Nonlinear Analysis 74 (2011) 3694–3702

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Abel-like differential equations with a unique limit cycle

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ARTICLE INFO

Article history:
Received 23 June 2010
Accepted 17 February 2011
Accepting Editor: Ravi Agarwal

MSC:
34C25

Keywords:
Periodic solutions
Abel equation

ABSTRACT

For the family of scalar Abel-like equations $x' = A(t)x^n + bB(t)x^m$, where $A(t) = \sum_{l=1}^{k} a_l \sin^{i_l}(t)\cos^{j_l}(t)$, $B(t) = \sin^{i_b}(t)\cos^{i_b}(t)$, $a_l, b \in \mathbb{R}$, $n, m, k, i_l, j_l, i_b, j_b \in \mathbb{Z}^+$, $n, m \geq 2$, and $k \geq 1$, we characterize the existence of non-trivial limit cycles (periodic solutions that are isolated in the set of periodic solutions different from the trivial $x(t) \equiv 0$) in terms of $n, m, k, i_l, j_l, i_b, j_b$.

1. Introduction

Consider the generalized Abel equation

$$x' = \sum_{l=1}^{n} a_l(t)x^l, \tag{1.1}$$

where $a_l(t),\ 1 \leq l \leq n$ are trigonometric polynomials. We shall say that a periodic solution of (1.1) is a limit cycle if it is isolated in the set of all periodic solutions of (1.1), and that the origin of (1.1) is a centre if there exists a punctured neighbourhood of it such that every orbit contained in that neighbourhood is periodic.

The problem of bounding the number of limit cycles of (1.1) only in terms of n turned out to be impossible since Lins Neto [1] proved that, for a fixed $n \geq 3$, there are functions $a_l(t)$ such that Eq. (1.1) has k limit cycles for all k. Consequently, a new problem arose consisting of bounding the number of limit cycles of (1.1), but now depending on n and on the degrees of $a_l(t),\ 1 \leq l \leq n$. This problem has been proposed by several authors, including Pugh and Lins Neto [1] and Yu. Ilyashenko [2]. Besides its own intrinsic interest, it is strongly related to Hilbert's 16th problem since some planar systems can be transformed into an equation of the form (1.1) (see [1] for more details on the quadratic case and [3–5] for other planar systems).

Nevertheless, bounding the number of limit cycles of Eq. (1.1) has turned out to be very hard, and even in the simple case $n = 3$, $a_1(t) \equiv 0$, and $a_2(t), a_3(t)$ of degree one, the problem is unsolved. There are only some partial results (see, for instance, [6,7]).

In general, the usual way to study this problem is to consider a certain family of generalized Abel equations. The Hilbert number of this family is the maximum of the number of limit cycles that the equations of the family can have.

This problem has been extensively studied, mainly for the classical Abel equation $x' = A(t)x^3 + B(t)x^2 + C(t)x$, obtaining that when $A(t)$ or $B(t)$ have definite sign then the equation has at most three limit cycles (see [8,9]). These results have been

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doi:10.1016/j.na.2011.02.049