A note on a family of non-gravitational central force potentials in dimension one

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Abstract

In this work we study a one-parameter family of differential equations and the different scenarios that arise with the change of parameter. We remark that these are not bifurcations in the usual sense but a wider phenomenon related with changes of continuity or differentiability. We offer an alternative point of view for the study for the motion of a system of two particles which will always move in some fixed line, we take $\mathbb{R}$ for the position space. If we fix the center of mass at the origin, so the system reduces to that of a single particle of unit mass in a central force field. We take the potential energy function $U(x) = |x|^\beta$, where x is the position of the single particle and β some positive real number.

Keywords: Singularities; collisions; non-gravitational interactions.

1. Introduction

In 1981, R. McGehee [1] investigated geometrically the regularization of binary collisions of classical particle systems with non-gravitational interactions. R. McGehee considered the motion near a collision of a particle in the vector field given by the homogeneous potential $U(x) = -|x|^{-\alpha}$, where $x \in \mathbb{R}^2$ is the position of a single particle and α is a positive real number. McGehee showed by appropriate coordinate transformations that the singularity corresponding to a double collision ($x = 0$) is blown up to a collision manifold, after that the time variable is rescaled appropriately, and finally the vector field is extended smoothly to this manifold. He noted that there exits a bifurcation at $\alpha = 2$.

More recently, Xia and Jardón-Kojakhmetov [2] investigated the topological structure of the same system as α varies along the entire real line $\mathbb{R}$. This study

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β	$U_\beta(x)$	$U'_\beta(x)$
$0<\beta<1$	continuous, not Lipschitz	not continuous
$\beta=1$	continuous, Lipschitz	not continuous
$1<\beta<2$	differentiable	continuous, not Lipschitz
$2\leq\beta$	differentiable	differentiable

Table 1: The properties of the potential $U_\beta(x)$ and $U'_\beta(x)$ at $x=0$.

recovers the results of the previous reference and tries to extend the analysis to $\alpha\leq 0$. They use the McGehee techniques without considering that for $\alpha\leq 0$ the potential $U(x)$ is defined at $x=0$ and claimed the occurrence of bifurcations at $\alpha=0$ and $\alpha=2$. The aim of this paper is to show that the study of the flow near the origin is relevant in the global flow. This fact was ignored by Xia and Jardon-Kojakhmetov [2].

In order to fix the ideas we define the concept of bifurcation and remark that the different phase spaces that arise with the change of parameter $\alpha<0$ do not fulfill this definition. Let $X_\alpha(q)$ be a family of vector fields where $q\in D\subset\mathbb{R}^n$ and $\alpha\in\mathbb{R}$, a bifurcation is a change in the topological or analytical behavior of the flow when the parameter α passes a value $\alpha_0\in\mathbb{R}$. We note that D, the domain of the vector field is the same for all the family.

We are going to study the differentiability and continuity properties of the family of central force Hamiltonians and potentials of [1, 2] only for the case $n=1$. We omit proofs because they are straightforward and adds nothing to the paper.

2. Equations of motion

Let us consider the Hamiltonian:

$$H(x,y)=\frac{y^2}{2}-U_\beta(x),$$

where x, $y\in\mathbb{R}$, $\beta>0$ and $U_\beta(x)=|x|^\beta$. The associated Hamiltonian equations are:

$$\dot{x}=y, \qquad \dot{y}=U'_\beta(x)=\beta x|x|^{\beta-2}. \tag{1}$$

Let us remark that β corresponds to $-\alpha$ in [1, 2]. The next proposition gives the continuity and differentiability properties of $U_\beta(x)$ and $U'_\beta(x)$ depending on the value of β.

Proposition 1. *The properties of the potentials $U_\beta(x)=|x|^\beta$ and Hamiltonian vector fields $X(x,y)=(X_1(y),X_2(x))=\left(y,U'_\beta(x)\right)$ where $\beta>0$ are:*

1. *If $x\neq 0$ the potentials $U_\beta(x)$ and their associated Hamiltonian vector fields $X(x,y)$ are differentiable.*
2. *$X_1(-y)=-X_1(y)$, $X_2(-x)=-X_2(x)$ and $U_\beta(x)=U_\beta(-x)$.*

β	$U'_\beta(x)$	$U''_\beta(x)$
$0<\beta<1$	$x>0$: $U'_\beta(x)>0$, $\lim\limits_{x\to 0^+} U'_\beta(x)=\infty$ $x<0$: $U'_\beta(x)<0$, $\lim\limits_{x\to 0^-} U'_\beta(x)=-\infty$	$x\neq 0$: $U''_\beta(x)<0$ $\lim\limits_{x\to 0} U''_\beta(x)=-\infty$
$\beta=1$	$x>0$: $U'_\beta(x)=1$ $x<0$: $U'_\beta(x)=-1$	$x\neq 0$: $U''_\beta(x)=0$
$1<\beta<2$	$U'_\beta(0)=0$, $x>0$: $U'_\beta(x)>0$ $x<0$: $U'_\beta(x)<0$	$x\neq 0$: $U''_\beta(x)>0$ $\lim\limits_{x\to 0} U''_\beta(x)=+\infty$
$2\leq\beta$	$U'_\beta(0)=0$, $x>0$: $U'_\beta(x)>0$ $x<0$: $U'_\beta(x)<0$	$x\neq 0$: $U''_\beta(x)>0$ $U''_\beta(0)=0$

Table 2: The properties of $U'_\beta(x)$ near $x=0$.

