Nonlinear Analysis 71 (2009) 923–934

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Configurations of critical points in complex polynomial differential equations

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ARTICLE INFO

Article history:
Received 5 February 2008
Accepted 4 November 2008

MSC:
primary 34C05
secondary 34A34
32A10
37C10

Keywords:
Polynomial vector field
Holomorphic vector field
Configuration of singularities
Euler–Jacobi formula
Center type critical points

ABSTRACT

In this work we focus on the configuration (location and stability) of simple critical points of polynomial differential equations of the form $\dot{z} = f(z), z \in \mathbb{C}$. The case where all the critical points are of center type is studied in more detail finding several new center configurations. One of the main tools in our approach is the 1-dimensional Euler–Jacobi formula.

1. Introduction and main results

Consider the first order differential equation

$$\dot{z} := \frac{dz}{dt} = f(z), \quad t \in \mathbb{R}, z \in \mathbb{C}, \tag{1.1}$$

where f is a complex polynomial of degree n. It is well-known that this type of equation presents only three kinds of simple critical points, all of them of index $+1$: foci, centers and nodes (see Theorem 2.1). Furthermore, the centers of this equation are all isochronous and equations of the form (1.1) cannot have limit cycles (see [3,5,12,15,16,19,21]). Indeed this result is even true for differential equations defined by meromorphic functions f, see again [12,15,16].

Our interest lies in showing some connections between the geometrical distribution in $\mathbb{C}$ of the critical points of Eq. (1.1) and their types. Our main motivations are Berlinskiĭ's Theorem, see [2,10] and some results of [5].

Recall that Berlinskiĭ's Theorem classifies the critical points of real quadratic systems depending on their distribution in the plane. It turns out that not all configurations are possible. For instance, if a quadratic system has four critical points located at the vertices of a convex quadrilateral, then a couple of opposite critical points are saddles (index -1) while the other two are anti-saddles (index $+1$). Afterwards, these types of results were extended to cubic systems in [8,23].

In [5,6] similar properties are studied but for holomorphic vector fields. We point out that there is an essential difference between the complex and the planar (real) case because, as we have already said, all the simple critical points in holomorphic

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0362-546X/$ – see front matter
doi:10.1016/j.na.2008.11.018