LIMIT CYCLES OF DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY LINEAR CENTERS IN $\mathbb{R}^2$ AND SEPARATED BY TWO CIRCLES

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Abstract. We show that discontinuous planar piecewise differential systems formed by linear centers and separated by two concentric circles can have at most three limit cycles. Usually is a difficult problem to provide the exact upper bound that a class of differential systems can exhibit. Here we also provide examples of such systems with zero, one, two, or three limit cycles.

1. Introduction and statement of the main result

In the qualitative theory of the differential systems in $\mathbb{R}^2$ one of the main difficult objects to study are the limit cycles. Recall that a *limit cycle* is an isolated periodic solution in the set of all periodic solutions of the differential systems, see for instance the second part of the famous 16th Hilbert problem [5, 7, 9].

The study of piecewise linear discontinuous differential systems started with Andronov, Vitt and Khaikin in [1]. Due to the fact that these systems model many real phenomena and different modern devices, they have became a topic of great interest these last twenty years. For more details see for instance the books [2, 19] and the references therein.

In recent years many authors have been widely interested in solving the second part of the 16-th Hilbert's problem for continuous and discontinuous piecewise linear differential systems in $\mathbb{R}^2$, that is to determine an upper bound for the maximum number of limit cycles for these class of differential systems.

The easiest continuous piecewise linear differential systems are formed by two linear differential systems separated by a straight line. It is known that such systems have at most one limit cycle, see [4,13,17,18]. But if both linear differential systems are linear centers, then it is known that the continuous piecewise linear differential systems have no limit cycles, see [11, 15].

Also the easiest discontinuous piecewise linear differential systems are formed by two linear differential systems separated by a straight line. It is known that this class of differential systems can have three limit cycles, but at this moment it is unknown if three is the maximum number of limit cycles that this class can

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exhibit, see for instante [8, 10, 14]. Other works on the limit cycles of discontinuous piecewise linear differential systems can be found in [6, 12].

A natural further step is to know which is the maximum number of limit cycles of continuous or discontinuous piecewise linear differential systems separated by other objects rather than a straight line and in particular if these linear systems are centers. This is the goal of this paper.

More precisely, in this paper we deal with discontinuous piecewise differential systems formed by two linear centers separated by two concentric circles. Without loss of generality we can assume that these circles are

$$\mathbb{S}_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \quad \text{and} \quad \mathbb{S}_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = s^2\},$$

with $s > 1$. We use the Filippov conventions for defining the discontinuous piecewise differential system on $\mathbb{S}_1$ and on $\mathbb{S}_2$, see [3]. We denote by (C) the class of planar discontinuous piecewise differential systems formed by three linear centers and separated by two circles $\mathbb{S}_1$ and $\mathbb{S}_2$.

We note that if we try to obtain continuous piecewise differential systems formed by three linear centers separated by the circles $\mathbb{S}_1$ and $\mathbb{S}_2$, then we obtain a global linear center in the plane, and consequently we do not have limit cycles. Therefore the interesting piecewise differential systems formed by three linear centers separated by the circles $\mathbb{S}_1$ and $\mathbb{S}_2$ are the discontinuous ones, the ones that we shall study.

Here a *crossing periodic orbit* for a system in (C) is a periodic orbit which has exactly four points of discontinuity, that is, two points in the circle $\mathbb{S}_1$ and the other two points in the circle $\mathbb{S}_2$. A *crossing limit cycle* for a system in (C) is a crossing periodic orbit isolated in the set of all crossing periodic orbits of the system.

We remark that we are not interested in studying the crossing limit cycles for the systems in (C) having only two intersection points with either the circle $\mathbb{S}_1$ or the circle $\mathbb{S}_2$, because in [16] it was proved that such systems can have at most two crossing limit cycles intersecting a circle in two points, and that there are examples exhibiting two such crossing limit cycles.

The objective of this paper is to study the number of crossing limit cycles that systems in the class (C) can have intersecting in two points each of the circles $\mathbb{S}_1$ and $\mathbb{S}_2$.

In what follows when we talk about a crossing limit cycle, it will be a crossing limit cycle that intersects in two points each of the circles $\mathbb{S}_1$ *and* $\mathbb{S}_2$.

Our main result is the following.

Theorem 1. *The following statements hold.*

(a) *Every system in (C) has at most three crossing limit cycles.*
(b) *There are systems in (C) having three crossing limit cycles.*
(c) *There are systems in (C) having exactly two crossing limit cycles.*
(d) *There are systems in (C) having exactly one crossing limit cycle.*
(e) *There are system in (C) without crossing limit cycles.*

The proof of Theorem 1 is given in the next section.

2. Proof of Theorem 1

To prove Theorem 1 we fist recall the following lemma, proved in [15] which provides a normal form for an arbitrary linear differential system having a center.

Lemma 2. *A linear differential system having a center can be written as*

$$\dot{x} = -bx - \frac{4b^2+\omega^2}{4a}y + d, \quad \dot{y} = ax + by + c, \tag{1}$$

with $a > 0$ and $\omega > 0$.

We prove each statement of Theorem 1 separately.

Proof of statement (*a*) *of Theorem* 1*.* In the bounded region R_1 limited by the circle $\mathbb{S}_1$ we consider the arbitrary linear differential center (1), which has the first integral

$$H_1(x,y) = 4(ax+by)^2 + 8a(cx-dy) + \omega^2 y^2.$$

In the bounded region R_2 limited by the circles $\mathbb{S}_1$ and $\mathbb{S}_2$ we consider the arbitrary linear differential center

$$\dot{x} = -Bx - \frac{4B^2+\Omega^2}{4A}y + D, \quad \dot{y} = Ax + By + C, \tag{2}$$

with $A > 0$ and $\Omega > 0$. This system has the first integral

$$H_2(x,y) = 4(Ax+By)^2 + 8A(Cx-Dy) + \Omega^2 y^2.$$

Finally in the region R_3 outside the circle $\mathbb{S}_2$ we consider the arbitrary linear differential center

$$\dot{x} = -\beta x - \frac{4\beta^2+\mathrm{w}^2}{4\alpha}y + \delta, \quad \dot{y} = \alpha x + \beta y + \gamma, \tag{3}$$

with $\alpha > 0$ and $\mathrm{w} > 0$, that has first integral

$$H_3(x,y) = 4(\alpha x+\beta y)^2 + 8\alpha(\gamma x-\delta y) + \mathrm{w}^2 y^2.$$

Take the scaling of the times $\sigma = at$ in R_1, $\tau = At$ in R_2 and $\nu = \alpha t$ in R_3. These three scalings change the velocity in which the orbits of system (1), (2) and (3) are traveled, but they do not change the orbits, and consequently they do not change the crossing limit cycles that the discontinuous piecewise linear differential systems may have. Note that after these scalings of the time we can always assume, without loss of generality, that $a = A = \alpha = 1$, with the dot in the systems denoting derivative with respect to the new times σ, τ and ν.

The proof of statement (a) will be done by contradiction. We thus assume that the discontinuous piecewise linear differential system formed by the three linear centers (1), (2) and (3) have four crossing periodic orbits that intersect the circle $\mathbb{S}_1$ in the points (x_i, y_i) for $i = 2, 3$ and the circle $\mathbb{S}_2$ in the points (x_i, y_i) for $i = 1, 4$, so these points must satisfy the system

$$
\begin{aligned}
&f_1 = H_2(x_1,y_1) - H_2(x_2,y_2) = 0,\\
&f_2 = H_1(x_2,y_2) - H_1(x_3,y_3) = 0,\\
&f_3 = H_2(x_3,y_3) - H_2(x_4,y_4) = 0,\\
&f_4 = H_3(x_4,y_4) - H_3(x_1,y_1) = 0,\\
&f_5 = x_1^2 + y_1^2 - s^2 = 0,\ f_6 = x_4^2 + y_4^2 - s^2 = 0,\\
&f_7 = x_2^2 + y_2^2 - 1 = 0,\ f_8 = x_3^2 + y_3^2 - 1 = 0,
\end{aligned}
\tag{4}
$$

or equivalently

(5)

$$
\begin{aligned}
f_1 =&4(By_1+x_1)^2 - 4(By_2+x_2)^2 + 8C(x_1-x_2) - 8D(y_1+y_2) + \Omega^2(y_1^2-y_2^2) = 0,\\
f_2 =&4(by_2+x_2)^2 - 4(by_3+x_3)^2 + 8c(x_2-x_3) - 8d(y_2+y_3) + \omega^2(y_2^2-y_3^2) = 0,\\
f_3 =&4(By_3+x_3)^2 - 4(By_4+x_4)^2 + 8C(x_3-x_4) - 8D(y_3+y_4) + \Omega^2(y_3^2-y_4^2) = 0,\\
f_4 =&4(\beta y_4+x_4)^2 - 4(\beta y_1+x_1)^2 + 8\gamma(x_4-x_1) - 8\delta(y_4+y_1) + \mathrm{w}^2(y_4^2-y_1^2) = 0,\\
f_5 =&0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 0.
\end{aligned}
$$

Assume that the four solutions of this system are (p_1,p_2,p_3,p_4), (q_1,q_2,q_3,q_4), (t_1,t_2,t_3,t_4) and (w_1,w_2,w_3,w_4) with $p_i,q_i,t_1,w_i \in \mathbb{R}^2$, $p_i \neq p_j$, $q_i \neq q_j$, $t_i \neq t_j$, and $w_i \neq w_j$ for $i \neq j$ and $i,j = 1,2,3,4$. We rewrite these points in the following way

$$
p_i = (x_i,y_i),\ q_i = (k_i,l_i),\ t_i = (m_i,n_i),\ w_i = (h_i,j_i),
\tag{6}
$$

for $i = 1,2,3,4$.