3. *The properties of $U_\beta(x)$ and $U'_\beta(x)$ at $x=0$ are given in the Table 1.*
4. *The properties of $U'_\beta(x)$ near $x=0$ are described in the Table 2.*
5. *There is only one minimum of $U_\beta(x)$ at $x=0$: $U(0)=0$.*

Let $\varphi(t)=(x(t),y(t))$ be the solution of the Hamiltonian vector field $X(x,y)$ with initial conditions (x_0,y_0), $H(x_0,y_0)=h$ and $x_0\neq 0$.

The next proposition describes the flow if $x\neq 0$.

Proposition 2. *The solution $\varphi(t)$ satisfies*

1. *If $y(t)>0$ then the component $x(t)$ is increasing, and if $y(t)<0$ then the component $x(t)$ is decreasing.*
2. *If $x(t)>0$ then the component $y(t)$ is increasing, and if $x(t)<0$ then the component $y(t)$ is decreasing.*

3. Analysis of the different scenarios

In this section we analyse the behaviour of $\varphi(t)$ as the parameter β changes.

3.1. $0<\beta\leq 1$

In this case the vector field is not defined in $x=0$.

Proposition 3. *If $x_0<0$ and $h\geq 0$, then there exists $t_1=t_1(x_0,|y_0|)>0$ and one of the following statements hold.*

1. *$h\geq 0$ and $y_0>0$ then $\lim\limits_{t\to t_1^-}\varphi(t)=(0,\sqrt{2h})$.*
2. *$h\geq 0$ and $y_0<0$ then $\lim\limits_{t\to -t_1^+}\varphi(t)=(0,-\sqrt{2h})$.*

If $h<0$ the solution is defined for all $t\in\mathbb{R}$, see Figure 1.

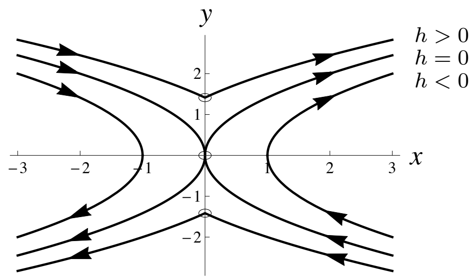

Figure 1: Distinct solutions of (1) with $0 < \beta \leq 1$ depending on the value of h.

There is a similar proposition mutatis mutandis for the right half plane.

We observed that although the solutions in the left half plane and $y > 0$ with energy $h > 0$ has maximal interval $(-\infty, t_1)$, they can be identified with the orbits in the right half plane with the same energy and $y > 0$. With this identification all solutions will be defined for all time. Moreover, they preserve the continuous dependence with respect to initial conditions. The curves just found are not differentiable in the crossing with the y-axis. In a similar way we can work with the orbits with energy $h > 0$ and $y < 0$.

In the case of zero energy there are two possible choices for the continuation of the curves. In one hand, neither one will preserve the continuous dependence with respect to initial conditions, in the other hand one of the resulting curves will be differentiable.

3.2. $1 < \beta < 2$

In this case the vector field is defined in $x = 0$, but is not Lipschitz. Now, when $h > 0$ the solutions that start in $x_0 < 0$ and $y_0 > 0$ pass the line $x = 0$ and enter into the first quadrant in a differential way. In a similar way we can identify the orbits with $y_0 < 0$. For $h = 0$ we have the following result:

Proposition 4. *If $x_0 < 0$ and $h = 0$, then there exists $t_1 = t_1(x_0, |y_0|)$ such that*

1. *if $y_0 > 0$ then $x(t_1) = 0$, $y(t_1) = 0$.*
2. *if $y_0 < 0$ then $x(-t_1) = 0$, $y(-t_1) = 0$.*
3. *$x(t) = y(t) = 0$ is a solution for $t \in \mathbb{R}$.*

There is an analogous proposition for the right half plane.

Let us observe that we can obtain solutions passing through the origin with the following procedure: we start for instance with a solution of the form 1 of Proposition 4 (i.e. a solution with $x_0 < 0$ and $y_0 > 0$), we follow it until it reaches the origin, we stay at the origin as many time as we wish and leave the origin either following a solution of the form 2 of Proposition 4 (i.e. a solution with $x_0 < 0$ and $y_0 < 0$), or following a solution of the analogous of Proposition 4 with $x_0 > 0$ and $y_0 > 0$. Depending on the different choices we have different solutions. See Figure 3.

