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MONODROMY AND STABILITY FOR NILPOTENT CRITICAL POINTS

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We give a new and short proof of the characterization of monodromic nilpotent critical points. We also calculate the first generalized Lyapunov constants in order to solve the stability problem. We apply the results to several families of planar systems obtaining necessary and sufficient conditions for having a center. Our method also allows us to generate limit cycles from the origin.

Keywords: Nilpotent critical point; limit cycle.

1. Introduction

The study of singular points in planar analytic vector fields $\mathbf{X}$, is a problem almost completely solved. In most cases one can know which is the behavior of the solutions in a neighborhood of a singular point. The only case that remains open is the monodromic one. In this case, the orbits turn around the singular point. Even more difficult is to distinguish when the orbits spiral toward or backward the critical point (i.e. the origin is a focus) and when there exists a punctured neighborhood of the point where all the orbits are periodic (i.e. the origin is a center). See [Mañosa, 2002] for a nice survey on this problem.

When the eigenvalues of the matrix $D\mathbf{X}(p)$ at the critical point p are imaginary, we know that the origin is monodromic. If their real part is different from zero then the critical point is a focus, while if their real part is zero the critical point may be a center or a focus. This last case is the classical Lyapunov–Poincaré center problem and was solved by both authors. The study of this problem for a concrete family of differential equations passes through the calculation of the so-called Lyapunov constants, which give the necessary conditions for center. The problem of how many constants are needed to completely solve the stability of a given $\mathbf{X}$ is still open.

When the matrix $D\mathbf{X}(p)$ has its two eigenvalues equal to zero but the matrix is not identically null, it is said that p is a nilpotent critical point. The monodromy problem in this case was solved in [Andreev, 1958] and the center problem in [Moussu, 1982]. Nevertheless, in practice, given an analytic system with a nilpotent monodromic critical point it is not an easy task to know if it is a focus or a

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