Substituting the solutions (p_1,p_2,p_3,p_4) and (q_1,q_2,q_3,q_4), with p_i,q_i given in (6) inside (5) we can determine the parameters d, δ, c and γ in function of the coordinates of the points p_i,q_i as follows

(7)

$$
\begin{aligned}
d =&\frac{(4b^2+\omega^2)(y_2^2-y_3^2) + 8x_2(by_2+c) - 8x_3(by_3+c) + 4x_2^2 - 4x_3^2}{8(y_2-y_3)},\\
\delta =&\frac{\left(4\beta^2+\mathrm{w}^2\right)\left(y_1^2-y_4^2\right) + 4x_1^2 + 8x_1(\gamma+\beta y_1) - 4x_4^2 - 8x_4(\gamma+\beta y_4)}{8(y_1-y_4)},\\
c =&\frac{1}{8(l_2-l_3)(x_2-x_3) - 8(k_2-k_3)(y_2-y_3)}\left(4\left(l_2\left(b^2y_3^2 + 2by_3(x_3-k_2) + 2bk_2y_2\right.\right.\right.\\
&\left.-(by_2+x_2)^2 + x_3^2\right) + b^2l_2^2(y_2-y_3) - b^2l_3y_3^2 + y_3\left((bl_3+k_3)^2 - 2bl_3x_3 - k_2^2\right)\\
&\left.+l_3\left(-by_2(bl_3+2k_3)\right) + (by_2+x_2)^2 - x_3^2\right) + 4y_2(k_2-k_3)(k_2+k_3)\\
&\left.+\omega^2(l_2-l_3)(y_2-y_3)(l_2+l_3-y_2-y_3)\right),\\
\gamma =&\frac{1}{8(l_1-l_4)(x_1-x_4) - 8(k_1-k_4)(y_1-y_4)}\left(8\beta(k_1l_1(y_1-y_4) + k_4l_4(y_4-y_1)\right.\\
&-(l_1-l_4)(x_1y_1-x_4y_4)) + 4(k_1-k_4)(k_1+k_4)(y_1-y_4)\\
&+(l_1-l_4)\left(\mathrm{w}^2(y_1-y_4)(l_1+l_4-y_1-y_4) - 4x_1^2 + 4x_4^2\right)\\
&\left.+4\beta^2(l_1-l_4)(y_1-y_4)(l_1+l_4-y_1-y_4)\right).
\end{aligned}
$$

We assume that the points (t_1,t_2,t_3,t_4), (w_1,w_2,w_3,w_4) with t_i, w_i as in (6) satisfies system (5), then we can obtain the remaining parameters, namely, ω^2 and w^2. In fact, from equation f_2 we obtain $\omega^2 = S/T$, where

$$\begin{aligned}S = {} & 4(-(k_2-k_3)(y_3(-b^2n_2^2-2bm_2n_2+(bn_3+m_3)^2+2bx_3(n_2-n_3)-m_2^2)+b^2n_2^2y_2\\
&+b^2y_3^2(n_2-n_3)-b^2n_2y_2^2-b^2n_3^2y_2+b^2n_3y_2^2+2bm_2n_2y_2-2bm_3n_3y_2-2bn_2x_2y_2\\
&+2bn_3x_2y_2+k_2((n_2-n_3)(x_2-x_3)-(m_2-m_3)(y_2-y_3))+k_3((n_2-n_3)(x_2-x_3)\\
&-(m_2-m_3)(y_2-y_3))+y_2(m_2-m_3)(m_2+m_3)-n_2x_2^2+n_2x_3^2+n_3x_2^2-n_3x_3^2)\\
&+l_2(m_2(b^2(y_3^2-y_2^2)+2b(k_2y_2-k_2y_3+n_2x_2-n_2x_3-x_2y_2+x_3y_3)-x_2^2+x_3^2)\\
&+m_3(b^2(y_2-y_3)(y_2+y_3)+2b(-k_2y_2+k_2y_3-n_3x_2+n_3x_3+x_2y_2-x_3y_3)\\
&+(x_2-x_3)(x_2+x_3))+b(n_2-n_3)(x_2-x_3)(b(n_2+n_3)-2k_2)+m_2^2(x_2-x_3)\\
&+m_3^2(x_3-x_2))+l_3(m_2(b^2(y_2-y_3)(y_2+y_3)+2b(-k_3y_2+k_3y_3+n_2(x_3-x_2)\\
&+x_2y_2-x_3y_3)+(x_2-x_3)(x_2+x_3))+m_3(b^2(y_3^2-y_2^2)+2b(k_3y_2-k_3y_3+n_3(x_2-x_3)\\
&-x_2y_2+x_3y_3)-x_2^2+x_3^2)+b(n_2-n_3)(x_3-x_2)(b(n_2+n_3)-2k_3)+m_2^2(x_3-x_2)\\
&+m_3^2(x_2-x_3))+l_2^2b^2((m_2-m_3)(y_2-y_3)+(n_3-n_2)(x_2-x_3))+l_3^2b^2((n_2-n_3)(x_2-x_3)\\
&-(m_2-m_3)(y_2-y_3))),\end{aligned}$$

and

$$\begin{aligned}T = {} & (k_2-k_3)(n_2-n_3)(y_2-y_3)(n_2+n_3-y_2-y_3)+l_2^2((n_2-n_3)(x_2-x_3)\\
&-(m_2-m_3)(y_2-y_3))+l_2((m_2-m_3)(y_2-y_3)(y_2+y_3)+(n_3-n_2)(n_2+n_3)(x_2-x_3))\\
&+l_3^2((m_2-m_3)(y_2-y_3)+(n_3-n_2)(x_2-x_3))+l_3((n_2-n_3)(n_2+n_3)(x_2-x_3)\\
&-(m_2-m_3)(y_2-y_3)(y_2+y_3)).\end{aligned}$$

Analogously from equation f_4 we obtain $\mathrm{w}^2 = R/U$, where

$$\begin{aligned}R = {} & 4(-(k_1-k_4)(k_1((n_1-n_4)(x_1-x_4)-(m_1-m_4)(y_1-y_4))-k_4(m_1-m_4)(y_1-y_4)\\
&+k_4(n_1-n_4)(x_1-x_4)+2\beta(m_1n_1(y_1-y_4)+m_4n_4(y_4-y_1)-(n_1-n_4)(x_1y_1-x_4y_4))\\
&+(m_1-m_4)(m_1+m_4)(y_1-y_4)-(n_1-n_4)(x_1-x_4)(x_1+x_4)+\beta^2(n_1-n_4)(y_1-y_4)\\
&\times(n_1+n_4-y_1-y_4))+l_1(2\beta(k_1(m_1-m_4)(y_1-y_4)-k_1(n_1-n_4)(x_1-x_4)\\
&+m_1(n_1(x_1-x_4)-x_1y_1+x_4y_4)+m_4(n_4(x_4-x_1)+x_1y_1-x_4y_4))+\beta^2((n_1-n_4)\\
&\times(n_1+n_4)(x_1-x_4)-(m_1-m_4)(y_1-y_4)(y_1+y_4))+(m_1-m_4)(x_1-x_4)\\
&\times(m_1+m_4-x_1-x_4))+l_4(2\beta(-k_4(m_1-m_4)(y_1-y_4)+k_4(n_1-n_4)(x_1-x_4)\\
&+m_1(n_1(x_4-x_1)+x_1y_1-x_4y_4)+m_4(n_4(x_1-x_4)-x_1y_1+x_4y_4))\\
&+\beta^2((m_1-m_4)(y_1-y_4)(y_1+y_4)+(n_4-n_1)(n_1+n_4)(x_1-x_4))\\
&+(m_1-m_4)(x_1-x_4)(-m_1-m_4+x_1+x_4))+\beta^2l_1^2((m_1-m_4)(y_1-y_4)\\
&+(n_4-n_1)(x_1-x_4))+\beta^2l_4^2((n_1-n_4)(x_1-x_4)-(m_1-m_4)(y_1-y_4))),\end{aligned}$$

and

$$\begin{aligned}U = {} & (k_1-k_4)(n_1-n_4)(y_1-y_4)(n_1+n_4-y_1-y_4)+l_1^2((n_1-n_4)(x_1-x_4)\\
&-(m_1-m_4)(y_1-y_4))+l_1((m_1-m_4)(y_1-y_4)(y_1+y_4)+(n_4-n_1)(n_1+n_4)(x_1-x_4))\\
&+l_4^2((m_1-m_4)(y_1-y_4)+(n_4-n_1)(x_1-x_4))+l_4((n_1-n_4)(n_1+n_4)(x_1-x_4)\\
&-(m_1-m_4)(y_1-y_4)(y_1+y_4)).\end{aligned}$$

Substituting p_i, q_i, t_i, w_i as in (6) in equations f_2 and f_4, we obtain the expressions for b and β. In order to simplify these expressions, we use the relations

$$x_i^2 = s_i^2 - y_i^2,\ k_i^2 = s_i^2 - l_i^2,\ m_i^2 = s_i^2 - n_i^2,\ h_i^2 = s_i^2 - j_i^2$$

with $s_i = 1$ for $i = 2, 3$ and $s_i = r$ for $i = 1, 4$. Doing this we get

$$\begin{aligned} b = &\left(h_3^2 + j_3^2 - 1\right)(-(k_2 - k_3)(-k_2(m_2 - m_3)(y_2 - y_3) + k_2(n_2 - n_3)(x_2 - x_3) \\ &- k_3(m_2 - m_3)(y_2 - y_3) + k_3(n_2 - n_3)(x_2 - x_3) + (m_2 - m_3)(m_2 + m_3)(y_2 - y_3) \\ &- (n_2 - n_3)(x_2 + x_3)(x_2 - x_3)) + l_2(m_2 - m_3)(x_2 - x_3)(m_2 + m_3 - x_2 - x_3) \\ &- l_3(m_2 - m_3)(x_2 - x_3)(m_2 + m_3 - x_2 - x_3)), \end{aligned}$$

$$\begin{aligned} \beta = &\left(h_1^2 + j_1^2 - r^2\right)(-(k_1 - k_4)(-k_1(m_1 - m_4)(y_1 - y_4) + k_1(n_1 - n_4)(x_1 - x_4) \\ &- k_4(m_1 - m_4)(y_1 - y_4) + k_4(n_1 - n_4)(x_1 - x_4) + (m_1 - m_4)(m_1 + m_4)(y_1 - y_4) \\ &- (n_1 - n_4)(x_1 + x_4)(x_1 - x_4)) + l_1(m_1 - m_4)(x_1 - x_4)(m_1 + m_4 - x_1 - x_4) \\ &- l_4(m_1 - m_4)(x_1 - x_4)(m_1 + m_4 - x_1 - x_4)). \end{aligned}$$

