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A CHARACTERIZATION OF THE UNIQUELY ERGODIC ENDOMORPHISMS OF THE CIRCLE

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ABSTRACT. We characterize the uniquely ergodic endomorphisms of the circle in terms of their periodic orbits.

Let $f: \mathbf{S}^1 \to \mathbf{S}^1$ be a continuous endomorphism of the circle $\mathbf{S}^1$. Denote by $F: \mathbf{R} \to \mathbf{R}$ its lifting to the universal covering space. Then, for all $x \in \mathbf{R}$, F satisfies $F(x+1) = F(x) + k$ for some $k \in \mathbf{Z}$. The number k is called the *degree* of f (when necessary this number will also be called the degree of F). Denote by $\mathscr{L}$ the class of all liftings of continuous endomorphisms of the circle and by $\mathscr{L}_1$ the class of maps from $\mathscr{L}$ having degree 1.

Let $F \in \mathscr{L}_1$. For each $x \in \mathbf{R}$ we define the *rotation number* of x, denoted by $\rho_F(x)$, as (see [NPT]) $\limsup_{n\to\infty} F^n(x)/n$.

From [I] it follows that the set $L_F = \{\rho_F(x) : x \in \mathbf{R}\}$ is a closed interval (or perhaps a point). This interval is called the *rotation interval* of F. In what follows, whenever L_F degenerates to a point it will be denoted by $\rho(F)$. In such a case we shall talk about the *rotation number of* F. It is well known that when F is nondecreasing, it has degenerate rotation interval.

A continuous endomorphism of the circle f is said to be *uniquely ergodic* if there exists a unique f-invariant probability measure on $\mathbf{S}^1$. From [H] it follows that every homeomorphism of the circle of degree one with irrational rotation number is uniquely ergodic. The aim of this paper is to extend this result to the endomorphisms of the circle. In fact, we shall give necessary and sufficient conditions for an endomorphism of the circle to be uniquely ergodic. Our main result is the following

Theorem. *A circle endomorphism f is uniquely ergodic if and only if it has at most one periodic orbit.*

To prove the above theorem we shall use the following results. For them we have to introduce some notation.

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