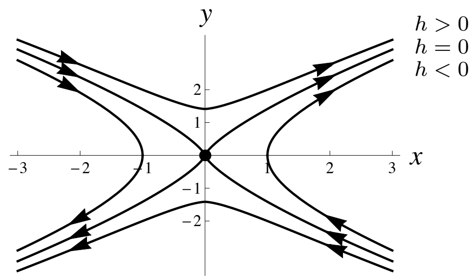

Figure 2: Distinct solution of (1) with $1 < \beta < 2$ depending on the value of h.

Proposition 5. *The initial value problem (1) with $\varphi(0) = (0,0)$ has infinitely many solutions, see Figure 3.*

The Proposition 5 does not contradict the Existence and Uniqueness Theorem of Ordinary Differential Equations since the vector field is not Lipschitz in $(0,0)$, see [3].

3.3. $2 \leq \beta$

This case is differentiable, so the usual results and techniques of Hamiltonian dynamical systems can be used to describe the flow. Clearly the phase space is similar to the one of a hyperbolic fixed point. All solution are unique and defined for all time. In particular the solution with zero energy tends to the origin back or forward but in a infinity time.

4. Conclusions

In a differential equation with parameters defined in a domain it loosely said that a bifurcation occurs in a specific parameter if the behavior of its solutions changes. However, in the Hamiltonian system (1) associated to the potential $U_\beta(x) = |x|^\beta$ where $\beta \in \mathbb{R}$ there are several troubles. First, there are different domains of definition when the parameter β varies. Second, as it is shown the Proposition 1 there are different possibilities in the behavior of the solutions. For example, the solutions with energy $h = 0$ and that start in the second quadrant and if $0 < \beta \leq 1$ reach the origin in a finite time and they can not be continued. If $1 < \beta < 2$ the solutions reach the origin in a finite time and they can be continued in a not unique form. If $2 \leq \beta$ the solutions never reach the origin. However the change of behavior of the flow does not correspond to bifurcations.

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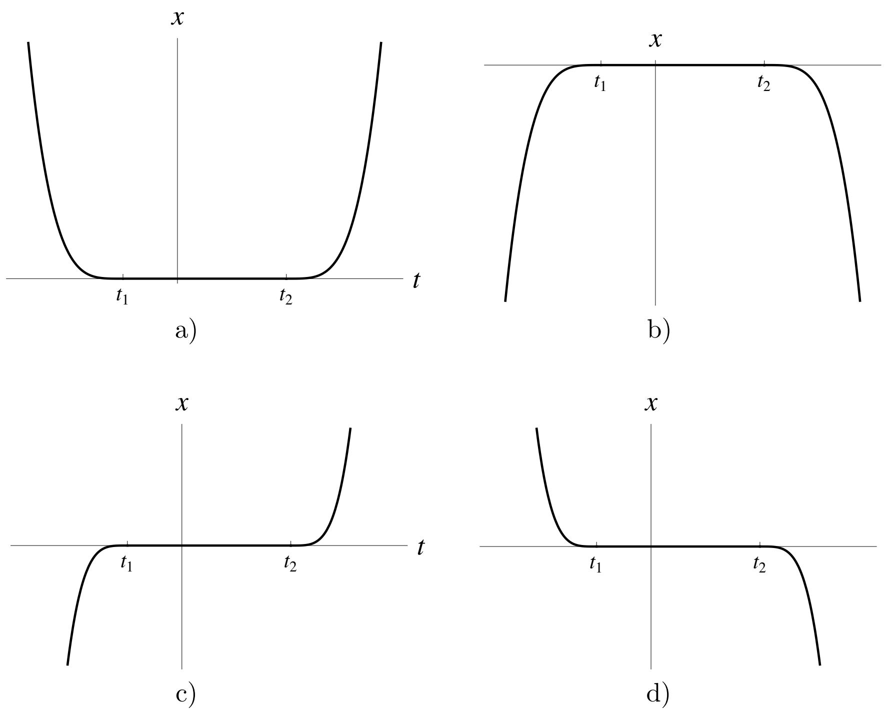

Figure 3: Examples of solutions of (1) with $\varphi(t) = (0, 0)$ for some $t \in \mathbb{R}$. a) Solution with $x(t) > 0$, $y(t) < 0$ for $t < t_1$, $x(t) = 0$, $y(t) = 0$ for $t \in [t_1, t_2]$, and $x(t) > 0$, $y(t) > 0$ for $t > t_2$. b) Solution with $x(t) < 0$, $y(t) > 0$ for $t < t_1$, $x(t) = 0$, $y(t) = 0$ for $t \in [t_1, t_2]$, and $x(t) < 0$, $y(t) < 0$ for $t > t_2$. c) Solution with $x(t) < 0$, $y(t) > 0$ for $t < t_1$, $x(t) = 0$, $y(t) = 0$ for $t \in [t_1, t_2]$, and $x(t) > 0$, $y(t) > 0$ for $t > t_2$. d) Solution with $x(t) > 0$, $y(t) < 0$ for $t < t_1$, $x(t) = 0$, $y(t) = 0$ for $t \in [t_1, t_2]$, and $x(t) < 0$, $y(t) < 0$ for $t > t_2$.

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