Since $h_3^2 + j_3^2 - 1 = 0$ and $h_1^2 + j_1^2 - r^2 = 0$ (because they are points on the circles of radius 1 and r, respectively), we obtain that $b = 0$ and $\beta = 0$. Therefore we get that the piecewise linear differential center is formed by the linear differential centers

$$\dot{x} = -y, \quad \dot{y} = x,$$

in the regions R_1 and R_3. This is a contradiction because in the region R_1 inside the circle of radius 1 we have the first integral $H_1(x, y) = x^2 + y^2$. Then using this first integral it is not possible to generate crossing limit cycles. Therefore there is no discontinuous piecewise linear differential systems (1), (2) and (3) having more than three crossing limit cycles. □

Proof of statement (*b*) *of Theorem* 1. We construct a system in (C) having three crossing limit cycles. In the bounded region R_1 limited by the circle $\mathbb{S}_1$ we consider the linear differential center

$$\dot{x} = 4x - 11.9153y + 2.59286, \quad \dot{y} = 8x - 4y - 7.70389, \tag{8}$$

with the first integral

$$\begin{aligned} H_1(x, y) = &-7.703888863058826x - 2.59286097618567y + 4x^2 - 4xy \\ &+ 5.957636997088071y^2. \end{aligned}$$

In the bounded region R_2 limited by the circles $\mathbb{S}_1$ and $\mathbb{S}_2$ with $s = 2$ we consider the linear differential center

$$\dot{x} = -2.48171x - 3.50967y + 5.2645, \quad \dot{y} = 8x + 2.48171y - 12.2078, \tag{9}$$

with the first integral

$$\begin{aligned} H_2(x, y) = &-12.207843699316152x - 5.264498126903808y + 4x^2 \\ &+ 2.481708216718382xy + 1.7548327089679396y^2. \end{aligned}$$

Finally in the region R_3 outside the circle $\mathbb{S}_2$ we consider the linear differential center

$$\dot{x} = -8x + 4y + 12.2222, \quad \dot{y} = 8x - 4y - 12.2222, \tag{10}$$

with the first integral

$$H_3(x,y) = -12.222157553827293x - 3.9748620974752176y + 4x^2 - 4xy + 6.143143553839666y^2.$$

The discontinuous piecewise differential system formed by the previous three linear differential centers has three crossing limit cycles (p_1,p_2,p_3,p_4), (q_1,q_2,q_3,q_4) and (t_1,t_2,t_3,t_4) with p_i, q_i, t_i

$$p_i = (x_i,y_i) = (s_i\cos\rho_i, s_i\sin\rho_i),\ \ q_i = (k_i,l_i) = (s_i\cos\nu_i, s_i\sin\nu_i),$$
$$t_i = (m_i,n_i) = (s_i\cos u_i, s_i\sin u_i),$$

where $\rho_i, \nu_i, u_i \in [0,2\pi)$, $i = 1,2,3,4$, $s_i = 1$ for $i = 2,3$ and $s_i = 2$ for $i = 1,4$. Moreover, $\rho_1 = \rho_2 = \pi/4$, $\rho_3 = \rho_4 = 1/10$, $\nu_1 = \nu_2 = \pi/2$, $\nu_3 = \nu_4 = -\pi/4$, $u_1 = 2$, $u_2 = 2.2778851991902505$, $u_3 = 4$ and $u_4 = 4.760510529471427$. See the three crossing limit cycles of this system in Figure 1. □

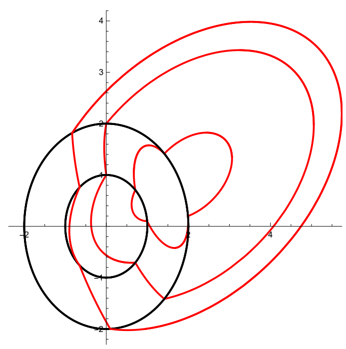

Figure 1: The three crossing limit cycles of the discontinuous piecewise differential systems formed by the linear centers (8), (9) and (10).

Proof of statement (*c*) *of Theorem* 1. We construct a system in (C) having exactly two crossing limit cycles. In the bounded region R_1 limited by the circle $\mathbb{S}_1$ we consider the linear differential center

$$\dot{x} = \frac{2}{5} - y, \quad \dot{y} = x - 1, \tag{11}$$

with the first integral

$$H_1(x,y) = 4x^2 + 8\left(-x - \frac{2y}{5}\right) + 4y^2.$$

In the bounded region R_2 limited by the circles $\mathbb{S}_1$ and $\mathbb{S}_2$ with $s = 2$ we consider the linear differential center

$$\dot{x} = \frac{11x}{19} + \frac{18292y}{6859} - 4, \quad \dot{y} = -\frac{19x}{2} - \frac{11y}{19} - 10, \tag{12}$$

with the first integral

$$H_2(x,y) = 4\left(-\frac{19x}{2} - \frac{11y}{19}\right)^2 - 76(4y - 10x) + 100y^2.$$

And in the region R_3 outside the circle $\mathbb{S}_2$ we consider the linear differential center

$$\dot{x} = 16y + \frac{1}{5}, \quad \dot{y} = 1 - \frac{x}{4}, \tag{13}$$

with the first integral

$$H_3(x,y) = \frac{x^2}{4} - 2\left(x - \frac{y}{5}\right) + 16y^2.$$

This discontinuous piecewise differential system formed by the previous linear centers (11), (12) and (13) has two crossing limit cycles, which are the unique two real solutions $(p_1^j, p_2^j, p_3^j, p_4^j)$ with $j = 1, 2$ satisfying (4). Since we want to use Groebner bases, we use the rational parametrization of the circle for obtain polynomial equations, so we write the points $p_i = (x_i, y_i)$ as

$$(x_i, y_i) = \left(\frac{R(1-t_i^2)}{1+t_i^2}, \frac{2Rt_i}{1+t_i^2}\right), \tag{14}$$

with $t_i \neq t_j$, $i \neq j$ for $i = 1, 2, 3, 4$. Doing this system (4) reduces to the polynomial system

(15)

$$\begin{aligned}
f_1 =& 116603t_1^4t_4^4 - 566048t_1^4t_4^3 - 169146t_1^4t_4^2 - 311904t_1^4t_4 + 1214043t_1^4 + 251256t_1^3t_4^4 \\
&+ 502512t_1^3t_4^2 + 251256t_1^3 + 59434t_1^2t_4^4 - 1132096t_1^2t_4^3 - 685836t_1^2t_4^2 - 623808t_1^2t_4 \\
&+ 2254314t_1^2 + 187720t_1t_4^4 + 375440t_1t_4^2 + 187720t_1 - 432117t_4^4 - 566048t_4^3 \\
&- 1266586t_4^2 - 311904t_4 + 665323, \\
f_2 =& 2t_1t_2 + 5t_1 + 5t_2 - 2, \\
f_3 =& -116603t_2^4t_3^4 + 566048t_2^4t_3^3 + 169146t_2^4t_3^2 + 311904t_2^4t_3 - 1214043t_2^4 - 251256t_2^3t_3^4 \\
&- 502512t_2^3t_3^2 - 251256t_2^3 - 59434t_2^2t_3^4 + 1132096t_2^2t_3^3 + 685836t_2^2t_3^2 + 623808t_2^2t_3 \\
&- 2254314t_2^2 - 187720t_2t_3^4 - 375440t_2t_3^2 - 187720t_2 + 432117t_3^4 + 566048t_3^3 \\
&+ 1266586t_3^2 + 311904t_3 - 665323, \\
f_4 =& 2t_3^3t_4^3 + 305t_3^3t_4^2 + 2t_3^3t_4 - 10t_3^3 + 305t_3^2t_4^3 - 2t_3^2t_4^2 - 10t_3^2t_4 - 2t_3^2 + 2t_3t_4^3 - 10t_3t_4^2 \\
&+ 2t_3t_4 - 325t_3 - 10t_4^3 - 2t_4^2 - 325t_4 - 2, \\
f_5 =& 0, \ f_6 = 0, \ f_7 = 0, \ f_8 = 0.
\end{aligned}$$

Doing the Groebner basis using Mathematica we can eliminate the variables t_2, t_4 using the equations (f_2, f_3) and (f_1, f_4), respectively. Moreover we get two polynomials: p_1 of degree 8 and p_2 of degree 24 with respect to variables t_1 and t_3. Solving the system obtained for these two polynomials we obtain the following six real solutions

$$\begin{aligned}
&(t_1^1, t_3^1) = (3.113683889, -3.345831873), \ (t_1^2, t_3^2) = (-1.208512958, 3.444180431), \\
&(t_1^3, t_3^3) = (-0.07918973421, -2.003557002), \ (t_1^4, t_3^4) = (0.7258741121, -0.9412890517), \\
&(t_1^5, t_3^5) = (-0.2525471397, 0.8580640103), \ (t_1^6, t_3^6) = (0.4948650262, 0.5345302181).
\end{aligned}$$

Therefore (15) has the next six solutions

$$
\begin{aligned}
(t_1^1,t_2^1,t_3^1,t_4^1) &= (3.113684, -1.208513, -3.345832, 3.444180),\\
(t_1^2,t_2^2,t_3^2,t_4^2) &= (-1.208513, 3.113684, 3.444180, -3.345832),\\
(t_1^3,t_2^3,t_3^3,t_4^3) &= (0.7258741, -0.2525471, -0.941289, 0.8580640),\\
(t_1^4,t_2^4,t_3^4,t_4^4) &= (-0.2525472, 0.7258742, 0.8580642, -0.941289),\\
(t_1^5,t_2^5,t_3^5,t_4^5) &= (-0.07919058, 0.4948659, -2.003559, 0.5345301),\\
(t_1^6,t_2^6,t_3^6,t_4^6) &= (0.4948641, -0.07918876, 0.5345306, -2.003555).
\end{aligned}
$$

