CONFIGURATIONS OF CRITICAL POINTS IN COMPLEX POLYNOMIAL DIFFERENTIAL EQUATIONS

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Abstract. In this work we focus on the configuration (location and stability) of simple critical points of polynomial differential equations of the form $\dot{z} = f(z)$, $z \in \mathbb{C}$. The case where all the critical points are of center type is studied in more detail finding several new center configurations. One of the main tools in our approach is the 1-dimensional Euler-Jacobi formula.

1. Introduction and main results

Consider the first order differential equation

$$\dot{z} := \frac{dz}{dt} = f(z), \quad t \in \mathbb{R}, \quad z \in \mathbb{C}, \tag{1.1}$$

where f is a complex polynomial of degree n. It is well-known that this type of equation presents only three kind of simple critical points, all of them of index +1: foci, centers and nodes (see Theorem 2.1). Furthermore, the centers of this equation are all isochronous and equations of the form (1.1) can not have limit cycles (see [3, 5, 12, 15, 16, 19, 21]). Indeed this result is even true for differential equations defined by meromorphic functions f, see again [12, 15, 16].

Our interest is to show some connections between the geometrical distribution in $\mathbb{C}$ of the critical points of equation (1.1) and their type. Our main motivations are the Berlinskiĭ's Theorem, see [2, 10] and some results of [5].

Recall that Berlinskiĭ's Theorem classifies the critical points of real quadratic systems depending on their distribution in the plane. It turns out that not all configurations are possible. For instance, if a quadratic system has four critical points located at the vertices of a convex quadrilateral, then a couple of opposite critical points are saddles (index -1) while the other two are anti-saddles (index +1). Afterwards, this type of results was extended to cubic systems in [8] and [23].

In [5, 6] similar properties are studied but for holomorphic vector fields. We point out that there is an essential difference between the complex and the planar (real) case because, as we have already said, all the simple critical points in holomorphic differential equations have index +1. In this work we continue studying some connections between the distribution and the class of the critical points of equation (1.1).

Our main results are the following:

Theorem 1.1. *Consider equation $\dot{z} = f(z)$, with f a complex polynomial and assume that all their critical points are simple. Then:*

1. *If all the critical points are foci, then any geometrical distribution in $\mathbb{C}$ can be achieved.*
2. *Not all the critical points have the same stability.*

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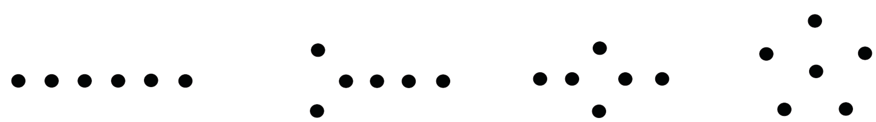
(3) If all the critical points are collinear, then all of them are of the same type.
(4) If all the critical points are collinear and they are not centers, then they have alternated stability.

In Section 4, we give more detailed results that the ones stated in previous theorem when the degree of f is three or four. Next theorem deals with the case of $\dot{z} = f(z)$ having all its critical points of center type.

Theorem 1.2. *Consider equation $\dot{z} = f(z)$, with f a complex polynomial of degree n and assume that all its critical points are centers. Then, for any n, the following configurations exist (see also Figure 1):*
(1) n aligned critical points.
(2) $n-2$ aligned critical points and the other two symmetric with respect to this line. Moreover, the convex hull defined by all the critical points is either:
(i) a triangle ($n \geq 4$), or
(ii) a quadrilateral ($n \geq 5$)
(3) All the critical points, except one, located at the vertices of a regular polygon and the last one at its center ($n \geq 4$).

Moreover, for $n \geq 6$ there are many different configurations to the ones listed above.



Collinear Triangle Quadrilateral Regular polygon

FIGURE 1. Some of the center configurations existing for any $n \geq 5$. In the figure, $n = 6$.

It is already known that when $n \leq 3$ the unique center configuration is the one given in item (1) and when $n = 4$, apart from the aligned one, only the triangular configuration given in item (2i) exists[1], see [5] and [6]. Moreover, in this last case the point inside the triangle is its orthocenter, as it is also proved in [6]. Configurations stated in items (1) and (3) are already known to exist for any n. The above theorem shows that for $n \geq 5$ the situation is much more complicated. As we will see along its proof, the method that we use to give new configurations for $n = 6, 7, 8, 9$ can be easily extended for bigger values of n.

We remark that the knowledge of different center configurations is useful to study different configurations of limit cycles obtained by studying perturbations of (1.1) of the form $\dot{z} = f(z) + \varepsilon g(z, \bar{z}, t)$, see for instance [9, 13].

It is easy to see that Theorem 1.2 also holds when instead of assuming that all the critical points are of center type we assume that all them are nodes, see Remark 2.2. Moreover in this case, the stability of the critical points can also be taken into account when we study the possible configurations. From these facts, we obtain the following result.

Corollary 1.3. *Consider equation $\dot{z} = f(z)$, with f a complex polynomial of degree n and assume that all its critical points are nodes. Then, for any n, the following configurations exist:*

[1]Notice that when $n = 4$ the configuration given in item (3) is a particular case of the one given in item (2i).

(1) n aligned critical points alternating stability.
(2) $n-2$ aligned critical points alternating stability and the other two symmetric with respect to this line and sharing stability. Moreover, the convex hull defined by all the critical points can be either a triangle ($n \geq 4$) or a quadrilateral ($n \geq 5$).
(3) All the critical points, except one, located at the vertices of a regular polygon and with the same stability and the last one at its center with opposite stability ($n \geq 4$).

Moreover, for $n \geq 6$ there are many different configurations to the ones listed above.

In the proof of the above result there are more details about the stabilities of the nodes for all the configurations given in it.

The paper is organized as follows: In Section 2 we give several general results on equation (1.1). In particular, Theorem 1.1.(1) is proved in Proposition 2.3, Theorem 1.1.(2) is proved in Proposition 2.6(d) and Theorem 1.1.(3) and (4) in Proposition 2.7. The results about center and node configurations are presented in Section 3 (proofs of Theorem 1.2 and Corollary 1.3). In Section 4 we study in more detail the concrete cases $n = 3$ and $n = 4$. Finally, in Section 5 we give some examples of phase portraits of this type of equations on the Poincaré disc, specially regarding to the critical points configuration.

2. Preliminary results and proof of Theorem 1.1

Recall that given a holomorphic function f in a point $z = z_0$, the behavior of the solutions of the differential equation $\dot{z} = f(z)$ near this point is well-known, see for instance [4, 11, 15, 16]. In next theorem we recall this result.

