Universitat Autònoma de Barcelona

Critical points and periodic orbits of planar differential equations

María Jesús Álvarez Torres

Departament de Matemàtiques
Universitat Autònoma de Barcelona

Director: Armengol Gasull Embid

Puntos críticos y órbitas periódicas de ecuaciones diferenciales en el plano

María Jesús Álvarez Torres

Certifico que la presente memoria ha sido realizada por María Jesús Álvarez Torres bajo mi supervisión y constituye su Tesis para aspirar al grado de Doctor en Matemáticas por la Universitat Autònoma de Barcelona

Dr. Armengol Gasull Embid

A mis padres, mis hermanas y Toni

Contents

Agradecimientos

Y si es difícil resumir el trabajo de estos últimos 6 años en una memoria de 150 páginas, imaginaos lo que supone condensar en poco más de una las vivencias y, sobre todo, las personas que han pasado por mi vida durante este tiempo.

Por supuesto la primera persona que tiene que aparecer en estos agradecimientos es Armengol Gasull porque, aunque sea un tópico, sin él nunca hubiera hecho esta tesis. Le quiero dar las gracias por muchísimas cosas: por su paciencia infinita, por la dedicación incondicional que me ha prestado, por la cantidad de conocimientos que ha puesto a mi disposición,... pero sobre todo por su pasión por las Matemáticas. Es imposible no contagiarse por sus ganas de aprender y eso es lo que más me ha motivado durante todo este tiempo.

Quiero también agradecer a todos los profesores que han trabajado conmigo en esta tesis doctoral: a Xavi Jarque, a Jiang Yu (thank you!), a Héctor Giacomini (¡gracias por invitarme a Tours!) y a Rafel Prohens. Todos me han enseñado mucho y con ellos he aprendido a trabajar de muchas maneras distintas.

Evidentemente también quiero darle las gracias a todo el grupo de *sistemes dinàmics* de la UAB que desde el principio me acogieron como una más. Realmente se puede aprender por ósmosis, y este grupo es un ejemplo de ello: pasear por los pasillos del *Departament de Matemàtiques* de la UAB es una forma de aprender.

Gracias, por supuesto, a todos mis compañeros del *Departament de Matemàtiques i Informàtica* de la UIB, sobre todo a Ana Belén y a Rut que me aguantan todas las mañanas cuando voy a hacerles una visita y con las que he compartido tantos problemas y alegrías.

¿Qué decir del grupo de *sistemes dinàmics* de la UIB? Tomeu y Rafel están absolutamente comprometidos en el empeño de que este grupo que estamos empezando tenga una continuidad y por ello luchan por nosotros de manera incondicional. Muchísimas gracias por estar siempre a mi disposición para ayudarme en lo que haga falta. Y, desde luego, unas gracias enormes a Antonio, no sólo por ser el mejor compañero de despacho del mundo sino por su forma de ser, por sus canciones y conversaciones, que hacen que la vida sea mucho más alegre.

Aún me queda mucha gente que he conocido a través de la Matemáticas y que tienen que estar aquí: Maite (con la que he recorrido medio mundo y con la que espero recorrer el otro medio), Anna, Sara, $X\alpha\rho\alpha$, Toni, Mar...

¿Y qué sería de la vida sin amigos? Además de los que ya he mencionado, quiero darle las gracias especialmente a mis amigas de Palma con las que he compartido los momentos más importantes de mi vida.

He dejado para el final a las personas más importantes: ¡¡¡mi familia!!! Mis padres, mis hermanas, mis cuñados, mi sobrino (y la que está por llegar) y Toni. Todos estos años ellos han *sufrido* esta tesis, con todos los inconvenientes que conlleva el vivir en dos ciudades separadas por el Mar Mediterráneo, pero en todo momento me han apoyado. No existen palabras adecuadas para expresar todo lo que les tengo que decir, así que simplemente les diré GRACIAS.

