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# Limit cycles for two families of cubic systems

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ABSTRACT

In this paper we study the number of limit cycles of two families of cubic systems introduced in previous papers to model real phenomena. The first one is motivated by a model of star formation histories in giant spiral galaxies and the second one comes from a model of Volterra type. To prove our results we develop a new criterion on the non-existence of periodic orbits and we extend a well-known criterion on the uniqueness of limit cycles due to Kuang and Freedman. Both results allow to reduce the problem to the control of the sign of certain functions that are treated by algebraic tools. Moreover, in both cases, we prove that when the limit cycles exist they are non-algebraic.


## 1. Introduction

In this work we study the number of limit cycles of two 2-parameter families of planar cubic systems introduced in previous papers to model real phenomena. The first one is the planar system

$$\begin{cases}\dot{x} = A(1-x-y) - Bxy^2,\\ \dot{y} = -y(1-x-y) + Bxy^2,\end{cases} \tag{1.1}$$

where $A$ and $B$ are positive real parameters. It turns out to be one of the simplest models for the formation of spiral galaxies. In Section 4 we briefly summarize how to derive it, following the explanation of [1]. The study developed in [1] is mainly numeric. In that work, it is proved that a limit cycle appears, via an Andronov–Hopf bifurcation, when crossing the curve $B = (1-2A)/A^2$ and it always exists for $B < (1-2A)/A^2$. Moreover some numerical evidences are presented to illustrate the situation. Here we make a complete analytic study of the system, characterizing the existence and uniqueness of the limit cycle. We prove the following theorem.

**Theorem A.** *Consider system (1.1) in the first quadrant with $A > 0$, $B > 0$. It has a periodic orbit if and only if $B < (1-2A)/A^2$. Moreover in this case it is unique, stable and non-algebraic.*

The second system is a predator–prey one of Volterra type that in dimensionless variables writes as

$$\begin{cases}\dot{x} = x(x(1-x) - (x+n)y),\\ \dot{y} = y(x+n)(x-m),\end{cases} \tag{1.2}$$

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