Theorem 2.1. *Let f be a holomorphic map at $z = z_0$ and assume that $f(z_0) = 0$. Then the critical point z_0 of the differential equation $\dot{z} = f(z)$ is:*

(a) A center if and only if $0 \neq f'(z_0) \in i\mathbb{R}$. Moreover in this case the center is isochronous and its period is $2\pi/|f'(z_0)|$.
(b) A stable (resp. unstable) node if and only if $f'(z_0) < 0$ (resp. $f'(z_0) > 0$).
(c) A stable (resp. unstable) focus if and only if $\mathrm{Re}(f'(z_0))\,\mathrm{Im}(f'(z_0)) \neq 0$ and $\mathrm{Re}(f'(z_0)) < 0$ (resp. $\mathrm{Re}(f'(z_0)) > 0$).
(d) The union of $2(m-1)$ elliptic sectors if and only if $f(z) = (z - z_0)^m g(z)$ for some $m \in \mathbb{N}$, $m > 1$ and $g(z_0) \neq 0$.

Remark 2.2. *Observe that z_0 is a center of the equation $\dot{z} = f(z)$ if and only if z_0 is a node of equation $\dot{z} = if(z)$. This fact is useful to reduce some proofs of the properties concerning nodes to the analogous but for centers.*

In next result we prove that any distribution of critical points of focus type exists. Regarding their possible stabilities the problem has a different answer. As we will see in Propositions 2.7 and 4.1, not all the situations are possible.

Proposition 2.3. *Given n arbitrary different complex points: $z_1, \ldots, z_n$ there exists an equation of the form (1.1) having all them as critical points of focus type.*

Proof. Consider the equation

$$\dot{z} = f_\alpha(z) := (\alpha + i) \prod_{j=1}^{n} (z - z_j).$$

We only need to prove that there exists $\alpha \in \mathbb{R}$ such that all the critical points are of focus type.

Observe that, for any $k = 1, \ldots, n$,

$$f'_\alpha(z_k) = (\alpha + i) \prod_{\substack{j=1,\\ j \neq k}}^{n} (z_k - z_j) =: (\alpha + i) A_k,$$

with $0 \neq A_k \in \mathbb{C}$. By Theorem 2.1.(c) it suffices to check for any $k = 1, \ldots, n$, $\operatorname{Re}(f'_\alpha(z_k)) \operatorname{Im}(f'_\alpha(z_k)) \neq 0$. Note that

$$f'_\alpha(z_k) = (\alpha + i)A_k = \alpha \operatorname{Re}(A_k) - \operatorname{Im}(A_k) + i(\alpha \operatorname{Im}(A_k) + \operatorname{Re}(A_k)).$$

Consider the polynomial

$$Q(\alpha) := \prod_{k=1}^{n} (\alpha \operatorname{Re}(A_k) - \operatorname{Im}(A_k))(\alpha \operatorname{Im}(A_k) + \operatorname{Re}(A_k)) \not\equiv 0.$$

By taking a value $\alpha^* \in \mathbb{R}$ such that $Q(\alpha^*) \neq 0$ the result follows. $\square$

In the proof of Berlinskiĭ's Theorem given in [8], as well as in the paper [14], one of the key ingredients to relate the configuration of the critical points with their types is the 2-dimensional version of the well-known Euler-Jacobi formula. Its classical m-dimensional version goes to [17] and more modern formulations appear in [1, 18, 22]. Although in our paper we only will use the 1-dimensional version, for the sake of completeness, we state below an m-dimensional version and a 1-dimensional version, in a suitable notation for our purposes. Again by completeness we include a sketch of a proof of the easiest case $m = 1$.

Theorem 2.4. *(Euler-Jacobi Formula) Let $f = (f_1, \ldots, f_m)$ be a polynomial map from $\mathbb{C}^m$ to $\mathbb{C}^m$ and assume the set $V(f) := \{f_1 = \ldots = f_m = 0\} \subset \mathbb{C}^m$ has cardinal* $\deg(f_1) \cdots \deg(f_m)$. *Then for any complex polynomial g such that* $\deg(g) < \deg(\det(Df(w))) = \sum_{j=1}^{m} \deg(f_j) - m$,

$$\sum_{w \in V(f)} \frac{g(w)}{\det(Df(w))} = 0,$$

where Df stands for the differential of the map f.

Theorem 2.5. *(1-dimensional Euler-Jacobi Formula) Let f be a polynomial map from $\mathbb{C}$ to $\mathbb{C}$ and assume the set $V(f) := \{f = 0\} \subset \mathbb{C}$ has cardinal $n :=$* $\deg(f)$. *Then for any complex polynomial g such that* $\deg(g) < \deg(f') = \deg(f) - 1$,

$$\sum_{w \in V(f)} \frac{g(w)}{f'(w)} = \sum_{k=1}^{n} \frac{g(w_k)}{f'(w_k)} = 0,$$

where $V(f) = \{w_1, w_2, \ldots, w_n\}$.

Sketch of the proof of Theorem 2.5. Consider the meromorphic map $h(z) = \frac{g(z)}{f(z)}$ and let Γ_r be a circumference of radius r surrounding all the zeroes of f. Because of the Residues Theorem,

$$\int_{\Gamma_r} h(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(h(z), w_k) = 2\pi i \sum_{k=1}^{n} \frac{g(w_k)}{f'(w_k)}.$$

On the other hand,

$$\Big| \int_{|z|=r} \frac{g(z)}{f(z)}\, dz \Big| = \Big| \int_0^{2\pi} \frac{g(re^{i\theta})}{f(re^{i\theta})} rie^{i\theta} d\theta \Big| \leq \int_0^{2\pi} \frac{|g(re^{i\theta})r|}{|f(re^{i\theta})|} d\theta.$$

If we let r tend to infinity, since $\deg(g) + 1 < \deg(f)$, the above expression goes to zero and the result follows. $\square$

By using the 1-dimensional Euler-Jacobi formula we obtain next result. Indeed items (a) and (c) are already proved in [5]. Recall that if $z = z_k$ is a center of equation (1.1) then it is isochronous and its period is $T_k = 2\pi/|f'(z_k)|$. We define its *signed period* as the real number $\tau_k := 2\pi i/f'(z_k)$. Notice that $T_k = |\tau_k|$.

Proposition 2.6. *Consider equation (1.1) and assume that all its critical points, z_k for $k = 1, \dots, n$, are simple. Then:*

(a) If $z_1, \dots, z_{n-1}$ are centers, then z_n is also a center.
(b) If each of the critical points $z = z_k$ is a center and its corresponding signed period is τ_k, then $\sum_{k=1}^{n} \tau_k = 0$.
(c) If $z_1, \dots, z_{n-1}$ are nodes, then z_n is also a node.
(d) If not all the critical points are centers, then there exist at least two of them that have different stability.