Introduction

The theory of ordinary differential equations (ODE) began in the 17th century with I. Newton and G. W. Liebnitz. Despite the efforts and advances of a big number of excellent mathematicians in the course of more than three centuries, the fact is that the number of ODE's that can be solved by means of quadratures is very small. Because of that, one of the biggest achievements in the field was the appearance of the qualitative theory of differential equations. This took place during the last quarter of the 19th century and it was mainly developed by H. Poincaré and A. M. Lyapunov. This approach consists in knowing the behavior of the solutions of an ODE (or of a system of ODE's) without computing them explicitly. This knowledge comes only from the properties of the vector fields that define the ODE.

In this work we are going to use the qualitative theory of ordinary differential equations to deal with planar systems of the form

$$\begin{cases} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{cases} \tag{0.1}$$

where $P(x,y)$ and $Q(x,y)$ are analytic functions. We say that (x_0, y_0) is a *critical point* of (0.1) if $P(x_0,y_0) = Q(x_0,y_0) = 0$. We also say that $\gamma(t)$ is a *periodic orbit* of (0.1) if it is a non-constant solution and there exists a real number $T > 0$ such that $\gamma(0) = \gamma(T)$; an isolated periodic orbit is called *limit cycle*.

By the Poincaré-Bendixson Theorem, see [11, 62], we know that the global topological structure of the previous system is completely determined by the configuration of its singular solutions, that is, critical points, periodic orbits and polycycles (sets of solutions formed by critical points and orbits joining them).

Concerning the critical points, we can classify them into hyperbolic, that are the ones for which the determinant of their linear part does not vanish, nilpotent, the ones for which their linear part is not identically zero but the determinant vanishes, and degenerate (or linearly zero), for which their linear part identically vanishes. One of the main problems about critical points is the center-focus problem. It consists in distinguishing when the orbits arriving to a critical point spiral toward or backward it (*i.e.* the origin is a focus) and when there exists a punctured neighborhood of the point where all the orbits are periodic (*i.e.* the origin is a center). For the hyperbolic critical points the previous problem was solved by Poincaré, see [62] and Lyapunov, see [52], by means of the so-called Lyapunov constants, and for the nilpotent ones was solved by Moussu, see [58]. Nevertheless, although for these two kind of critical points the problem is theoretically solved, in the practice it is still very difficult to distinguish between a center and a focus. Moreover, there are still some open problems related to the previous one for hyperbolic and nilpotent singularities, for instance, fixed a class of planar systems, to know how many Lyapunov constants are necessary to solve the center-focus problem. For the degenerate critical points very few is known about the center-focus problem; in fact, even the monodromy problem (the one consisting in distinguishing if there is any orbit tending to the critical point with a concrete slope or if all the orbits turn around it) is very difficult, see for instance [57].

Except for the center-focus problem, the Hartman Theorem completely classifies the hyperbolic critical points. Avoiding this problem, the nilpotent ones have also been classified (see [5]), but the degenerate ones are far from being well-known. This is another one of the challenging problems about critical points and a big number of contributions has been published in the last years, see for instance [10, 34, 37, 54].

Concerning the periodic orbits, the main problem of the subject is the second part of the Hilbert's 16th problem, one of the only two non-solved problems of its famous list, see [42]. It asks for an uniform upper bound, $H(n)$, for the number of limit cycles of all polynomial vector fields of a given degree n. Moreover, the problem also asks for the configuration of these limit cycles. The history of this problem has been very complicated along all the 20th century and although it has been shown that it is very difficult to attack directly, a very large number of partial contributions has been published. The main result in this sense is the Finiteness Theorem, first stated by Dulac (see [26]) with a gap in his proof and finally proved by Il'yashenko and Ecalle in an independent way, see [28, 44]. This theorem proves that for any analytic vector field on the real plane, its number of limit cycles is finite. But it is still not known if the asked uniform upper bound exists or not, even in the simplest case $n = 2$.

