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ON THE ROTATION SETS FOR NON–CONTINUOUS CIRCLE MAPS

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Abstract. We give some examples of non-continuous circle maps whose rotation sets lack the good properties they have in the case of continuous maps.

1. Introduction

Let $e\colon \mathbf{R} \longrightarrow \mathbf{S}^1$ be the natural projection defined by $e(x) = \exp(2\pi i x)$. A map $F\colon \mathbf{R} \longrightarrow \mathbf{R}$ is called a **lifting** of a map $f\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ if $e \circ F = f \circ e$ and there is a $d \in \mathbf{Z}$ such that $F(x+1) = F(x) + d$ for all $x \in \mathbf{R}$. This d is called the **degree** of the lifting F. Note that since we do not impose continuity for F, every f has liftings of all degrees.

We shall only consider maps (liftings) of degree one. Following [**4**], a map $F\colon \mathbf{R} \longrightarrow \mathbf{R}$ is called an **old** map (old is a mnemonic for 'degree one lifting') if $F(x+1) = F(x) + 1$ for all $x \in \mathbf{R}$. Clearly, F is an old map if and only if there exists $f\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ such that F is a lifting of f of degree one. It is easy to see that in this case $F(x+k) = F(x) + k$ for all $x \in \mathbf{R}$ and $k \in \mathbf{Z}$, and that iterates of an old map are also old maps.

A point $x \in \mathbf{R}$ is called **periodic** mod 1 of **period** $q \in \mathbf{N}$ and **rotation number** p/q for an old map F if $F^q(x) - x = p \in \mathbf{Z}$ and $F^i(x) - x \notin \mathbf{Z}$ for $i = 1, 2, \dots, q-1$. Similarly, for any $x \in \mathbf{R}$ we can define its **orbit** mod 1 under F as the set $e^{-1}(\mathrm{Orb}_f(e(x))) = \mathrm{Orb}_F(x) + \mathbf{Z} = \cup_{n\ge 0}(F^n(x) + \mathbf{Z})$. The orbit mod 1 of a periodic mod 1 point is also called a **lifted cycle** of F.

For an old map F and a point $x \in \mathbf{R}$ we define $\overline{\rho}_F(x)$ and $\underline{\rho}_F(x)$ (or $\overline{\rho}(x)$ and $\underline{\rho}(x)$ if no confusion seems possible) as

$$\overline{\rho}_F(x) = \limsup_{m\to\infty} \frac{F^m(x) - x}{m} \quad \text{and} \quad \underline{\rho}_F(x) = \liminf_{m\to\infty} \frac{F^m(x) - x}{m}\,.$$

When $\overline{\rho}_F(x) = \underline{\rho}_F(x)$ we write $\rho_F(x)$ or $\rho(x)$ to denote both $\overline{\rho}_F(x)$ and $\underline{\rho}_F(x)$. The number $\rho_F(x)$ (if it exists) is called the **rotation number of x with respect**

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