Proof. (a-b) As we are assuming that the critical points $z_1, \dots, z_{n-1}$ are centers, we know that $f'(z_k) = ib_k$, with $0 \neq b_k \in I\!\!R$ for $k = 1, \dots, n-1$. Then, we apply the Euler-Jacobi formula (see Theorem 2.5) with $g(z) \equiv 1$ and, for some $\beta \in I\!\!R$, we get

$$\sum_{k=1}^{n} \frac{1}{f'(z_k)} = \sum_{k=1}^{n-1} \frac{1}{ib_k} + \frac{1}{f'(z_n)} = i\beta + \frac{1}{f'(z_n)} = 0 \Rightarrow f'(z_n) \in iI\!\!R.$$

Hence, by Theorem 2.1.(a), z_n is a center and (a) follows. The above formula also implies that $\sum_{k=1}^{n} \tau_k = 0$ and therefore proves item (b).

(c) The proof reduces to the previous one by using Remark 2.2.

(d) We apply again the Euler-Jacobi formula with $g(z) \equiv 1$. If, for $k = 1, \dots, n$, we denote $f'(z_k) = \alpha_k + i\beta_k$, then it turns out that

$$\sum_{k=1}^{n} \frac{g(z_k)}{f'(z_k)} = \sum_{k=1}^{n} \frac{\alpha_k - i\beta_k}{\alpha_k^2 + \beta_k^2} = 0.$$

Taking the real part in former equality, we get

$$\sum_{k=1}^{n} \frac{\alpha_k}{\alpha_k^2 + \beta_k^2} = 0.$$

As we are assuming that not all of the critical points are centers, then not all the non-zero α_k have the same sign and by Theorem 2.1.(b) and (c) the result follows. □

In next two results we study the type of critical points that equation (1.1) can have when some of them are aligned.

Proposition 2.7. *Consider equation (1.1) with n simple critical points and assume that for some $k \geq 0$ it has $n - 2k$ of them located on a straight line $\mathcal{L}$ and the other $2k$ points symmetric with respect to this line. Then*

(a) All the points on $\mathcal{L}$ are of the same type and if they are not centers, then they have alternated stability.
(b) If all the points on $\mathcal{L}$ are of center type then each pair of symmetric points with respect to $\mathcal{L}$ is formed by two points of the same type and if they are not centers then they have opposite stabilities.
(c) If all the points on $\mathcal{L}$ are of node type then each pair of symmetric points with respect to $\mathcal{L}$ is formed by two points of the same type and if they are not centers then they have the same stability.

Proof. (a) Without loss of generality, we can assume that all the critical points $z_j = a_j$ for $j = 1, \dots, n-2k$ are real and the other $2k$ appear in couples of complex conjugated numbers, w_j and $\overline{w_j}$, $j = 1, \dots, k$. Then, equation (1.1) can be written in the following way:

$$\dot{z} = (\alpha + i\beta) \prod_{j=1}^{n-2k} (z - a_j) \prod_{j=1}^{k} (z - w_j)(z - \overline{w_j}),$$

with $a_j < a_{j+1}$, for all $j = 1, \ldots, n-2k-1$, all the w_j are different and $\mathrm{Im}(w_j) \neq 0$, $j = 1, \ldots k$.

We define

$$g(z) = \prod_{j=3}^{n-2k} (z - a_j) \prod_{j=1}^{k} (z - w_j)(z - \overline{w_j}),$$

and apply the Euler-Jacobi formula with this function $g(z)$, see Theorem 2.5. It turns out that

$$\sum_{j=1}^{n} \frac{g(z_j)}{f'(z_j)} = \frac{g(a_1)}{f'(a_1)} + \frac{g(a_2)}{f'(a_2)} = 0. \tag{2.1}$$

As $g(a_j) \in I\!R$, for $j = 1, 2$, then

- $f'(a_1) \in I\!R$ if and only if $f'(a_2) \in I\!R$, and then a_1 is a node if and only if a_2 is also a node.
- $f'(a_1) \in iI\!R$ if and only if $f'(a_2) \in iI\!R$, and then a_1 is a center if and only if a_2 is also a center.
- a_1 is a focus if and only if a_2 is also a focus.

If we repeat an analogous reasoning but changing a_2 by either $a_3, \ldots, a_{n-2k}$, then we get that all the critical points on $\mathcal{L}$ are of the same type.

To prove the second assertion of this item, we assume that all the critical points are either of node or focus type. As $\mathrm{sgn}(g(a_1)) = \mathrm{sgn}(g(a_2))$, taking real part in the expression (2.1), we get that $\mathrm{sgn}(\mathrm{Re}(f'(a_1))) \neq \mathrm{sgn}(\mathrm{Re}(f'(a_2)))$ and, consequently, by Theorem 2.1.(b) and (c), the points a_1 and a_2 have opposite stability.

An analogous reasoning can be done with any pair of consecutive critical points on $\mathcal{L}$ and, as a consequence, we get item (a).

(b-c) Fix some j, $1 \leq j \leq k$. Then it is easy to check that

$$f'(\overline{w_j}) = \frac{\alpha + i\beta}{\alpha - i\beta} f'(w_j). \tag{2.2}$$

If we assume that all the points on $\mathcal{L}$ are of center type we obtain that $\alpha = 0$ and thus equation (2.2) tells us that $f'(\overline{w_j}) = -f'(w_j)$. From this equality and Theorem 2.1 we get statement (b). The proof of item (c) follows the same steps, but in this case $\beta = 0$ and then $f'(\overline{w_j}) = f'(w_j)$ for each $j = 1, \ldots, k$. □

Remark 2.8. Notice that when all the points on $\mathcal{L}$ are of focus type, a similar statement to (b) or (c) does not hold as we can see in next example. Consider equation

$$\dot{z} = (1+i)z(z-1)(z-2)(z-(x+i))(z-(x-i)),$$

being x any of the solutions of $-2x^3 + 14x - 8 = 0$. We note that this equation has 3 aligned focus, a node and a center as its critical points.

Proposition 2.9. *Consider equation (1.1). Assume that it has n simple critical points, $n \geq 3$, and $n-1$ of them are collinear. Then the following statements hold:*

(a) If 2 or more of the collinear critical points are centers, then the rest of the critical points are as well centers and aligned.

(b) If 2 or more of the collinear critical points are nodes, then the rest of the critical points are as well nodes and aligned.