In order to estimate $H(n)$, some lower bounds have been given. For instance, it is known that $H(2) \geq 4$, $H(3) \geq 12$ and $H(n) \geq \frac{1}{2}(n+1)^2(\log_2(n+1) - 3) + 3n$, see [20, 47, 49]. The techniques for getting these bounds are mainly two: bifurcations (especially of Hopf type) from a weak focus, and perturbations of integrable systems (problem that involves Abelian integrals).

The previous mentioned two techniques leads us to two important open problems that join critical points and periodic orbits. The first one is determining the number of limit cycles that can bifurcate from a weak focus (called the cyclicity problem) and the second problem is determining how many limit cycles can bifurcate from a continuous of periodic orbits (usually associated with a center).

This present work deals with these two types of special solutions, the critical points and the periodic orbits, and some of the problems stated above. The first part of this Thesis, consisting of three chapters, is devoted to the study of the degenerate critical points, focusing on the nilpotent ones. The second part, consisting of four chapters, is devoted to the study of periodic orbits.

Concerning the structure of the work, each chapter has been written independently of the others, with all the necessary concepts introduced along each one. Because of that, the order of the chapters in the Thesis is not the unique that can be followed, although we think it is the natural one.

In the following we describe the two parts of the Thesis and each one of the chapters, stating the most important result of each one of them. For a more detailed presentation, see the corresponding introductions.

In the first part we review the topological classification of the critical points of a planar system, focusing our attention on the nilpotent ones. In the first two chapters we also deal with the center-focus and cyclicity problems for this kind of points. We summarize each chapter of this first part in the following.

In Chapter 1 we give a new and short proof of the characterization of monodromic nilpotent critical points, first proved by Andreev in [5]. We also introduce for this kind of points the generalized Lyapunov constants as, essentially, the coefficients of the Taylor expansion at the point of the Poincaré map in a suitable coordinates system. We calculate the first one of these coefficients for a monodromic nilpotent critical point in order to determine its stability. Our main theorem of this chapter in the next one. We remark that any planar vector field having a monodromic nilpotent critical point can be easily written in the form (0.2).

Theorem. Consider next system

$$\left\{ \begin{array}{rcl} \dot{x} & = & y(-1 + X_1(x,y)), \\ \dot{y} & = & f(x) + \phi(x)y + Y_0(x,y)y^2, \end{array} \right. \tag{0.2}$$

with the origin being monodromic, and introduce the following notation

$$X_1(x,y) = \sum_{i+j\geq 1} d_{ij}x^iy^j, \qquad f(x) = x^{2n-1} + \sum_{i\geq 0} a_i x^{2n+i},$$
$$\phi(x) = bx^\beta + \sum_{i\geq 1} b_i x^{\beta+i}, \qquad Y_0(x,y) = \sum_{i+j\geq 0} e_{ij}x^iy^j.$$

Then the origin is a stable (resp. unstable) monodromic critical point when $\Delta < 0$ (resp. $\Delta > 0$), where:

(a) $\Delta = b$, if

 (i) either $\beta \in \{n-1, n\}$ and n is *even*,

 (ii) or $\beta = n+1$ and n is *odd*;

(b) $\Delta = (2n+1)b_1 + \big(-3e_{00} + (n-1)d_{10} - (n+2)a_0\big)b$,
if $\beta = n$ and n is odd;

(c) $\Delta = (2n+1)b_1 + \big(-5e_{00} + (n-2)d_{10} - (n+3)a_0\big)b +$
$5\big(d_{11} + 3e_{01} + d_{01}d_{10} + 2d_{01}e_{00}\big)\mathcal{X}_{\{n=2\}}$,
if $\beta = n+1$ and n is *even*.

