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LIMIT CYCLES FOR CUBIC SYSTEMS WITH A SYMMETRY OF ORDER 4 AND WITHOUT INFINITE CRITICAL POINTS

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ABSTRACT. In this paper we study those cubic systems which are invariant under a rotation of $2\pi/4$ radians. They are written as $\dot{z} = \varepsilon z + p\,z^2\bar{z} - \bar{z}^3$, where z is complex, the time is real, and $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $p = p_1 + ip_2$ are complex parameters. When they have some critical points at infinity, i.e. $|p_2| \le 1$, it is well-known that they can have at most one (hyperbolic) limit cycle which surrounds the origin. On the other hand when they have no critical points at infinity, i.e. $|p_2| > 1$, there are examples exhibiting at least two limit cycles surrounding nine critical points. In this paper we give two criteria for proving in some cases uniqueness and hyperbolicity of the limit cycle that surrounds the origin. Our results apply to systems having a limit cycle that surrounds either 1, 5 or 9 critical points, the origin being one of these points. The key point of our approach is the use of Abel equations.

1. Introduction and main results

This paper deals with the equation

$$\dot{z} = \frac{dz}{dt} = \varepsilon z + p\,z^2\bar{z} - \bar{z}^3, \tag{1.1}$$

where z is a point in the complex plane, t is real, $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $p = p_1 + ip_2$ are complex parameters, and $p_2 \ge 0$. In Section 2 we will see that this last inequality can be assumed without loss of generality. We are interested in the number and location of the limit cycles of this equation that surround the origin and, eventually, other critical points.

The previous equation (1.1) is the particular case for $q = 4$ of a general family, the one with a rotational invariance of $2\pi/q$ radians ($q \in \mathbb{N}, q > 2$). In [2] it is proved that the differential equations which are invariant for a rotation of an angle of $2\pi/q$ radians have the normal form

$$\dot{z} = z\mathcal{A}(|z|^2) + \mathcal{B}\bar{z}^{q-1} + O(|z|^{q+1}),\ q > 2, \tag{1.2}$$

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