Proof. Without loss of generality, equation (1.1) can be written as

$$\dot{z} = (\alpha + i\beta)(z - z_n) \prod_{k=1}^{n-1} (z - a_k), \; a_k \in I\!R. \tag{2.3}$$

Assume that two of the aligned critical points are centers, say a_1 and a_2, and define the following function

$$g(z) = (z - z_n) \prod_{k=3}^{n-1} (z - a_k).$$

By applying the Euler-Jacobi formula with the above function g we get

$$\begin{aligned}\sum_{k=1}^{n} \frac{g(z_k)}{f'(z_k)} &= \frac{\prod_{k=3}^{n-1}(a_1 - a_k)}{i\beta_1}(a_1 - z_n) + \frac{\prod_{k=3}^{n-1}(a_2 - a_k)}{i\beta_2}(a_2 - z_n) = \\ &=: \; iA_1(a_1 - z_n) + iA_2(a_2 - z_n) = 0,\end{aligned}$$

where $A_1, A_2 \in I\!R$. The only way for this quantity to be equal zero is when $(a_1 - z_n)$ and $(a_2 - z_n)$ are proportional. Hence, z_n must be aligned with a_1 and a_2. Consequently, $z_n \in I\!R$.

Applying now Proposition 2.7 with $k = 0$, since the n critical points are collinear and two of them are centers, all the other ones are also centers.

The proof of the second statement can be reduced to the former one by using Remark 2.2. □

As we will see in the example given in equation (3.1) of next section, Proposition 2.9 can not be generalized when it is assumed that less than $n-1$ critical points are aligned, even in the case when all the critical points are of the same type.

3. Center and node configurations

In order to study the relationship between location and stability of the centers in the complex plane, we introduce the definition of center configuration for equation (1.1).

Definition 3.1. Consider equation (1.1) with n critical points of center type, $\mathcal{C}_1 := \{z_1, z_2, ...z_n\}$. We will say that it has the center configuration $\langle k_1, k_2, \ldots, k_m \rangle$ if the following holds: the boundary of the convex hull of the n points contains exactly k_1 centers. After removing these points we obtain another set $\mathcal{C}_2$ containing $n - k_1$ centers. Then k_2 is the number of centers that belong to the boundary of the convex hull of $\mathcal{C}_2$. By continuing this procedure we obtain $k_3, k_4, \ldots$ Clearly, we stop when $k_1 + k_2 + \cdots + k_m = n$. See Figure 2 for some examples when $n = 9$. Moreover when the last k_m points are aligned we write this number in boldface font.

Observe that when $k_m \le 2$ these points are always aligned, nevertheless for aesthetical reasons we will not use the boldface font for them.

In the above notation, items (1), (2) and (3) of Theorem 1.2, proved below, imply that the following center configurations always exist: $\langle \mathbf{n} \rangle$ and $\langle n-1, 1 \rangle$ for any $n \ge 2$, $\langle 3, \mathbf{n-3} \rangle$ for $n \ge 4$ and $\langle 4, \mathbf{n-4} \rangle$ for $n \ge 5$.

Proof of Theorem 1.2. (1) An example realizing the aligned configuration is,

$$\dot{z} = i(z - a_1)(z - a_2) \cdots (z - a_n),$$

where $a_k \in I\!R$, for all $k = 1, \ldots, n$.

(2) Fix $a_1 < a_2 < \ldots < a_{n-2}$. Consider equation

$$\dot{z} = f(z) := i(z - a_1)(z - a_2) \cdots (z - a_{n-2})(z - \overline{z}_n)(z - z_n), \tag{3.1}$$

where $z_n = x + i \in \mathbb{C}$ and $x \in I\!R$. We want to find conditions on x to ensure that all the critical points are centers. Recall that, by Theorem 2.1.(a), it suffices to show that f' at them takes pure imaginary values. It is easy to check that $f'(a_k) \in iI\!R$, for all

n=6	n=7	n=8	n=9
$\langle \mathbf{6} \rangle$	$\langle \mathbf{7} \rangle$	$\langle \mathbf{8} \rangle$	$\langle \mathbf{9} \rangle$
$\langle 5,1 \rangle$	$\langle 6,1 \rangle$	$\langle 7,1 \rangle$	$\langle 8,1 \rangle$
$\langle 4,2 \rangle$	$\langle 5,2 \rangle$	$\langle 6,2 \rangle$	$\langle 7,2 \rangle$
$\langle 3,3 \rangle$	$\langle 4,3 \rangle$	$\langle 5,3 \rangle$	$\langle 6,3 \rangle$
$\langle 3,\mathbf{3} \rangle$	$\langle 4,\mathbf{3} \rangle$	$\langle 5,\mathbf{3} \rangle$	$\langle 6,\mathbf{3} \rangle$
	$\langle 3,\mathbf{4} \rangle$	$\langle 4,\mathbf{4} \rangle$	$\langle 5,\mathbf{4} \rangle$
			$\langle 5,4 \rangle$
		$\langle 3,\mathbf{5} \rangle$	$\langle 4,\mathbf{5} \rangle$
		$\langle 4,3,1 \rangle$	$\langle 5,3,1 \rangle$
		$\langle 3,4,1 \rangle$	$\langle 4,3,2 \rangle$
		$\langle 3,3,2 \rangle$	$\langle 3,5,1 \rangle$

TABLE 1. Center configurations found for $6 \le n \le 9$.

$k = 1,\dots,n-2$. To compute $f'(x \pm i)$, let us write $x \pm i - a_k = r_k(x)e^{\pm i\theta_k(x)}$. With this notation, the conditions $f'(x \pm i) \in i\mathbb{R}$ are equivalent to the set of equations

$$\sum_{k=1}^{n-2} \theta_k(x) = (2\ell+1)\frac{\pi}{2}, \quad \ell \in \mathbb{Z}. \tag{3.2}$$

Since $h(x) := \sum_{k=1}^{n-2} \theta_k(x)$ is a monotonous function that varies between 0 and $(n-2)\pi$, it is not difficult to see that there are exactly $n-2$ values of ℓ, $\ell = 0,1,\dots,n-3$ for which the corresponding equation (3.2) has a unique solution, say x_0, $x_1,\dots,x_{n-3}$, respectively. Moreover, it is easy to see that for $n \ge 5$ we can choose $a_1,\dots,a_{n-2}$ in such a way that the solutions x_0, x_k and x_{n-3}, for some k, $0 < k < n-3$, satisfy $x_{n-3} \in (-\infty, a_1]$, $x_k \in (a_1, a_{n-2})$ and $x_0 \in [a_{n-2}, +\infty)$. Notice that the first and last solutions give rise to a triangular configuration and prove (2i), while the middle one exhibits a quadrilateral configuration, proving (2ii). For instance, by taking $n=5$ and the system

$$\dot z = iz(z-1)(z-2)(z-(x-i))(z-(x+i)),$$

these center configurations appear when $x \in \{-1,1,3\}$.

(3) The configuration corresponding to this item is given by equation

$$\dot z = z\left(z^{n-1} - \frac{i}{n-1}\right). \tag{3.3}$$

To end the proof we list in Table 1 the center configurations for $n = 6,7,8$ and 9 that we have obtained. In this table we use the notation introduced at the beginning of this section. To get it we follow similar ideas to the ones used to prove item (2). As example, we give the computations for some of them.