It is ought to say that one case has resisted this approach (the case when $\beta = n-1$, $b^2 - 4n < 0$ and n odd). Nevertheless, we can apply the results of this chapter to several families of planar systems, obtaining necessary and sufficient conditions for having a center at the origin. Using our method and standard tools for studying degenerate Hopf bifurcations, we also study how many limit cycles bifurcate from the origin in these families. The results of this chapter have appeared in [1].

In Chapter 2 we apply the normal form theory to compute the first generalized Lyapunov constant for monodromic nilpotent singularities in all cases (even the one that has resisted in Chapter 1), and hence to determine the stability of this kind of singularities. We state the main result in the following. We remark that the existence of the analytic change of coordinates given in next result is proved in [68].

Theorem. Consider an analytic planar system having a monodromic nilpotent critical point. Then there exists an analytic change of variables such that it writes as

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= x^{2n-1} + yb(x), \end{cases}$$

being $b(x) \equiv 0$ or $b(x) = \sum_{j\geq\beta} b_j x^j$, with $b_\beta \neq 0$, and satisfying one of the following conditions:

(a) $\beta > n-1$,

(b) $\beta = n-1$, and ${b_\beta}^2 - 4n < 0$.

Furthermore:

(i) If $b(x) = b^o(x) + x^{2\ell}\left(b_{2\ell} + O(x)\right)$, with $b_{2\ell} \neq 0$, being $b^o(x) := (b(x) - b(-x))/2$, then its first significant generalized Lyapunov constant is

(I) $V_{2-n+2\ell} = Kb_{2\ell}$ when either $\beta > n-1$, or $\beta = n-1$ and β is odd. Here $K = K(n, \ell, b_{n-1})$ is a positive constant given in the proof.

(II) $V_1 = \exp\left(\frac{2b_\beta\pi}{n\sqrt{4n-b_\beta^2}}\right)$ when $\beta = 2\ell = n-1$.

(ii) The origin is a center if and only if $b^e(x) := b(x) - b^o(x) \equiv 0$.

We apply our results to several families of planar systems obtaining necessary and sufficient conditions for having a center at the origin. In this chapter we see how the normal form theory can also be applied to generate limit cycles from nilpotent singularities. The results of this chapter have appeared in [2].

In Chapter 3, the last one of the first part, we deal with the called weighted blow up or $k-$blow up technique , see for instance [13]. The usual blow up method used to study the degenerate singularities consists in exploding them to a line (or to a circle) in order to better understand the orbits (if there are any) tending to them, in forward or backward time. The number of blow up's needed to completely desingularize the point depends on its degeneracy. The k–blow up technique consists in doing a bunch of usual blow up's at once. In this chapter we give an algorithmic approach to the study of degenerate singularities by using the k–blow up technique. As an application we prove the Nilpotent Theorem due to Andreev in [5]. The results of this chapter have been done in collaboration with Xavier Jarque.

In the second part of this work we focus our attention on the number of periodic orbits of some families of differential equations on the cylinder of the form

$$\begin{cases} \frac{dr}{dt} = & \alpha(\theta)\, r \quad + \quad \beta(\theta)\, r^{k+1} \quad + \quad \gamma(\theta)\, r^{2k+1}, \\ \frac{d\theta}{dt} = & b(\theta) \quad + \quad c(\theta)\, r^{k}, \end{cases} \tag{0.3}$$

where t is real, $k \in \mathbb{Z}^+$ and all the above functions are real, smooth and $2\pi-$periodic. Note that this family includes the famous Abel equations,

$$\frac{dr}{dt} = A(t)r^3 + B(t)r^2 + C(t)r, \tag{0.4}$$

as well as the polar expression of several types of planar polynomial systems given by the sum of three homogeneous vector fields. Recall also that it is proved in [18] that some planar systems can be transformed, after an adequate change of variables, into Abel equations. Moreover, by the mentioned change of variables, the limit cycles of the planar system are transformed into $2\pi-$periodic orbits of (0.4). Because of that, all the results obtained for the differential equations (0.3) or (0.4) can be applied to some planar systems and thus can be useful to advance in the knowledge of the Hilbert's 16th problem. As we have said at the beginning of the introduction, this huge problem is far from being solved, see [67] and because of that, some simplified versions of it, including the study of the Abel equations, have been proposed in the literature. In the following we summarize the last four chapters and state the most important result of each one of them.