The solutions $(t_1^j,t_2^j,t_3^j,t_4^j)$ for $j=2,4,6$, define the same solutions as for $j=1,3,4$. Therefore the solutions of system (4) in the variables (x_i,y_i) are

$$
\begin{aligned}
(p_1^1,p_2^1,p_3^1,p_4^1) =&(0.880088, -0.474811, 0.309849, 0.950786, 0.303784, 1.97679,\\
&0.120863, -1.99634),\\
(p_1^2,p_2^2,p_3^2,p_4^2) =&(-0.187158, -0.98233, -0.812997, 0.582268, -1.68902, 1.07109,\\
&-1.67199, -1.09748),\\
(p_1^3,p_2^3,p_3^3,p_4^3) =&(0.987536, -0.157392, 0.606566, 0.795033, -1.20227, -1.59829,\\
&1.11109, 1.66297).
\end{aligned}
$$

The third solution does not define a crossing limit cycle. See the two crossing limit cycles in Figure 2. □

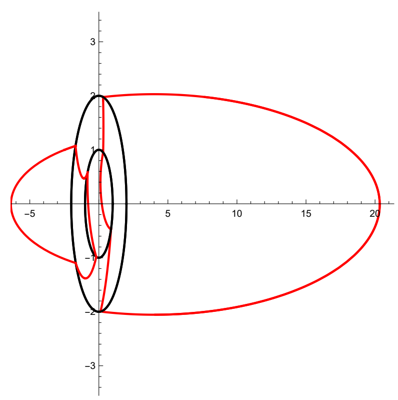

Figure 2: The two crossing limit cycles of the discontinuous piecewise differential systems formed by the linear centers (11), (12) and (13).

Proof of statement (d) of Theorem 1. We construct a system in (C) having exactly one crossing limit cycle. In the bounded region R_1 limited by the circle $\mathbb{S}_1$ we consider the linear differential center

$$
\dot{x} = -\frac{x}{3} + \frac{2849y}{1917} + \frac{34}{25}, \quad \dot{y} = -\frac{213x}{77} + \frac{y}{3} + 1, \tag{16}
$$

with the first integral

$$H_1(x,y) = 4\left(\frac{y}{3} - \frac{213x}{77}\right)^2 - \frac{1704}{77}\left(x - \frac{34y}{25}\right) + 16y^2.$$

In the bounded region R_2 limited by the circles $\mathbb{S}_1$ and $\mathbb{S}_2$ with $s = 2$ we consider the linear differential center

$$\dot{x} = \frac{11x}{19} + \frac{18292y}{6859} - 4, \quad \dot{y} = -\frac{19x}{2} - \frac{11y}{19} - 10, \tag{17}$$

with the first integral

$$H_2(x,y) = 4\left(-\frac{19x}{2} - \frac{11y}{19}\right)^2 - 76(4y - 10x) + 100y^2.$$

Finally in the region R_3 outside the circle $\mathbb{S}_2$ we consider the linear differential center

$$\dot{x} = \frac{23x}{22} - \frac{9117y}{5324} + \frac{2}{5}, \quad \dot{y} = \frac{11x}{9} - \frac{23y}{22} - \frac{131}{88}, \tag{18}$$

with the first integral

$$H_3(x,y) = 4\left(\frac{11x}{9} - \frac{23y}{22}\right)^2 + \frac{88}{9}\left(-\frac{131x}{88} - \frac{2y}{5}\right) + 4y^2.$$

This discontinuous piecewise differential system formed by the linear centers (16), (17) and (18) has exactly one crossing limit cycle, coming from the unique real solution (p_1, p_2, p_3, p_4) with $p_i = (x_i, y_i)$, $p_i \neq p_j$ for $i \neq j$ and $i, j = 1, 2, 3, 4$ that satisfies system (4). Again since we want to use Groebner bases we use the rational parametrization (14) and system (4) reduces to the polynomial system

(19)

$$\begin{aligned}
f_1 =& 116603t_1^4t_4^4 - 566048t_1^4t_4^3 - 169146t_1^4t_4^2 - 311904t_1^4t_4 + 1214043t_1^4 + 251256t_1^3t_4^4\\
&+ 502512t_1^3t_4^2 + 251256t_1^3 + 59434t_1^2t_4^4 - 1132096t_1^2t_4^3 - 685836t_1^2t_4^2 - 623808t_1^2t_4\\
&+ 2254314t_1^2 + 187720t_1t_4^4 + 375440t_1t_4^2 + 187720t_1 - 432117t_4^4 - 566048t_4^3\\
&- 1266586t_4^2 - 311904t_4 + 665323,\\
f_2 =& 6248781t_1^3t_2^3 - 8413925t_1^3t_2^2 + 3788631t_1^3t_2 - 3690225t_1^3 - 8413925t_1^2t_2^3 - 8708931t_1^2t_2^2\\
&- 3690225t_1^2t_2 - 6248781t_1^2 + 3788631t_1t_2^3 - 3690225t_1t_2^2 + 1328481t_1t_2 + 1033475t_1\\
&- 3690225t_2^3 - 6248781t_2^2 + 1033475t_2 - 3788631,\\
f_3 =& -116603t_2^4t_3^4 + 566048t_2^4t_3^3 + 169146t_2^4t_3^2 + 311904t_2^4t_3 - 1214043t_2^4 - 251256t_2^3t_3^4\\
&- 502512t_2^3t_3^2 - 251256t_2^3 - 59434t_2^2t_3^4 + 1132096t_2^2t_3^3 + 685836t_2^2t_3^2 + 623808t_2^2t_3\\
&- 2254314t_2^2 - 187720t_2t_3^4 - 375440t_2t_3^2 - 187720t_2 + 432117t_3^4 + 566048t_3^3\\
&+ 1266586t_3^2 + 311904t_3 - 665323,\\
f_4 =& 810216t_3^3t_4^3 - 243515t_3^3t_4^2 - 1193544t_3^3t_4 - 713295t_3^3 - 243515t_3^2t_4^3 - 2813976t_3^2t_4^2\\
&- 713295t_3^2t_4 - 810216t_3^2 - 1193544t_3t_4^3 - 713295t_3t_4^2 - 3197304t_3t_4 - 1183075t_3\\
&- 713295t_4^3 - 810216t_4^2 - 1183075t_4 + 1193544,\\
f_5 =& 0, \;\; f_6 = 0, \;\; f_7 = 0, \;\; f_8 = 0.
\end{aligned}$$

Doing the Groebner basis using Mathematica we can eliminate the variables t_2, t_4 using the equations (f_2, f_3) and (f_1, f_4), respectively. Moreover, we get two polynomials p_3 and p_4 both of degree 24 with respect to variables t_1 and t_3. Solving the system formed by these two polynomials we obtain its two unique real solutions $(t_1^1, t_3^1) = (-0.2761369370, -5.638072382)$, $(t_1^2, t_2^2) = (-1.456914010, -0.9512182090)$. Therefore system (19) has the two solutions

$$(t_1^1, t_2^1, t_3^1, t_4^1) = (-1.4576899, -0.27613694, -0.95176617, -5.6380724),$$
$$(t_1^2, t_2^2, t_3^2, t_4^2) = (-0.27613694, -1.4576899, -5.6380724, -0.95176617),$$

From these two solutions we obtain the solutions of system (4) in the variables (x_i, y_i), i.e.

$$(p_1^1, p_2^1, p_3^1, p_4^1) = (-0.359971, -0.932963, 0.858302, -0.513146, 0.0987913, -1.99756, -1.878, -0.687824),$$
$$(p_1^2, p_2^2, p_3^2, p_4^2) = (0.858302, -0.513146, -0.359971, -0.932963, -1.878, -0.687824, 0.0987913, -1.99756).$$

But the both solution define the same crossing limit cycle. See this crossing limit cycle of the piecewise differential system in Figure 3. □

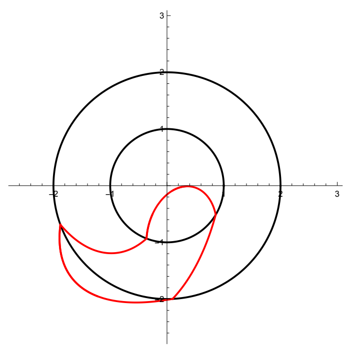

Figure 3: The crossing limit cycle of the discontinuous piecewise differential system formed by the linear centers (16), (17) and (18).