Configurations $\langle 5,1 \rangle$ *and* $\langle 4,2 \rangle$ *for* $n=6$: Consider

$$\dot z = iz(z-1)(z-(x-i))(z-(x+i))(z-(y-i))(z-(y+i)),$$

with x and y arbitrary real numbers. It is clear that for all values of x and y the differential equation has a center at the points $z=0$ and $z=1$. By imposing that the other points are four different centers we obtain that x and y, with $x \ne y$, have to be the real solutions of the system of equations

$$\begin{cases} x^3 - x^2y - x^2 + xy - 5x + y + 2 = 0, \\ xy^2 - y^3 + y^2 - xy + 5y - x - 2 = 0. \end{cases} \tag{3.4}$$

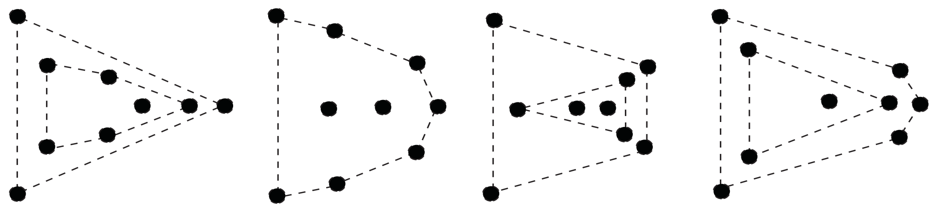


FIGURE 2. Different center configurations for family (3.5). From left to right, the configurations are: $\langle 3,5,1\rangle$, $\langle 7,2\rangle$, $\langle 4,3,2\rangle$ and $\langle 5,3,1\rangle$.

Their solutions are $(x,y) \in \{(0,-2),(-2,0),(1,3),(3,1),((1+\sqrt{13})/2,(1-\sqrt{13})/2),(1-\sqrt{13})/2,(1+\sqrt{13})/2)\}$, and their corresponding configurations are $\langle 5,1\rangle$ and $\langle 4,2\rangle$.

Configurations $\langle 3,5,1\rangle$, $\langle 7,2\rangle$, $\langle 4,3,2\rangle$ *and* $\langle 5,3,1\rangle$ *for* $n=9$: Consider

$$\dot z = iz(z-1)(z-2)(z-(x-i))(z-(x+i))(z-(y-2i))(z-(y+2i))(z-(t-3i))(z-(t+3i)), \tag{3.5}$$

with (x,y,t) taking the values

$$\{(-0.329..,-3.897..,-4.905..),(1.417..,0.123..,-2.694..),\\ (2.830..,8.943..,-9.996..),(1.324..,-3.231..,-4.232..)\}.$$

These numbers have been found by solving (numerically) a system of 3 equations with 3 unknowns, similar to (3.4). Their corresponding configurations are: $\langle 3,5,1\rangle$, $\langle 7,2\rangle$, $\langle 4,3,2\rangle$ and $\langle 5,3,1\rangle$, see Figure 2. $\square$

In order to simplify the last part of the proof of Corollary 1.3, we introduce the notion of signed node configuration. Essentially this definition is the same that the one of center configuration, but, in each level of critical points, instead of only counting the number of points we write a plus (resp. minus) for each repulsive (resp. attractive) node, following the counterclockwise orientation. When the points of the last level are aligned we follow the usual order and we type the signs in boldface font. For instance the configurations $\langle 5,3\rangle$ and $\langle 3,4,1\rangle$ showed in Figure 3 correspond to the signed configurations $\langle(-,-,-,+,+),(+,-,-)\rangle$ and $\langle(-,+,+),(-,-,-,-),+\rangle$.

Proof of Corollary 1.3. By Remark 2.2, it is clear that all the configurations existing for equation $\dot z = f(z)$ with all the critical points of center type also exist for some equation of the form (1.1) but with all the points of node type. Indeed the corresponding equation is $\dot z = if(z)$. Notice also that the stability of the nodes of this new equation is given by the orientation of the centers of $\dot z = f(z)$. More concretely, if $z=z_0$ is a center of $\dot z = f(z)$ for which the orbits turn counterclockwise (resp. clockwise) then the same point is an attracting (resp. a repelling) node for $\dot z = if(z)$. Thus our results on the stability of the node configurations can also be interpreted as refinements of the center configurations where the direction of rotation of the periodic orbits surrounding the centers is also taken into account.

The results about the stability of the node configurations obtained from the center ones and stated in items (1) and (2) are a direct consequence of Proposition 2.7. The proof of statement (3) is a consequence of Theorem 2.1. Note that this configuration is given by

$$\dot z = f(z) := iz\left(z^{n-1} - \frac{i}{n-1}\right),$$

see equation (3.3). Since $f'(0) = 1/(n-1)$ and for each $(n-1)$-root of $i/(n-1)$, w, it holds that $f'(w) = -1$, we have that the origin is a repelling node, while the points at the vertices of the regular polygon are attracting ones, as we wanted to see.

The proof of the last part of the corollary uses the equation

$$\dot{z} = -(z-a_1)(z-a_2)\cdots(z-a_{n-2})(z-\overline{z}_n)(z-z_n),$$

where $z_n = x+i \in \mathbb{C}$ and $x \in I\!R$, see also equation (3.1). Its node configurations are like the center configurations $\langle 3, \mathbf{n-3}\rangle$ for $n \geq 4$ and $\langle 4, \mathbf{n-4}\rangle$ for $n \geq 5$. It is not difficult to check that their corresponding signed node configurations are:

- $\langle(+,+,+),(\boldsymbol{-},\boldsymbol{+},\boldsymbol{-},\ldots,\boldsymbol{+},\boldsymbol{-})\rangle$ when $n \geq 4$ is even,
- $\langle(+,+,-),(\boldsymbol{-},\boldsymbol{+},\boldsymbol{-},\ldots,\boldsymbol{-},\boldsymbol{+})\rangle$ when $n \geq 5$ is odd,
- $\langle(+,+,+,-),(\boldsymbol{+},\boldsymbol{-},\boldsymbol{+},\ldots,\boldsymbol{+},\boldsymbol{-})\rangle$ when $n \geq 6$ is even,
- $\langle(-,-,-,-),(\boldsymbol{+},\boldsymbol{-},\boldsymbol{+},\ldots,\boldsymbol{-},\boldsymbol{+})\rangle$ when $n \geq 5$ is odd,
- $\langle(-,+,-,+),(\boldsymbol{+},\boldsymbol{-},\boldsymbol{+},\ldots,\boldsymbol{-},\boldsymbol{+})\rangle$ when $n \geq 7$ is odd,

Moreover all the configurations interchanging the minus signs by the plus signs and viceversa, are also clearly realizable.

Notice also that the signed configuration corresponding to the configuration $\langle n-1, 1\rangle$ of item (3) of Corollary 1.3 is $\langle(-,-,\ldots,-,-),+\rangle$.