In Chapter 4 we study the Abel equation $\dot{x} = A(t)x^3 + B(t)x^2$ where $A(t)$ and $B(t)$ are trigonometric polynomials. Many authors have worked in the question of finding bounds for the number of isolated periodic orbits of the previous equation, depending only on the degrees of the functions $A(t)$ and $B(t)$, see [45]. Even more, it is not known if this bound exists. In this chapter we study this problem for two special cases: the one in which $B(t)$ has degree one and the one in which is $A(t)$ that has degree one. For both cases, we give a lower bound for the number of isolated periodic orbits. These lower bounds are obtained by studying the perturbations of some Abel equations having a continuum of periodic orbits. We state the main theorem of this chapter in the following.

Theorem. Set $H(n,m)$ for the number of isolated periodic orbits of the Abel equation $\dot{x} = A(t)x^3 + B(t)x^2$, where $A(t)$ and $B(t)$ are two trigonometric polynomials of degrees n and m respectively. Then

(a) $H(n,0) = H(0,m) = 2$,

(b) $H(n,1) \geq n+2$,

(c) $H(1,m) \geq 2m+1$.

We have also studied two concrete Hilbert numbers, $H(3,1)$ and $H(2,2)$, by the method of computing several Lyapunov constants associated with the solution $x=0$. We can summarize our results in the Table 0.1.

$deg(A(t))$ / $deg(B(t))$	0	1	2	3	4	$\cdots$	n
0	2	2	2	2	2	$\cdots$	2
1	2	≥ 3	≥ 4	≥ 7	≥ 6	$\cdots$	$\geq n+2$
2	2	≥ 5	≥ 7				
3	2	≥ 7					
$\vdots$	$\vdots$	$\vdots$					
m	2	$\geq 2m+1$					

Table 0.1: Values of $H(n,m)$ got on Chapter 4. Note that the value of $H(3,1)$ given in the table is bigger than the general bound of $H(n,1)$.

The results of this chapter have been done in collaboration with Armengol Gasull and Jiang Yu.

Motivated for the problem of determining $H(1,1)$, in Chapter 5 we give a new criterion of uniqueness of non-zero periodic orbits of Abel equations of the form $\dot{x} = A(t)x^3 + B(t)x^2$. We prove,

Theorem. Consider the Abel equation $\dot{x} = A(t)x^3 + B(t)x^2$. Assume that there exist two real numbers a and b such that $aA(t)+bB(t)$ does not vanish identically and does not change sign. Then it has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.

This result extends the known criteria about the Abel equation that only refer to the cases where either $A(t)$ or $B(t)$ does not change sign, see [61] and [33]. We apply this new criterion to study the number of periodic solutions of two simple cases of Abel equations: the one where the functions $A(t)$ and $B(t)$ are trigonometric polynomials of degree one and the case where these two functions are polynomials with three monomials. Finally, we give an upper bound for the number of isolated periodic orbits of the general Abel equation (0.4) when $A(t), B(t)$ and $C(t)$ satisfy adequate conditions.

The results of this chapter have been done in collaboration with Armengol Gasull and Héctor Giacomini.

In Chapter 6 we give two criteria for bounding the number of non-contractible limit cycles of the family of differential equations (0.3). We consider the functions

$$\begin{aligned}\mathbf{A}(\theta) &= k(c(\theta)^2\alpha(\theta) + b(\theta)^2\gamma(\theta) - b(\theta)\beta(\theta)c(\theta)),\\ \mathbf{B}(\theta) &= -2kc(\theta)\alpha(\theta) + kb(\theta)\beta(\theta) + c(\theta)b'(\theta) - b(\theta)c'(\theta).\end{aligned}$$

The main theorem of this chapter is the following. The case where $b(\theta)$ does not vanish is also treated in this chapter.

