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PERIODIC ORBITS OF LARGE DIAMETER FOR CIRCLE MAPS

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ABSTRACT. Let f be a continuous circle map and let F be a lifting of f. In this paper we study how the existence of a large orbit for F affects its set of periods. More precisely, we show that, if F is of degree $d \geq 1$ and has a periodic orbit of diameter larger than 1, then F has periodic points of period n for all integers $n \geq 1$, and thus so has f. We also give examples showing that this result does not hold when the degree is nonpositive.

1. INTRODUCTION

One of the basic problems in topological dynamics in one dimension is the characterization of the sets of periods of all periodic points. This problem has its roots and motivation in Sharkovskiĭ's theorem [7]. A lot of effort has been spent in generalizing Sharkovskiĭ's theorem for more and more general classes of continuous self-maps on trees, and finally the characterization of the set of periods of general tree maps is given in [1]. While the set of periods of tree maps can be described with a finite number of orderings, circle maps display new features. The set of periods of a continuous circle map depends on the degree of the map (see, e.g., [2]). Consider a continuous map $f\colon \mathbb{S} \to \mathbb{S}$, where $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, and F is a lifting of f, that is, a continuous map $F\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ \pi = \pi \circ F$, where $\pi\colon \mathbb{R} \to \mathbb{S}$ is the canonical projection (F is uniquely defined up to the addition of an integer). The degree of f (or F) is the integer $d \in \mathbb{Z}$ such that $F(x+1) = F(x) + d$ for all $x \in \mathbb{R}$. If $|d| \geq 2$, then the set of periods is $\mathbb{N}$ (the case $\mathbb{N} \setminus \{2\}$ is also possible when $d = -2$). If $d = 0$ or $d = -1$, then the possible sets of periods are ruled by the Sharkovskiĭ order, as for continuous interval maps. The case $d = 1$ is the most complex one and requires rotation theory. Let F be a lifting of a degree 1 circle map f. The rotation number of a point $x \in \mathbb{R}$ is $\rho_F(x) = \lim_{n\to+\infty} \frac{F^n(x)-x}{n}$, when the limit exists. The set of all rotation numbers is a compact interval $[a, b]$, and the set of periods of f contains

$$\{q \in \mathbb{N} \mid \exists p \in \mathbb{Z},\ a < \frac{p}{q} < b\}.$$

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