Clearly the above description improves the statement of Corollary 1.3.

Similarly we could obtain all the signed configurations corresponding to the ones given in Table 1. Instead of this, we have preferred to give only a couple of examples. We present them in Figure 3. □

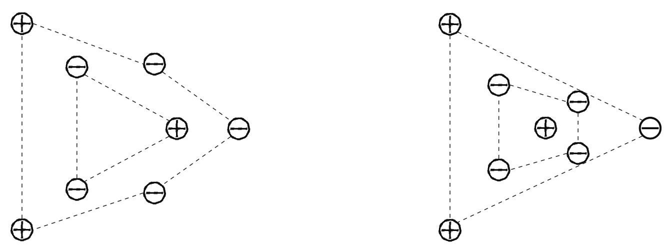


FIGURE 3. Examples of the signed node configurations $\langle 5, 3\rangle$ and $\langle 3, 4, 1\rangle$ existing for $n = 8$. In this figure each of the plus or minus signs represents a repelling or attracting node, respectively.

Remark 3.2. Notice that perturbing the signed node configurations we can obtain many different configurations with all the critical points of focus type and for which we know as well their stabilities, because these stabilities are inherited for the foci from the nodes.

4. Case of f having degree 3 or 4

In this section we study in more detail the cases in which the polynomial f in equation (1.1) has degree 3 or 4 and all the singularities are simple.

In the case of f having degree 3 it is not difficult to study which configurations of critical points are possible. For instance, from Proposition 2.9, if the three singularities are centers (or nodes) they have to be aligned. On the other hand if they are at the vertices of a triangle, by Proposition 2.6, not all of them have the same stability.

From Theorem 1.2 we know which distributions can adopt three or four critical points when they are centers (or nodes). Also, as we have previously proved in Proposition 2.3, when the critical points are foci, any location can be achieved. But, concerning stabilities not any configuration is allowed. For instance, as we have proved in Proposition 2.7 with $k = 0$, if the foci are collinear then they must have alternated stabilities.

In the following, we introduce some names for the geometrical distributions of four points and study which ones are possible when f has degree 4 and all the critical points of equation (1.1) are of focus type. Notice that because there are only 4 critical points, the description given below is more precise that the one used in Section 3.

Clearly, four points in the plane can take four different geometrical distributions (see Figure 4), defined as:

- Collinear: all of them are aligned,
- Triangle: three on the vertices of a triangle and the other one inside,
- Border: three aligned and the other one not,
- Quadrilateral: on the vertices of a quadrilateral.

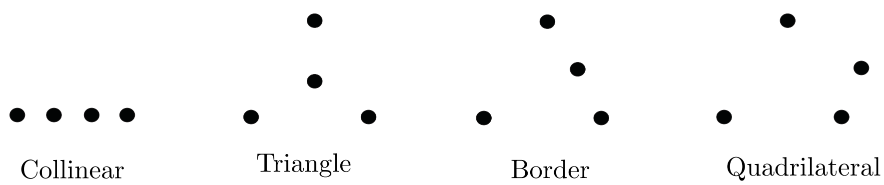


FIGURE 4. The four possible geometrical distributions of critical points for equation (1.1) when $n = 4$.

In order to do that with a simple notation, if a critical point of focus type is stable, then we will denote its stability by the "$-$" sign while if it is unstable, then we will use "$+$" sign. To denote the stability of all the critical points jointly, we will follow the following notation. First, we will use a letter to denote which of the four configurations are we dealing with: c for collinear, q for quadrilateral, t for triangle and b for border; followed by a colon. After that, we will put four signs (plus or minus) indicating the stability of each one of the four foci. More concretely, if $z_1, \ldots, z_4$ are the four simple critical points of equation (1.1) with $n = 4$ and, we define

$$s_i = \begin{cases} + & \text{if } \operatorname{Re}(f'(z_i))) > 0, \\ - & \text{if } \operatorname{Re}(f'(z_i))) < 0, \end{cases}$$

when $i = 1, \ldots, 4$, then we will denote the stability configuration by:

- $(c : s_1, s_2, s_3, s_4)$ in the collinear case. Here the points are ordered.
- $(q : s_1, s_2, s_3, s_4)$ in the quadrilateral case. Here the points are ordered counterclockwise.
- $(t : s_1, s_2, s_3; s_4)$ in the triangle case. Here, z_4 is the point inside the triangle and the other points are also ordered counterclockwise.
- $(b : s_1, s_2, s_3; s_4)$ in the border case. Here, z_4 stands for the non-aligned point and the other points are ordered.

We will also assume, by changing the sign of the time if necessary, that the number of plus signs is greater or equal than the number of minus signs.

With this notation we have next result where we prove which configurations are allowed and which ones are not, for equation (1.1) when $n = 4$ and all its critical points of focus type.

Proposition 4.1. *Consider equation (1.1) with $n = 4$ and assume that it has four simple critical points, all of them of focus type. Then, the only impossible stability configurations are:*

$$(b : +, +, +; *), \quad (c : +, +, *, *) \quad \textit{and} \quad (x; +, +, +, +),$$

where $x \in \{c, q, t\}$ *and* $*$ *denotes either* $+$ *or* $-$.

Proof. First we are going to prove that the stability configurations stated in the proposition are impossible. After that we will prove that all the rest configurations can be realized.

From Proposition 2.6(d), at least two different stabilities must exist. Hence, the third item is not possible. By Proposition 2.7, the aligned critical points must have alternated stability. Hence, the second stability configuration is also impossible.

To prove that the first configuration does not exist, suppose that there is an equation (1.1) with a border configuration having three aligned unstable foci plus another one, which is non-aligned. Hence, equation (1.1) can be written as

$$\dot{z} = (\alpha + i\beta)z(z-1)(z-a_2)(z-(a_3+ib_3)),$$

where $1 < a_2 \in I\!\!R$ and $b_3 \neq 0$. To study the stabilities of the three real critical points we compute:

$$\begin{array}{rcl} \mathrm{Re}(f'(0)) & = & a_2(b_3\beta - a_3\alpha) =: a_2\alpha_0, \\ \mathrm{Re}(f'(1)) & = & (a_2-1)(\alpha(a_3-1) - b_3\beta) = (a_2-1)(-\alpha_0 - \alpha), \\ \mathrm{Re}(f'(a_2)) & = & a_2(a_2-1)(\alpha(a_2-a_3) + b_3\beta) = a_2(a_2-1)(\alpha a_2 + \alpha_0). \end{array}$$

As we are assuming that $z = 0$ is unstable, then $\alpha_0 > 0$. As we are assuming that $z = 1$ is unstable too, then $\alpha < -\alpha_0 < 0$. Finally, the stability of a_2, is given by the sign of $\alpha a_2 + \alpha_0$; that is,

$$\alpha a_2 + \alpha_0 \leq -\alpha_0 a_2 + \alpha_0 = \alpha_0(1 - a_2) < 0.$$

Consequently, it is impossible to have three aligned foci with the same stability.