Proof of statement (*e*) *of Theorem* 1. To prove statement (e) it is sufficient consider the discontinuous piecewise linear differential system having the linear center $\dot{x} = -y$, $\dot{y} = x$ in the region R_2 limited by the circles $\mathbb{S}_1$ and $\mathbb{S}_2$ with first integral $H(x, y) = x^2 + y^2$, and two arbitrary different centers in the region R_1 limited by the circle $\mathbb{S}_1$ and in the region R_3 outside the circle $\mathbb{S}_2$. Then it is impossible to connect orbits of the region R_1 with orbits of the region R_3, so they do not exist crossing limit cycle for this discontinuous piecewise differential system. □

3. Appendix

The polynomials p_1, p_2, p_3 and p_4 of the proofs of statements (c) and (d),

$$p_1 = a_1 t_1^4 t_3^4 + a_2 t_1^4 t_3^3 + a_3 t_1^4 t_3^2 + a_4 t_1^4 t_3 + a_5 t_1^4 + a_6 t_1^3 t_3^4 + a_7 t_1^3 t_3^2 + a_8 t_1^3 + a_9 t_1^2 t_3^4 + a_{10} t_1^2 t_3^3 + a_{11} t_1^2 t_3^2 + a_{12} t_1^2 t_3 + a_{13} t_1^2 + a_{14} t_1 t_3^4 + a_{15} t_1 t_3^2 + a_{16} t_1 + a_{17} t_3^4 + a_{18} t_3^3 + a_{19} t_3^2 + a_{20} t_3 + a_{21}$$

where the coefficients a_i are

$a_1 = 54607$	$a_2 = -16415392$	$a_3 = -11558994$	$a_4 = -9045216$
$a_5 = 31880367$	$a_6 = -10198280$	$a_7 = -20396560$	$a_8 = -10198280$
$a_9 = -4653694$	$a_{10} = -32830784$	$a_{11} = -32643804$	$a_{12} = -18090432$
$a_{13} = 58997826$	$a_{14} = -20969016$	$a_{15} = -41938032$	$a_{16} = -20969016$
$a_{17} = -7078753$	$a_{18} = -16415392$	$a_{19} = -25825714$	$a_{20} = -9045216$
$a_{21} = 24747007$			

$$
\begin{aligned}
p_2 =& b_1 t_3^{12} t_1^{12} + b_2 t_3^{11} t_1^{12} + b_3 t_3^{10} t_1^{12} + b_4 t_3^9 t_1^{12} + b_5 t_3^8 t_1^{12} + b_6 t_3^7 t_1^{12} + b_7 t_3^6 t_1^{12} + b_8 t_3^5 t_1^{12} \\
&+ b_9 t_3^4 t_1^{12} + b_{10} t_3^3 t_1^{12} + b_{11} t_3^2 t_1^{12} + b_{12} t_3 t_1^{12} + b_{13} t_1^{12} + b_{14} t_3^{12} t_1^{11} + b_{15} t_3^{11} t_1^{11} \\
&+ b_{16} t_3^{10} t_1^{11} + b_{17} t_3^9 t_1^{11} + b_{18} t_3^8 t_1^{11} + b_{19} t_3^7 t_1^{11} + b_{20} t_3^6 t_1^{11} + b_{21} t_3^5 t_1^{11} + b_{22} t_3^4 t_1^{11} \\
&+ b_{23} t_3^3 t_1^{11} + b_{24} t_3^2 t_1^{11} + b_{25} t_3 t_1^{11} + b_{26} t_1^{11} + b_{27} t_3^{12} t_1^{10} + b_{28} t_3^{11} t_1^{10} + b_{29} t_3^{10} t_1^{10} \\
&+ b_{30} t_3^9 t_1^{10} + b_{31} t_3^8 t_1^{10} + b_{32} t_3^7 t_1^{10} + b_{33} t_3^6 t_1^{10} + b_{34} t_3^5 t_1^{10} + b_{35} t_3^4 t_1^{10} + b_{36} t_3^3 t_1^{10} \\
&+ b_{37} t_3^2 t_1^{10} + b_{38} t_3 t_1^{10} + b_{39} t_1^{10} + b_{40} t_3^{12} t_1^9 + b_{41} t_3^{11} t_1^9 + b_{42} t_3^{10} t_1^9 + b_{43} t_3^9 t_1^9 \\
&+ b_{44} t_3^8 t_1^9 + b_{45} t_3^7 t_1^9 + b_{46} t_3^6 t_1^9 + b_{47} t_3^5 t_1^9 + b_{48} t_3^4 t_1^9 + b_{49} t_3^3 t_1^9 + b_{50} t_3^2 t_1^9 + b_{51} t_3 t_1^9 \\
&+ b_{52} t_1^9 + b_{53} t_3^{12} t_1^8 + b_{54} t_3^{11} t_1^8 + b_{55} t_3^{10} t_1^8 + b_{56} t_3^9 t_1^8 + b_{57} t_3^8 t_1^8 + b_{58} t_3^7 t_1^8 + b_{59} t_3^6 t_1^8 \\
&+ b_{60} t_3^5 t_1^8 + b_{61} t_3^4 t_1^8 + b_{62} t_3^3 t_1^8 + b_{63} t_3^2 t_1^8 + b_{64} t_3 t_1^8 + b_{65} t_1^8 + b_{66} t_3^{12} t_1^7 + b_{67} t_3^{11} t_1^7 \\
&+ b_{68} t_3^{10} t_1^7 + b_{69} t_3^9 t_1^7 + b_{70} t_3^8 t_1^7 + b_{71} t_3^7 t_1^7 + b_{72} t_3^6 t_1^7 + b_{73} t_3^5 t_1^7 + b_{74} t_3^4 t_1^7 + b_{75} t_3^3 t_1^7 \\
&+ b_{76} t_3^2 t_1^7 + b_{77} t_3 t_1^7 + b_{78} t_1^7 + b_{79} t_3^{12} t_1^6 + b_{80} t_3^{11} t_1^6 + b_{81} t_3^{10} t_1^6 + b_{82} t_3^9 t_1^6 + b_{83} t_3^8 t_1^6 \\
&+ b_{84} t_3^7 t_1^6 + b_{85} t_3^6 t_1^6 + b_{86} t_3^5 t_1^6 + b_{87} t_3^4 t_1^6 + b_{88} t_3^3 t_1^6 + b_{89} t_3^2 t_1^6 + b_{90} t_3 t_1^6 + b_{91} t_1^6 \\
&+ b_{92} t_3^{12} t_1^5 + b_{93} t_3^{11} t_1^5 + b_{94} t_3^{10} t_1^5 + b_{95} t_3^9 t_1^5 + b_{96} t_3^8 t_1^5 + b_{97} t_3^7 t_1^5 + b_{98} t_3^6 t_1^5 + b_{99} t_3^5 t_1^5 \\
&+ b_{100} t_3^4 t_1^5 + b_{101} t_3^3 t_1^5 + b_{102} t_3^2 t_1^5 + b_{103} t_3 t_1^5 + b_{104} t_1^5 + b_{105} t_3^{12} t_1^4 + b_{106} t_3^{11} t_1^4 \\
&+ b_{107} t_3^{10} t_1^4 + b_{108} t_3^9 t_1^4 + b_{109} t_3^8 t_1^4 + b_{110} t_3^7 t_1^4 + b_{111} t_3^6 t_1^4 + b_{112} t_3^5 t_1^4 + b_{113} t_3^4 t_1^4 \\
&+ b_{114} t_3^3 t_1^4 + b_{115} t_3^2 t_1^4 + b_{116} t_3 t_1^4 + b_{117} t_1^4 + b_{118} t_3^{12} t_1^3 + b_{119} t_3^{11} t_1^3 + b_{120} t_3^{10} t_1^3 \\
&+ b_{121} t_3^9 t_1^3 + b_{122} t_3^8 t_1^3 + b_{123} t_3^7 t_1^3 + b_{124} t_3^6 t_1^3 + b_{125} t_3^5 t_1^3 + b_{126} t_3^4 t_1^3 + b_{127} t_3^3 t_1^3 \\
&+ b_{128} t_3^2 t_1^3 + b_{129} t_3 t_1^3 + b_{130} t_1^3 + b_{131} t_3^{132} t_1^2 + b_{133} t_3^{11} t_1^2 + b_{134} t_3^{10} t_1^2 + b_{135} t_3^9 t_1^2 \\
&+ b_{136} t_3^8 t_1^2 + b_{137} t_3^7 t_1^2 + b_{138} t_3^6 t_1^2 + b_{139} t_3^5 t_1^2 + b_{140} t_3^4 t_1^2 + b_{141} t_3^3 t_1^2 + b_{142} t_3^2 t_1^2 \\
&+ b_{143} t_3 t_1^2 + b_{144} t_1^2 + b_{145} t_3^{12} t_1 + b_{146} t_3^{11} t_1 + b_{147} t_3^{10} t_1 + b_{148} t_3^9 t_1 + b_{149} t_3^8 t_1 \\
&+ b_{150} t_3^7 t_1 + b_{151} t_3^6 t_1 + b_{152} t_3^5 t_1 + b_{153} t_3^4 t_1 + b_{154} t_3^3 t_1 + b_{155} t_3^2 t_1 + b_{156} t_3 t_1 \\
&+ b_{157} t_1 + b_{158} t_3^{12} + b_{159} t_3^{11} + b_{159} t_3^{10} + b_{160} t_3^9 + b_{161} t_3^8 + b_{162} t_3^7 + b_{163} t_3^6 \\
&+ b_{164} t_3^5 + b_{165} t_3^4 + b_{166} t_3^3 + b_{167} t_3^2 + b_{168} t_3 + b_{169},
\end{aligned}
$$

