervals. Given a tree T and a finite set $A \subset T$, we shall say that (T, A) is a *pointed tree*. Giving a continuous map $f: T \longrightarrow T$ such that A is f-invariant, the corresponding combinatorial structure can be stated by means of the abstract notion of *pattern*, as it is done in [1]. We shall denote this pattern by $([T,A],[f|_A])$ or $(\mathcal{T},\Theta)$ (if any reference to a particular map is necessary) and we shall say that f *exhibits* the pattern $(\mathcal{T},\Theta)$ (over A, if necessary). Furthermore, there is an extension of the notion of "connect-the-dots" map from interval maps to tree maps. This notion is what we call an *A-monotone map*, that is, a monotone map between consecutive points of A.

Now we have an important difference between interval and tree maps: If f is a continuous tree map of the pointed tree (T, A) exhibiting a pattern over A, it is not always the case that there exists a continuous A-monotone tree map of the **same** pointed tree (T, A) with the same pattern over A, as it happens in the interval case. Fortunately our notion of pattern for tree maps is such that we may associate a monotone map to each pattern (changing of tree if necessary).

Then we can obtain similar properties to the ones obtained for patterns in the interval (see [1]). More precisely, for instance, given a pattern $(\mathcal{T},\Theta)$ on trees:

(A) We may associate (in a constructive way) a tree T, a finite set $A \subset T$ an a (non-unique) A-monotone continuous map $f: T \longrightarrow T$ such that f exhibits $(\mathcal{T},\Theta)$ over A.

(B) Any A-monotone map f satisfies that $h(f) \le h(g)$ for all g exhibiting the pattern $(\mathcal{T},\Theta)$. That is, $h(f) = h((\mathcal{T},\Theta)) =: \inf\{h(g) : g \text{ exhibits } (\mathcal{T},\Theta)\}$.

(C) The topological entropy of such maps f can be computed directly from a pattern invariant called the *path transition matrix* and denoted by $M_P(\mathcal{T},\Theta)$. Actually, we have

$$h(f) = \log\max\{\rho(M_P(\mathcal{T},\Theta)), 1)\}$$

where $\rho(.)$ denotes the spectral radius.

In this setting it remains open the following problem:

[P1] "To minimize the topological entropy among the class of maps which exhibit the pattern $(\mathcal{T},\Theta)$ in a fixed pointed tree".

This is a similar question to the one stated in [2]: "Given a map from a finite subset of the reals into the reals, how do you connect the dots in the graph of the map in order to minimize the topological entropy of the resulting continuous map of the interval". The answer of our problem will be obtained by using a similar approach to the one used in [2].

Next we formalize the necessary notions in order to state and answer precisely [P1]. Given a pointed tree (T, A) and a map $\theta: A \longrightarrow A$, we say that θ is an *action on* (T, A). To any action on (T, A) correspond a pattern $([T,A],[\theta])$. For any continuous tree map f **from** T **into itself** which exhibits a pattern $([T,A],[\theta])$, we shall also say that f is an *extension of* θ and the class of extensions of θ will be denoted by $\Gamma(T,A,\theta)$. The *entropy of an action* θ, denoted by $h(\theta)$, is defined as

$$h(\theta) = \inf\{h(f) : f \in \Gamma(T,A,\theta)\}.$$

Notice that, in general, $h(\theta) \ge h(([T,A],[f|_A]))$. But, if there exists an A-monotone extension f of θ, then $h(\theta) = h(f) = h(([T,A],[f|_A]))$.

Then [P1] is equivalent to the questions:

Q1. Is there some $f \in \Gamma(T,A,\theta)$ such that $h(\theta) = h(f)$?

Q2. How can we determine such a map f?

Notice that if T and θ are such that there exists an A-monotone extension f of θ, then from (B) it follows that $h(\theta) = h(f)$ and this together with (A) gives the answer to [Q1] and [Q2]. In particular, this happens when $V(T) \subset A$ (here $V(T)$ denotes the set of points of valence different from 2), because then an A-monotone map is obtained easily by "connecting-the-dots" in the graph of $f|_A$. Thus the control of the images of the points in $V(T)$ becomes the main goal in the answer to [Q1] and [Q2]. This answer is resumed in the next theorem.

Given a pointed tree (T, A), a nonempty set $Q \subset A$ is said to be a *discrete component* of (T, A) if either $|A| = 1$ or $|Q| > 1$ and there exists a connected component C of $T \setminus A$ such that $Q = A \cap Cl(C)$. Given $B \subset T$ we denote by $\langle B \rangle$ the smallest closed connected set containing B. We say that a vertex b is *outmapped by* f if $f(b) \notin \langle f(Q) \rangle$, where Q is the discrete component such that $b \in \langle Q \rangle$.

Theorem *Given an action theta on* (T, A). *There exists an action* η *on* $(T, A \cup V(T))$ *such that* $\eta|_A = \theta$, *no vertex is outmapped by* η *and* $h(\theta) = h(\eta)$.

Remarks

(1) Since η maps $V(T)$ over $A \cup V(T)$, as we have noticed before, by "connecting-the-dots" in the graph of η we obtain an $(A \cup V(T))$-monotone continuous map $f_\eta : T \longrightarrow T$ which exhibits $([T, A \cup V(T)],[f|_{A\cup V(T)}])$, thus $h(f) = h(\eta) = h(\theta)$ and it minimizes the topological entropy.

(2) From the proof of the theorem it follows that there is a constructive procedure to obtain η consisting on computing the entropies of all the actions on $(T, A \cup V(T))$ coinciding with θ over A and no outmapping vertices.

(3) This is not a question about minimum dynamics. In this context there is no sense to talk about minimum dynamics as it can be done for the interval or for patterns, where the orbits of an A-monotone map f exhibiting a given pattern are realized for any other map exhibiting the same pattern.

References

[1] L. Alsedà, J. Guaschi, J. Los, F. Mañosas and P. Mumbrú, *Canonical representatives for patterns of tree maps*, preprint, 1995.

[2] J. Ashley, E.M. Coven and W. Geller, *Entropy of semipatterns or how to connect the dots to minimize entropy*. Proc. Amer. Math. Soc. **113** (1991), 1115-1121.