Now we have to prove that all the other eleven stability configurations exist. More concretely, we are going to give examples of three stability configurations and prove the existence of the rest by perturbing these three ones.

(I) Case $(c : +, -, +, -)$: $\dot{z} = (-1+i)z(z-1)(z-2)(z-3)$,
(II) Case $(b : +, -, +; +)$: $\dot{z} = (-2+3i)z(z-1)(z-2)(z-(1+3i))$,
(III) Case $(b : +, +, -; +)$: $\dot{z} = (-2-3i)z(z-1)(z-2)(z-(2+i))$.

The remaining stability configurations are:

$$\begin{array}{lll} (q : +, -, +, -), & (q : +, +, -, -), & (q : +, +, +, -), \\ (t : +, +, -; -), & (t : +, -, +; +), & (t : +, +, +; -), \\ (b : +, +, -; -), & (b : +, -, +; -). & \end{array}$$

Note that the foci are structurally stable and hence, small perturbations in the coefficients of the differential equation do not change neither the type of the critical point nor its stability.

As an example we will get the case $(t : +, +, -; -)$ from the equation fulfilling case (I). Consider

$$\dot{z} = (-1+i)z(z-(1+\varepsilon i))(z-2)(z-(3+4\varepsilon i)), \tag{4.1}$$

with $\varepsilon > 0$ and small enough. By plotting the critical points and using the continuity with respect to ε is easy to see that its stability configuration is the desired one. □

5. Global phase portraits

In this section, to better understand the global dynamics of equation (1.1), we study its dynamics at infinity in the Poincaré compactification, see for instance [7, chap. 5] or [20, chap. 3.10]. This behavior at infinity, even for meromorphic rational functions f is already known, see for instance [12], but in order to be self-contained, in this section we present a new and simple proof in the case of f being a polynomial of degree n, see Theorem 5.1.

Also, in this section, we plot on the Poincaré disc an example of each possible configuration of equation (1.1) with $\deg(f) = 4$. Concretely, we draw examples of collinear, triangular and border configurations and an evolution of a family having the quadrilateral configuration. See Figure 5.

Theorem 5.1. *Consider equation (1.1) with* $\deg(f) = n$. *Then, it has exactly* $n-1$ *critical points at infinity, all of them of saddle type.*

Proof. Recall that the critical points at infinity of the Poincaré compactification of a polynomial planar vector field of degree n, $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, are given by the directions where the homogeneous polynomial of degree $n+1$, $R(x, y) := xQ_n(x, y) - yP_n(x, y)$ vanishes, being P_n and Q_n the homogeneous parts of degree n of P and Q, respectively. In our case we have that

$$\begin{aligned} R(x,y) &= x\,\mathrm{Im}(\gamma_n(x+iy)^n) - y\,\mathrm{Re}(\gamma_n(x+iy)^n) \\ &= \mathrm{Im}((x-iy)\gamma_n(x+iy)^n) = (x^2+y^2)\,\mathrm{Im}(\gamma_n(x+iy)^{n-1}), \end{aligned}$$

where $f(z) = \gamma_0 + \gamma_1 z + \cdots + \gamma_n z^n$, $\gamma_j \in \mathbb{C}$, $j = 0, \ldots, n$. Clearly the equation $R(x, y) = 0$ has exactly $n-1$ simple directions, which correspond with the $n-1$ critical points at infinity of the Poincaré compactification of $\dot{z} = f(z)$. Moreover the fact that the characteristic directions are simple implies that at least one of the eigenvalues associated to the critical points at infinity is not zero, *i.e.* that all the critical points are either hyperbolic or semi-hyperbolic. Hence these points have either index 1, 0 (saddle-nodes) or -1 (saddles), see for instance [7].

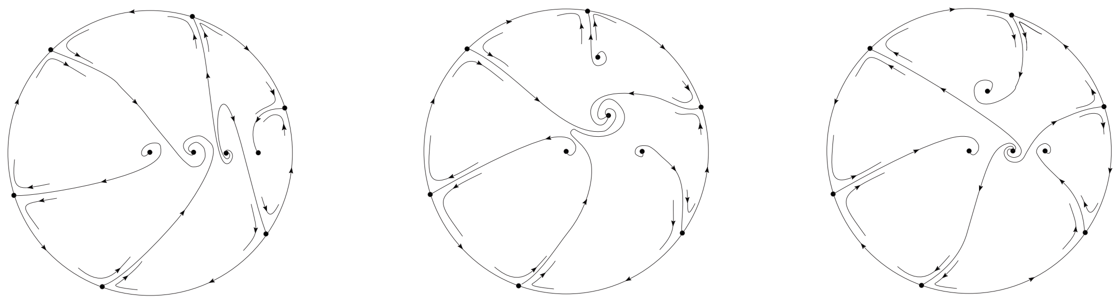
On the other hand, as a corollary of Theorem 2.1, it is clear that if we denote by Σ_F the sum of the indices of all the (finite) singularities of equation $\dot{z} = f(z)$ then $\Sigma_F = \deg(f) = n$ because the index of a singularity coincides with its multiplicity as a zero of f.

Recall also how looks a planar vector field when it is transported to the sphere through the Poincaré compactification: the north and the south hemispheres of $\mathbb{S}^2$ contain two exact copies of the planar vector field and the infinity is represented by its equator. Moreover each singularity appears twice in the equator at the two points associated with each of the vanishing directions of R, and for this reason points diametrally opposite are usually identified. By projecting one of the hemispheres into a disc, we can represent the flow of the plane in a disc, called the Poincaré disc, where now the infinity is its boundary, $\mathbb{S}^1$.

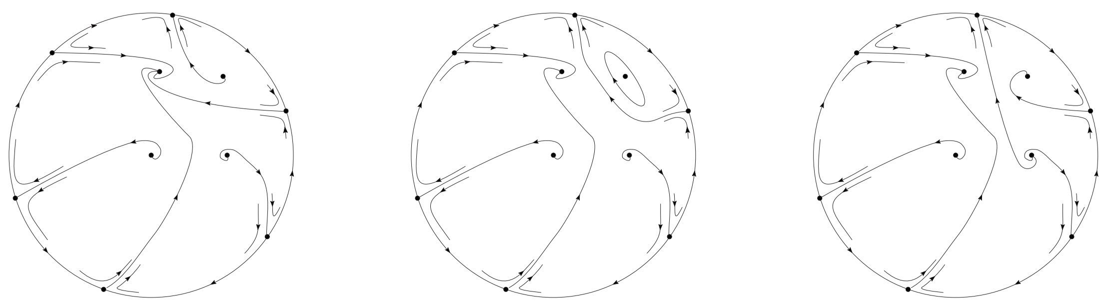
Finally we will need the well-known Poincaré-Hopf Theorem. It implies that the sum of the indices of all the singularities of a vector field defined on a sphere is equal to its Euler's characteristic, 2. Hence by the above considerations we know that $2 = 2\Sigma_F + \Sigma_\infty$, where Σ_∞ denotes the sum of the indices of all the singularities in the equator of the sphere. Moreover Σ_∞ is equal to two times the sum of all indices of the $n-1$ singularities at infinity. Since $\Sigma_F = n$ we get that $\Sigma_\infty = 2(1-n)$. By using that the only possibilities for the indices of the singularities at infinity are $\{1, 0, -1\}$ we obtain that all them have index -1. Hence we have proved that all the infinity singularities are of saddle type, as we wanted to see.