where the coefficients b_i are

$b_1 = 303991592446001788228938487$	$b_2 = 140804523326289420040988726 4$
$b_3 = -6103483100454386099859795 26$	$b_4 = 1507948223997213572885992 64$
$b_5 = 3175292247340716691173055833$	$b_6 = -3207402080141637801499335 04$
$b_7 = -17048997639478497073775739 08$	$b_8 = -4055730642111867547325842 56$
$b_9 = 622769876296825615581171 13$	$b_{10} = -1756713669376096824733512 64$
$b_{11} = -9091658747190671268431001 66$	$b_{12} = 8410996908736764862587488$
$b_{13} = 6779069003572339585723496 7$	$b_{14} = 7699197687756043949836937 04$
$b_{15} = 6086148202981589308165555 20$	$b_{16} = 1006338413132290284840738864$
$b_{17} = 14313215551031035152870988 80$	$b_{18} = 1106920085504028821353449720$
$b_{19} = 1051042237498193081988510720$	$b_{20} = 1785102108481388009724969120$
$b_{21} = 274182185151143817728302080$	$b_{22} = 598987412480135128015248120$
$b_{23} = 77449776729327796556689920$	$b_{24} = -1418246114756035420321443 36$
$b_{25} = 316030942714324763463552 00$	$b_{26} = 173788643378306408181172104$
$b_{27} = 1588372403382159061063149654$	$b_{28} = 8092975591745815633602116544$
$b_{29} = -3248693041709167069185540444$	$b_{30} = 211406878520421069494121024$
$b_{31} = 19938833981071197564188426058$	$b_{32} = -2121423179067511352527553664$
$b_{33} = -10009180546277357760027249288$	$b_{34} = -2156318684700125763185882496$
$b_{35} = 1048296922418733508883452938$	$b_{36} = -905812191897963002200866624$
$b_{37} = -4698504379735191229617889884$	$b_{38} = 626466076769497503574577 28$
$b_{39} = 470711522393073093178291734$	$b_{40} = 3275446737045079891833146776$
$b_{41} = 2798389897743457301579527680$	$b_{42} = 3430073858498420097704336016$
$b_{43} = 6384456051429072119840709120$	$b_{44} = 3437215685906732064733269480$
$b_{45} = 4256413077567038680571489280$	$b_{46} = 7086432988050350508983168480$
$b_{47} = 655959555828558576937681920$	$b_{48} = 1805773207367510400930991080$
$b_{49} = 88554595955003083626094080$	$b_{50} = -13965786760107236036772447 84$
$b_{51} = 102941964007868368998720000$	$b_{52} = 601492540218724645512852376$
$b_{53} = 1863185952562665535144523113$	$b_{54} = 18368537004625498191672760800$
$b_{55} = -10118633311500150875666933562$	$b_{56} = -3645531639364227160097798880$
$b_{57} = 47742911997329850459275889735$	$b_{58} = -7908940207973105675684819520$
$b_{59} = -29927104493544736358479363180$	$b_{60} = -5586703187951216642169113280$
$b_{61} = 1322636002702229535320877735$	$b_{62} = -2216882120199644964482927520$
$b_{63} = -10708672793085961168931909562$	$b_{64} = 104935955560450798251925920$
$b_{65} = 823337441741604282540503113$	$b_{66} = 4610181176067349319811463504$
$b_{67} = 4934488104553140293973765120$	$b_{68} = 673222061320160933366852064$
$b_{69} = 10443232592303538229139911680$	$b_{70} = -1826353774138329805781401680$
$b_{71} = 5118958520150359554781009920$	$b_{72} = 6459378942855282670710355520$
$b_{73} = -1347774333705502370068101120$	$b_{74} = -1787454240730479998434739280$
$b_{75} = -951934381413631957788410880$	$b_{76} = -5963483656776835980091631136$
$b_{77} = 60539846918320318945536 00$	$b_{78} = 172744186200068108390253904$
$b_{79} = -1621708105402125393637763148$	$b_{80} = 20878158753803570951808551040$
$b_{81} = -17588058999912760831475813128$	$b_{82} = -12899199805848832415420900480$
$b_{83} = 5865326757991651102661236302 0$	$b_{84} = -15440556721247650813313611520$
$b_{85} = -50934444886457453503292294000$	$b_{86} = -7665732406157195645450753280$
$b_{87} = -1867587215720614217259928180$	$b_{88} = -2791367357827462082945909120$
$b_{89} = -12614307807012810344397195528$	$b_{90} = 511488224574987090606841 60$
$b_{91} = 374525371112801525600025652$	$b_{92} 231291319794971341883529297 6$
$b_{93} = 4148805890329529288800711680$	$b_{94} = -6389500990001080093169282784$
$b_{95} = 7559997156913701546733317120$	$b_{96} = -10609352208188163894569321520$
$b_{97} = 728772418724086020836321280$	$b_{98} = 438991117668126118107964 80$
$b_{99} = -4884992862282772005446830080$	$b_{100} = -620879231616013110054535992 0$
$b_{101} = -2460343804730642330472913920$	$b_{102} = -916170666266967463594222358 4$
$b_{103} = -2577697903079565621223680 00$	$b_{104} = -1002077214505360541021321424$

$b_{105} = -3971430621041392996112889447$	$b_{106} = 12374289928344354025078994400$
$b_{107} = -123440004585941149247824169 22$	$b_{108} = -16330806431518499543110022880$
$b_{109} = 4643711013330358539125118133 5$	$b_{110} = -14124913545548867479214291520$
$b_{111} = -42477362711277341585174494380$	$b_{112} = -3946994294230016748826649280$
$b_{113} = -333055096850542145835106 65$	$b_{114} = -1059408107241328466289647520$
$b_{115} = -4873852326922304202721184922$	$b_{116} = 977554530949338159917275 20$
$b_{117} = 378892493096299322127122553$	$b_{118} = 200730767420231523311578024$
$b_{119} = 1644632956482672247915461120$	$b_{120} = -5190506023107083703857888016$
$b_{121} = 2171512967117559308974241280$	$b_{122} = -5226483846546228429709648680$
$b_{123} = -2128889600225717310326983680$	$b_{124} = 3195078713173340252023634720$
$b_{125} = -4473735425890521681580139520$	$b_{126} = -528169211751130452219818280$
$b_{127} = -2097914965047015435859668480$	$b_{128} = -4390595229150387720190227216$
$b_{129} = -279949150017098125665292800$	$b_{130} = -832100248207003813406591576$
$b_{131} = -1962737440728361463430634346$	$b_{132} = 357164505462827499587211974 4$
$b_{133} = -1135972992384322472318023644$	$b_{134} = -8698405461151187684087549376$
$b_{135} = 2394617115184791361455323725 8$	$b_{136} = -4881243111994631139488859264$
$b_{137} = -16439941847018161873601156488$	$b_{138} = 1104550916440195157556415104$
$b_{139} = 2661394009267641937515912138$	$b_{140} = 862790136065306374490947776$
$b_{141} = 1979063068004556038527706916$	$b_{142} = 199041204126846535197342528$
$b_{143} = 760913305201756890114011734$	$b_{144} = -597893269569197400394965 20$
$b_{145} = 240540093260984882325542400$	$b_{146} = -865085042503842495052102320$
$b_{147} = 101613906181427386827225600$	$b_{148} = 442302423721227403531571400$
$b_{149} = -944074338868289374965273600$	$b_{150} = 3264005580186614462109154400$
$b_{151} = -1318294267990666805410790400$	$b_{152} = 1902466505411470364808179400$
$b_{153} = -600610360955680350244569600$	$b_{154} = -2511847840742990075612863 20$
$b_{155} = -87464244753745424300736000$	$b_{156} = -1372438485673050688044885 20$
$b_{157} = -286675820524507063672675993$	$b_{158} = 391907233944915164121144864$
$b_{159} = 531122523280024395014816794$	$b_{160} = -1560655139996593684095106336$
$b_{161} = 3760122027727895557551059433$	$b_{162} = -886290507736563702809799 04$
$b_{163} = -5171315387161144294026388308$	$b_{164} = 1221770371975293894648126144$
$b_{165} = -932340071814670759258887287$	$b_{166} = 617572180718464402590098336$
$b_{167} = 588807353130457320567680154$	$b_{168} = 101734228697173615509243488$
$b_{169} = 241273820025099567960612487$	

$$
\begin{aligned}
p_3 =\, & c_1 t_1^{12} t_3^{12} + c_2 t_1^{12} t_3^{11} + c_3 t_3^{10} t_1^{12} + c_4 t_3^{9} t_1^{12} + c_5 t_3^{8} t_1^{12} + c_6 t_3^{7} t_1^{12} + c_7 t_3^{6} t_1^{12} + c_8 t_3^{5} t_1^{12} \\
& + c_9 t_3^{4} t_1^{12} + c_{10} t_3^{3} t_1^{12} + c_{11} t_3^{2} t_1^{12} + c_{12} t_3 t_1^{12} + c_{13} t_1^{12} + c_{14} t_3^{12} t_1^{11} + c_{15} t_1^{11} t_3^{11} \\
& + c_{16} t_3^{10} t_1^{11} + c_{17} t_3^{9} t_1^{11} + c_{18} t_3^{8} t_1^{11} + c_{19} t_3^{7} t_1^{11} + c_{20} t_3^{6} t_1^{11} + c_{21} t_3^{5} t_1^{11} + c_{22} t_3^{4} t_1^{11} \\
& + c_{23} t_3^{3} t_1^{11} + c_{24} t_3^{2} t_1^{11} + c_{25} t_3 t_1^{11} + c_{26} t_1^{11} + c_{27} t_3^{12} t_1^{10} + c_{28} t_3^{11} t_1^{10} + c_{29} t_1^{10} t_3^{10} \\
& + c_{30} t_3^{9} t_1^{10} + c_{31} t_3^{8} t_1^{10} + c_{32} t_3^{7} t_1^{10} + c_{33} t_3^{6} t_1^{10} + c_{34} t_3^{5} t_1^{10} + c_{35} t_3^{4} t_1^{10} + c_{36} t_3^{3} t_1^{10} \\
& + c_{37} t_3^{2} t_1^{10} + c_{38} t_3 t_1^{10} + c_{39} t_1^{10} + c_{40} t_3^{12} t_1^{9} + c_{41} t_3^{11} t_1^{9} + c_{42} t_3^{10} t_1^{9} + c_{43} t_1^{9} t_3^{9} \\
& + c_{44} t_3^{8} t_1^{9} + c_{45} t_3^{7} t_1^{9} + c_{46} t_3^{6} t_1^{9} + c_{47} t_3^{5} t_1^{9} + c_{48} t_3^{4} t_1^{9} + c_{49} t_3^{3} t_1^{9} + c_{50} t_3^{2} t_1^{9} + c_{51} t_3 t_1^{9} \\
& + c_{52} t_1^{9} + c_{53} t_3^{12} t_1^{8} + c_{54} t_3^{11} t_1^{8} + c_{55} t_3^{10} t_1^{8} + c_{56} t_3^{9} t_1^{8} + c_{57} t_3^{8} t_1^{8} + c_{58} t_3^{7} t_1^{8} + c_{59} t_3^{6} t_1^{8} \\
& + c_{60} t_3^{5} t_1^{8} + c_{61} t_3^{4} t_1^{8} + c_{62} t_3^{3} t_1^{8} + c_{63} t_3^{2} t_1^{8} + c_{64} t_3 t_1^{8} + c_{65} t_1^{8} + c_{66} t_3^{12} t_1^{7} + c_{67} t_3^{11} t_1^{7}
\end{aligned}
$$