Theorem. Consider system (0.3) on the cylinder and suppose that the function $b(\theta)$ vanishes. Define the functions $\mathbf{A}(\theta)$ and $\mathbf{B}(\theta)$ as above. Then

(a) If one of the functions $\mathbf{A}(\theta)$ or $\mathbf{B}(\theta)$ does not change sign then the system has at most 2 non-contractible limit cycles if k is odd, or 4 non-contractible limit cycles if k is even. Furthermore both bounds are sharp.

(b) If one of the functions $b(\theta)\mathbf{A}(\theta)$ or $b(\theta)\mathbf{B}(\theta)$ does not change sign then the system has at most 3 non-contractible limit cycles if k is odd, or 6 non-contractible limit cycles if k is even.

Finally, in Chapter 7 we study the number of limit cycles of a classical problem of the qualitative theory of planar differential equations: the cubic systems with a symmetry of order 4, see [9, 19]. These systems are invariant under a rotation of $2\pi/4$ radians and can be written as $\dot{z} = \varepsilon z + p\, z^2\bar{z} - \bar{z}^3$, where z is complex, the time is real and $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $p = p_1 + ip_2$ are complex parameters. When they have some critical points at infinity, *i.e.* $|p_2| \leq 1$, it is well-known that they can have at most one (hyperbolic) limit cycle which surrounds the origin. On the other hand when they have no critical points at infinity, *i.e.* $|p_2| > 1$, there are examples exhibiting at least two limit cycles surrounding nine critical points. In this chapter we give two criteria for proving in some cases the uniqueness and hyperbolicity of the limit cycle that surrounds the origin. Our results apply to systems having a limit cycle which surrounds either 1, 5 or 9 critical points, being the origin one of these points. The key point of our approach is the use of Abel equations. We state our main result in the following.

Theorem. (a) Consider equation $\dot{z} = \varepsilon z + p\, z^2\bar{z} - \bar{z}^3$, with $\epsilon_2 \neq 0$, $p_2 > 1$ and define the following four numbers:

$$\Sigma_A^- = \frac{\varepsilon_2 p_1 p_2 - \sqrt{\varepsilon_2^2(p_1^2+p_2^2-1)}}{p_2^2-1}, \qquad \Sigma_A^+ = \frac{\varepsilon_2 p_1 p_2 + \sqrt{\varepsilon_2^2(p_1^2+p_2^2-1)}}{p_2^2-1},$$

$$\Sigma_B^- = \frac{\varepsilon_2 p_1 p_2 - \sqrt{\varepsilon_2^2(p_1^2+9p_2^2-9)}}{2(p_2^2-1)}, \qquad \Sigma_B^+ = \frac{\varepsilon_2 p_1 p_2 + \sqrt{\varepsilon_2^2(p_1^2+9p_2^2-9)}}{2(p_2^2-1)}.$$

If one of the conditions

$$(i) \quad \varepsilon_1 \notin (\Sigma_A^-, \Sigma_A^+), \qquad (ii) \quad \varepsilon_1 \notin (\Sigma_B^-, \Sigma_B^+),$$

is satisfied then it has at most one limit cycle surrounding the origin. Furthermore, when it exists it is hyperbolic.

(b) There are equations $\dot{z} = \varepsilon z + p\, z^2\bar{z} - \bar{z}^3$, under condition *(i)* having exactly one hyperbolic limit cycle surrounding either 1 or 5 critical points and equations under condition *(ii)* having exactly one limit cycle surrounding either 1, 5 or 9 critical points.

The results of these last two chapters have been done in collaboration with Armengol Gasull and Rafel Prohens.