We think that an advantage of our proof is that it has very few calculations. On the other hand it does not clarify whether the saddle points are hyperbolic or not. It turns out that all them are hyperbolic, as can be easily checked by passing through computations that need the use of local coordinates for the compactified vector field. □

To end the paper we plot some phase portraits of several examples depicting all the possible geometric configurations of equation (1.1) with $\deg(f) = 4$ and four foci. We draw examples of the collinear, triangular and border configurations and an evolution of a family exhibiting a quadrilateral configuration. See Figure 5.



(a) Collinear (b) Triangle (c) Border



(d) Quadrilateral, $a \lesssim a^+$ (e) Quadrilateral, $a = a^+$ (f) Quadrilateral, $a \gtrsim a^+$

FIGURE 5. Some phase portraits on the Poincaré disc of equation (1.1), when $\deg(f) = 4$.

The examples are:

(1) collinear configuration, see Figure 5(a), with stability $(c : +, -, +, -)$, given by
$$\dot{z} = (-1 + 3i)z(z - 1)(z - 4)(z - 8).$$

(2) triangular configuration, see Figure 5(b), with stability $(t : +, +, +; -)$, given by
$$\dot{z} = (-1 + 3i)z(z - (1 + 3i))(z - 2)(z - (3 + 12i)).$$

(3) border configuration, see Figure 5(c), with stability $(b : -, +, -; -)$, given by
$$\dot{z} = (-1 + i)z(z - 1)(z - 2)(z - (3 + 4i)).$$

(4) quadrilateral configuration, see last row of Figure 5. We show the one-parameter family
$$\dot{z} = (-1 + 2i)z(z - 3)(z - 2i)(z - (a + 2i)),$$
for some values of the parameter a. Set $a^{\pm} = (-5 \pm \sqrt{89})/2$. We point out that when $a = a^{\pm}$ the point $z = a + 2i$ is a center, when $a \in (a^-, a^+)$ it is an unstable focus and otherwise a stable focus. In Figure 5(d), 5(e) and 5(f), we plot the phase portraits for the values of the parameter: $a = 2$, $a = a^+ \simeq 2.2$ and $a = 3$, respectively.

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References

[1] I. A. Aĭzenberg and A. P. Yuzhakov, A. P. *Integral representations and residues in multidimensional complex analysis*, Translations of Mathematical Monographs **58**. AMS, Providence, RI, 1983.

[2] A.N. Berlinskiĭ, *On the behaviour of the integral curves of a differential equation,* Izv. Vyssh. Uchebn. Zaved. Mat **2** (1960), 3–18.

[3] H. E. Benzinger, *Plane autonomous systems with rational vector fields*, Trans. of the AMS **326** (1991), 465–484.

[4] L. Brickman and E.S. Thomas, *Conformal equivalence of analytic flows*, J. Differential Equations **25** (1977), 310–324.

[5] K.A. Broughan, *Holomorphic flows on simply connected regions have no limit cycles*, Meccanica, **38** (2003), 699–709.

[6] K.A. Broughan, *Corrigenda for Holomorphic flows on simply connected regions have no limit cycles*, Meccanica **42** (2007), 213.

[7] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative theory of planar differential systems.* Universitext. Springer-Verlag, Berlin, 2006.

[8] A. Cima, A. Gasull, F. Mañosas, *Some applications of the Euler-Jacobi formula to differential equations*, Proc. Am. Math. Soc. **118** (1993), 151–163.

[9] A. Cima, A. Gasull, F. Mañosas, *Periodic orbits in complex Abel equations*, J. Differential Equations **232** (2007), 314–328.

[10] W.A. Coppel, *A survey of quadratic systems*, J. Differential Equations **2** (1966), 293–304.

[11] A. Garijo, A. Gasull and X. Jarque, *Normal forms for singularities of one dimensional holomorphic vector fields*, Electron. J. Differential Equations **2004**, n. 122, 1–7.

[12] A. Garijo, A. Gasull, X. Jarque, *Local and global phase portrait of equation* $\dot{z} = f(z)$, Discrete Cont. Dyn. Syst. **17** (2007), 309–329.

[13] A. Garijo, A. Gasull, X. Jarque, *Simultaneous bifurcation of limit cycles from two nests of periodic orbits.* To appear in J. Math. Anal. Appl.

[14] A. Gasull, J. Torregrosa, *Euler-Jacobi formula for double points and applications to quadratic and cubic systems*, Bull. Belg. Math. Soc. **6** (1999), 337–346.

[15] O. Hájek, *Notes on meromophic dynamical systems I*, Czech. Mathematical Journal **91** (1966), 14–27.

[16] O. Hájek, *Notes on meromophic dynamical systems II*, Czech. Mathematical Journal **91** (1966), 28–35.

[17] C.G. Jacobi, *De relationibus, quae locum habere debent inter puncta intersectionis duarum curvarum vel trium superficierum algebraicarum dati ordinis, simul cum enodatione paradoxi algebraici*, Gesammelte Werke, Band III, 329–354.

[18] A. Khovanskii, *Newton polyhedra and the Euler-Jacobi formula*, Russian Math. Surveys **33** (1978), 237–238.

[19] D.J. Needham and A.C. King, *On meromorphic complex differential equations*, Dynam. Stability Systems **9** (1994), 99–122.

[20] L. Perko, *Differential equations and dynamical systems*, Springer, TAM (7), New York, 2001.

[21] R. Sverdlove, *Vector fields defined by complex functions*, J. of Differential Equations **34** (1978), 427–439.

[22] A. Vidras, A. Yger, *On some generalizations of Jacobi's residue formula*, Annales Scientifiques de Ecole Normale Superieure **34** (2001), 131–157

[23] Ye Yan-Qian, Ye Weiyin, *A generalization of Berlinskiĭ's theorem to cubic and quartic differential systems*, Ann. Differ. Equations **4** (1988), 503–509.

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