$$
\begin{aligned}
&+ c_{68}t_3^{10}t_1^7 + c_{69}t_3^9t_1^7 + c_{70}t_3^8t_1^7 + c_{71}t_3^7t_1^7 + c_{72}t_3^6t_1^7 + c_{73}t_3^5t_1^7 + c_{74}t_3^4t_1^7 + c_{75}t_3^3t_1^7\\
&+ c_{76}t_3^2t_1^7 + c_{77}t_3t_1^7 + c_{78}t_1^7 + c_{79}t_3^{12}t_1^6 + c_{80}t_3^{11}t_1^6 + c_{81}t_3^{10}t_1^6 + c_{82}t_3^9t_1^6 + c_{83}t_3^8t_1^6\\
&+ c_{84}t_3^7t_1^6 + c_{85}t_3^6t_1^6 + c_{86}t_3^5t_1^6 + c_{87}t_3^4t_1^6 + c_{88}t_3^3t_1^6 + c_{89}t_3^2t_1^6 + c_{90}t_3t_1^6 + c_{91}t_1^6\\
&+ c_{92}t_3^{12}t_1^5 + c_{93}t_3^{11}t_1^5 + c_{94}t_3^{10}t_1^5 + c_{95}t_3^9t_1^5 + c_{96}t_3^8t_1^5 + c_{97}t_3^7t_1^5 + c_{98}t_3^6t_1^5 + c_{99}t_3^5t_1^5\\
&+ c_{100}t_3^4t_1^5 + c_{101}t_3^3t_1^5 + c_{102}t_3^2t_1^5 + c_{103}t_3t_1^5 + c_{104}t_1^5 + c_{105}t_3^{12}t_1^4 + c_{106}t_3^{11}t_1^4\\
&+ c_{107}t_3^{10}t_1^4 + c_{108}t_3^9t_1^4 + c_{109}t_3^8t_1^4 + c_{110}t_3^7t_1^4 + c_{111}t_3^6t_1^4 + c_{112}t_3^5t_1^4 + c_{113}t_1^4t_3^4\\
&+ c_{114}t_3^3t_1^4 + c_{115}t_3^2t_1^4 + c_{116}t_3t_1^4 + c_{117}t_1^4 + c_{118}t_3^{12}t_1^3 + c_{119}t_3^{11}t_1^3 + c_{120}t_3^{10}t_1^3\\
&+ c_{121}t_3^9t_1^3 + c_{122}t_3^8t_1^3 + c_{123}t_3^7t_1^3 + c_{124}t_3^6t_1^3 + c_{125}t_3^5t_1^3 + c_{126}t_3^4t_1^3 + c_{127}t_1^3t_3^3\\
&+ c_{128}t_3^2t_1^3 + c_{129}t_3t_1^3 + c_{130}t_1^3 + c_{131}t_3^{12}t_1^2 + c_{132}t_3^{11}t_1^2 + c_{133}t_3^{10}t_1^2 + c_{134}t_3^9t_1^2\\
&+ c_{135}t_3^8t_1^2 + c_{136}t_3^7t_1^2 + c_{137}t_3^6t_1^2 + c_{138}t_3^5t_1^2 + c_{139}t_3^4t_1^2 + c_{140}t_3^3t_1^2 + c_{141}t_3^2t_1^2\\
&+ c_{142}t_3t_1^2 + c_{143}t_1^2 + c_{144}t_3^{12}t_1 + c_{145}t_3^{11}t_1 + c_{146}t_3^{10}t_1 + c_{147}t_3^9t_1 + c_{148}t_3^8t_1\\
&+ c_{149}t_3^7t_1 + c_{150}t_3^6t_1 + c_{151}t_3^5t_1 + c_{152}t_3^4t_1 + c_{153}t_3^3t_1 + c_{154}t_3^2t_1 + c_{155}t_1t_3\\
&+ c_{156}t_1 + c_{157}t_3^{12} + c_{158}t_3^{11} + c_{159}t_3^{10} + c_{160}t_3^9 + c_{161}t_3^8 + c_{162}t_3^7 + c_{163}t_3^6\\
&+ c_{164}t_3^5 + c_{165}t_3^4 + c_{166}t_3^3 + c_{167}t_3^2 + c_{168}t_3 + c_{169},
\end{aligned}
$$

where the coefficients c_i are

$c_1 = 4928270142598459152996082273195366377314$19	$c_2 = -2438080869483823805524125927423327101760864$
$c_3 = -5839673697297456914073327259820654811942$22	$c_4 = -14030753213599793266190916869868749229728928$
$c_5 = -21123988808889924406495752267366290981937$39	$c_6 = -20704537678061895220347710778183777494702272$
$c_7 = 16314059677623217238964322331384947587760444$	$c_8 = -2952515995403178561164076745475852546919488$
$c_9 = 3116265877699899683390076372632753149365346$1	$c_{10} = 2995100914173664161345254044563304380156448$
$c_{11} = 8694194157488590194646535078660583863610258$	$c_{12} = 252497957065961189248293575799739160341728$
$c_{13} = 1082715163783039276166468232443383736985339$	$c_{14} = 3492602981916068139510311697321894896336160$
$c_{15} = -10867722169103829718478279355801841222464$00	$c_{16} = 19104079234490622993495661251498388385943360$
$c_{17} = -6479921110772705738680288056214355877292800$	$c_{18} = 4751365142442668658960640032292336800894$0000
$c_{19} = -8645153532208056857462556794504613480307200$	$c_{20} = 7174386984230392770612893284876213138944$9600
$c_{21} = -4909549409521381783013605948130978364288$00	$c_{22} = 6144370562749140844738829235260701202670$0000
$c_{23} = 4467727356180986120302936646931141870124800$	$c_{24} = 23145629144821074004398951377060529369111360$
$c_{25} = 1706677658787390207974200567873797981292800$	$c_{26} = 1543618187781461527518541104468774212528160$
$c_{27} = -23238165618940208152918624123808761862289$62	$c_{28} = -15221232929930773458456472178578407675326784$
$c_{29} = -34262923829620036787064124222157609493388588$	$c_{30} = -8496737363513909270207592847347425357269676$8
$c_{31} = -87563593152089934929362358545649791350450174$	$c_{32} = -129428210040115971570675908069145369017890432$
$c_{33} = -3289163113523431857186036315376701628705256$	$c_{34} = -31343870431795582756934128868875937872224128$
$c_{35} = 11359508136452527806315388636004118291087302$6	$c_{36} = 6148329632500583640777991728611485795069888$
$c_{37} = 3147811259224765536899766931979209245563029$2	$c_{38} = -1689390982001220570825035447768473853226432$
$c_{39} = 7427447651187831461330285031989406759342558$	$c_{40} = 1338301220322790791782462130312145274710564$8
$c_{41} = -1421559284855548360829961693156575825875200$	$c_{42} = 7429437807395635562957699113631030451034908$8
$c_{43} = -13568022171936691683662440860954055825350400$	$c_{44} = 18144738021536335667380917707436563887210152$0
$c_{45} = -10409412470586033575120823305984180370649600$	$c_{46} = 26382372411290729360129901467598288039902816$0
$c_{47} = 2408429732629772627535982358485441471740160$0	$c_{48} = 21093802017740330501723746343040147910938152$0
$c_{49} = 3223253980088452834558216210962802637472640$0	$c_{50} = 6502057172712382528487736326749406644097308$8
$c_{51} = 988529289108191118179339943865869112861504$00	$c_{52} = -2629738682007095093117892728101319378238352$
$c_{53} = -8585634321747039597044812015468889237484059$	$c_{54} = -38138639261621573185878772093209972833203360$
$c_{55} = -99459097371239746361552224007534640828174994$	$c_{56} = -208662958402028599241398852262553107190714720$
$c_{57} = -250110400612387426498394945973529777602342645$	$c_{58} = -325004178824310003530726853593163716251753280$
$c_{59} = -58308874505884115504607589998842526587829820$	$c_{60} = -99112247874103864818271271760567769256669120$
$c_{61} = 24444236494324284359887763077507624463792535$5	$c_{62} = -6383670908668405183963835675976378354920480$
$c_{63} = 7740921526593103280601993359661081934594020$6	$c_{64} = -10500086996941854444601098016745126992493280$
$c_{65} = 2691764751126121107412835093351142613183674$1	$c_{66} = 1909404326786293371008565582538095059791379$2
$c_{67} = 277261040898751428413879221026997400408320$0	$c_{68} = 10735261588993800127340880776183653982995955$2
$c_{69} = -435715139505203306977082946120434998233600$	$c_{70} = 25579463902800785463697903840168274924171808$0
$c_{71} = 2505128140873567834781787819237397059023360$0	$c_{72} = 35602425376392665447236325795661541804955264$0
$c_{73} = 8364720577168176295006450735897329271293440$0	$c_{74} = 25780799245428997934581875693980891204523808$0
$c_{75} = 7653465471419810620807308730499306364181760$0	$c_{76} = 4852295988631900971593378975960557134637555$2
$c_{77} = 2114705589974473919694233329478415052143360$0	$c_{78} = -20796845209977102231177595688815082658982208$

