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Uniqueness of limit cycles for polynomial first-order differential equations ☆

M.J. Álvarez [a,*], J.L. Bravo [b], M. Fernández [b]

[a] Departament de Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122, Palma de Mallorca, Spain

[b] Departamento de Matemáticas, Universidad de Extremadura, 06071 Badajoz, Spain

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ABSTRACT

We study the uniqueness of limit cycles (periodic solutions that are isolated in the set of periodic solutions) in the scalar ODE $x' = \sum_{k=1}^{m} a_k \sin^{i_k}(t)\cos^{j_k}(t)x^{n_k}$ in terms of $\{i_k\}$, $\{j_k\}$, $\{n_k\}$. Our main result characterizes, under some additional hypotheses, the exponents $\{i_k\}$, $\{j_k\}$, $\{n_k\}$, such that for any choice of $a_1, \dots, a_m \in \mathbb{R}$ the equation has at most one limit cycle. The obtained results have direct application to rigid planar vector fields, thus, planar systems of the form $x' = y + xR(x, y)$, $y' = -x + yR(x, y)$, where $R(x, y) = \sum_{k=1}^{m} a_k x^{i_k} y^{j_k}$. Concretely, when the set $\{i_k + j_k\colon\ k = 1, \dots, m\}$ has at least three elements (or exactly one) and another technical condition is satisfied, we characterize the exponents $\{i_k\}$, $\{j_k\}$ such that the origin of the rigid system is a center for any choice of $a_1, \dots, a_m \in \mathbb{R}$ and also when there are no limit cycles surrounding the origin for any choice of $a_1, \dots, a_m \in \mathbb{R}$.

1. Introduction

In this work we are going to deal with scalar differential equations of the form

$$x' = \sum_{k=0}^{n} p_k(t)x^k, \tag{1.1}$$

where $t \in \mathbb{R}$. Concretely, we are going to focus in the case in which $p_1, \dots, p_n$ are trigonometric polynomials with real coefficients. Although this assumption may seem very restrictive, we are mainly interested in applying the obtained results to real planar polynomial rigid systems. This type of vector fields, when changing to polar coordinates, is brought into a scalar differential equation of the form (1.1).

Motivated by the previous consideration, we define a periodic solution of Eq. (1.1) as a solution u that is defined in the interval $[0, 2\pi]$ and satisfies $u(0) = u(2\pi)$. A periodic solution is called a limit cycle if it is isolated in the set of periodic solutions. Observe that the analyticity of the solutions with respect to the initial conditions implies that a periodic solution of Eq. (1.1) is not a limit cycle if and only if there exists a neighborhood of it where all the solutions are also periodic.

The previous non-autonomous equation has been widely studied in the literature. One of the main problems concerning it was stated by C. Pugh and it involves finding an upper bound for the number of limit cycles of Eq. (1.1) depending only on the degree n. For $n = 1$, (1.1) is a linear equation, consequently having at most one limit cycle, and for $n = 2$ it is a

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* Corresponding author.
E-mail addresses: chus.alvarez@uib.es (M.J. Álvarez), trinidad@unex.es (J.L. Bravo), ghierro@unex.es (M. Fernández).

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