$c_{79} = -984959335685067562905726591234945254866863 6$	$c_{80} = -490762283082962513392172656785428191549228 80$
$c_{81} = -1184774401376864121352604160800731534844409 36$	$c_{82} = -2654674767783395680556323568372033091907897 60$
$c_{83} = -2960550998286442081690019334246494639300596 20$	$c_{84} = -4195987977413869632731703243121058255618342 40$
$c_{85} = -4619430240135787597181552491786092804990584 0$	$c_{86} = -1418588351554202582532050931487700878015449 60$
$c_{87} = 3358026562310881626460874129197546188787243 80$	$c_{88} = -2702747510306884901787605145088107190613184 0$
$c_{89} = 1169345054131694085836114024863498743171206 64$	$c_{90} = -2004126159029135279302853902557756763055424 0$
$c_{91} = 4639511200394839847878205341146722638277776 4$	$c_{92} = 1307453506074768153888960678025395147742908 8$
$c_{93} = 6376784916240002535742343118761989697036800$	$c_{94} = 7460148224413576871018448501422620715297772 8$
$c_{95} = 1710998861317960176999272143284443508847360 0$	$c_{96} = 1716622402373614944281059155014014106583451 20$
$c_{97} = 5441843066637115145431707229033617528752640 0$	$c_{98} = 2244713422581565406113354950312955853561849 60$
$c_{99} = 1044291536095902785330446949291231158541056 00$	$c_{100} = 1362074298992521391638819069440162850228251 20$
$c_{101} = 8215904509158549710439734312480549771434240 0$	$c_{102} = -8910406882712120111403793589855870289838272$
$c_{103} = 2141511845142677079141934217193611175632640 0$	$c_{104} = -3078178757778112592076124277000572493927491 2$
$c_{105} = -5082172265702464585247355603705187123392459$	$c_{106} = -3412251255915697390675788841725848642840336 0$
$c_{107} = -7110732442153962808180595630198208960842539 4$	$c_{108} = -1840218929512461117483734418438111561811147 20$
$c_{109} = -1777787228764743588849252927291026084941686 45$	$c_{110} = -2931789855630792410178301726147095474613532 80$
$c_{111} = 5630357503192470717671281385280068729202180$	$c_{112} = -1006716929556299549601936843852627704950691 20$
$c_{113} = 2637584200177929893707789007074301356260993 55$	$c_{114} = -2649451256460591943000148056551851518852048 0$
$c_{115} = 9281224791922069480891787222947894838968980 6$	$c_{116} = -1785122905844040424972417604592855840209328 0$
$c_{117} = 3946049025681978918188304906750993950192834 1$	$c_{118} = 4321739489821064172804252764526646351858592$
$c_{119} = 4183362550647183634987602511988674689555200$	$c_{120} = 2511682455332910866018268475640362303175635 2$
$c_{121} = 1365463951099157541966836854130573125671040 0$	$c_{122} = 5494830615876758746550562179487916929524208 0$
$c_{123} = 3527113488523141785201000075213678472328960 0$	$c_{124} = 6516077583630687871083420973857095489257664 0$
$c_{125} = 5670865391408686420884196509480950162115840 0$	$c_{126} = 2631091403653868689534213434145557047396208 0$
$c_{127} = 4130564846320985836538938953848820065151360 0$	$c_{128} = -2272096062849207074388378054814328359646764 8$
$c_{129} = 1039685247401002022387665916649842718648960 0$	$c_{130} = -1802431992398342190651889495403009179319740 8$
$c_{131} = -1125002969761079454640643743261266930181922$	$c_{132} = -1209321780716056208024146284337633095878278 4$
$c_{133} = -2126557968695248552553559746005939231845034 8$	$c_{134} = -6564084028149121238221108107806604106760876 8$
$c_{135} = -5383436799568805452444188226322616549944057 4$	$c_{136} = -1046644690670291198077635109891323031229784 32$
$c_{137} = 1607937548129460302416602035706012618689154 4$	$c_{138} = -3321161575485575201617099249496178053257612 8$
$c_{139} = 1057346852004224806155095817609812555202826 26$	$c_{140} = -1021232613213984577804858385797693077273811 2$
$c_{141} = 3477476594465536930559947343828282729328853 2$	$c_{142} = -7617078681372297909152188388464117654314432$
$c_{143} = 1602234115664910722288528223253187887106959 8$	$c_{144} = 5469740520575276993453064720814441215738 08$
$c_{145} = 9139751113398607723661853610730507002304 00$	$c_{146} = 3254472818263915389281563228566178584332448$
$c_{147} = 3197036819470756287680716967080561118060800$	$c_{148} = 6589009804973044595065681967128584515453520$
$c_{149} = 7668244565973921463169073165654146916339200$	$c_{150} = 6556544600134649349878176201217168576365760$
$c_{151} = 1135145380892848417220059334499216592615680 0$	$c_{152} = -1408043118396024876804727171359220863064 80$
$c_{153} = 7916445641118925243785908255978882074387200$	$c_{154} = -6585722983165385395068723479751240032115552$
$c_{155} = 1950174690033467019439856470633352646739200$	$c_{156} = -3769885109957790462639499772041999422138192$
$c_{157} = -6868442399995250858573548254168856693442 1$	$c_{158} = -1691446355125215979754414262605771425664864$
$c_{159} = -2527061143852823572625761565219155128485262$	$c_{160} = -9360086128095368883956155412510591229536928$
$c_{161} = -6630305236928942947664202400225894960245339$	$c_{162} = -1480377478308091488270500638432645649489427 2$
$c_{163} = 4055345117805025211365989277907354279147644$	$c_{164} = -3677864516934594686681181621301413907687488$
$c_{165} = 1659021242774356837236574373587365617720186 1$	$c_{166} = -1208389043607450702526581383314927720515552$
$c_{167} = 4600219955329106189093067501246646716799218$	$c_{168} = -1248913155348573767928782374936212903850272$
$c_{169} = 2438796771237462754231229636119130017439499$	

$$
\begin{aligned}
p_4 =& d_1t_3^{12}t_1^{12} + d_2t_3^{11}t_1^{12} + d_3t_3^{10}t_1^{12} + d_4t_3^{9}t_1^{12} + d_5t_3^{8}t_1^{12} + d_6t_3^{7}t_1^{12} + d_7t_3^{6}t_1^{12} + d_8t_3^{5}t_1^{12} + d_9t_3^{4}t_1^{12} + d_{10}t_3^{3}t_1^{12} \\
&+ d_{11}t_3^{2}t_1^{12} + d_{12}t_3t_1^{12} + d_{13}t_1^{12} + d_{14}t_3^{12}t_1^{11} + d_{15}t_3^{11}t_1^{11} + d_{16}t_3^{10}t_1^{11} + d_{17}t_3^{9}t_1^{11} + d_{18}t_3^{8}t_1^{11} + d_{19}t_3^{7}t_1^{11} \\
&+ d_{20}t_3^{6}t_1^{11} + d_{21}t_3^{5}t_1^{11} + d_{22}t_3^{4}t_1^{11} + d_{23}t_3^{3}t_1^{11} + d_{24}t_3^{2}t_1^{11} + d_{25}t_3t_1^{11} + d_{26}t_1^{11} + d_{27}t_3^{12}t_1^{10} + d_{28}t_3^{11}t_1^{10} \\
&+ d_{29}t_3^{10}t_1^{10} + d_{30}t_3^{9}t_1^{10} + d_{31}t_3^{8}t_1^{10} + d_{32}t_3^{7}t_1^{10} + d_{33}t_3^{6}t_1^{10} + d_{34}t_3^{5}t_1^{10} + d_{35}t_3^{4}t_1^{10} + d_{36}t_3^{3}t_1^{10} + d_{37}t_3^{2}t_1^{10} \\
&+ d_{38}t_3t_1^{10} + d_{39}t_1^{10} + d_{40}t_3^{12}t_1^{9} + d_{41}t_3^{11}t_1^{9} + d_{42}t_3^{10}t_1^{9} + d_{43}t_3^{9}t_1^{9} + d_{44}t_3^{8}t_1^{9} + d_{45}t_3^{7}t_1^{9} + d_{46}t_3^{6}t_1^{9} + d_{47}t_3^{5}t_1^{9} \\
&+ d_{48}t_3^{4}t_1^{9} + d_{49}t_3^{3}t_1^{9} + d_{50}t_3^{2}t_1^{9} + d_{51}t_3t_1^{9} + d_{52}t_1^{9} + d_{53}t_3^{12}t_1^{8} + d_{54}t_3^{11}t_1^{8} + d_{55}t_3^{10}t_1^{8} + d_{56}t_3^{9}t_1^{8} + d_{57}t_3^{8}t_1^{8} \\
&+ d_{58}t_3^{7}t_1^{8} + d_{59}t_3^{6}t_1^{8} + d_{60}t_3^{5}t_1^{8} + d_{61}t_3^{4}t_1^{8} + d_{62}t_3^{3}t_1^{8} + d_{63}t_3^{2}t_1^{8} + d_{64}t_3t_1^{8} + d_{65}t_1^{8} + d_{66}t_3^{12}t_1^{7} + d_{67}t_3^{11}t_1^{7} \\
&+ d_{68}t_3^{10}t_1^{7} + d_{69}t_3^{9}t_1^{7} + d_{70}t_3^{8}t_1^{7} + d_{71}t_3^{7}t_1^{7} + d_{72}t_3^{6}t_1^{7} + d_{73}t_3^{5}t_1^{7} + d_{74}t_3^{4}t_1^{7} + d_{75}t_3^{3}t_1^{7} + d_{76}t_3^{2}t_1^{7} + d_{77}t_3t_1^{7} \\
&+ d_{78}t_1^{7} + d_{79}t_3^{12}t_1^{6} + d_{80}t_3^{11}t_1^{6} + d_{81}t_3^{10}t_1^{6} + d_{82}t_3^{9}t_1^{6} + d_{83}t_3^{8}t_1^{6} + d_{84}t_3^{7}t_1^{6} + d_{85}t_3^{6}t_1^{6} + d_{86}t_3^{5}t_1^{6} + d_{87}t_3^{4}t_1^{6} \\
&+ d_{88}t_3^{3}t_1^{6} + d_{89}t_3^{2}t_1^{6} + d_{90}t_3t_1^{6} + d_{91}t_1^{6} + d_{92}t_3^{12}t_1^{5} + d_{93}t_3^{11}t_1^{5} + d_{94}t_3^{10}t_1^{5} + d_{95}t_3^{9}t_1^{5} + d_{96}t_3^{8}t_1^{5} + d_{97}t_3^{7}t_1^{5}
\end{aligned}
$$

$$+ d_{98}t_3^6t_1^5 + d_{99}t_3^5t_1^5 + d_{100}t_3^4t_1^5 + d_{101}t_3^3t_1^5 + d_{102}t_3^2t_1^5 + d_{103}t_3t_1^5 + d_{104}t_1^5 + d_{105}t_3^{12}t_1^4$$
$$+ d_{106}t_3^{11}t_1^4 + d_{107}t_3^{10}t_1^4 + d_{108}t_3^9t_1^4 + d_{109}t_3^8t_1^4 + d_{110}t_3^7t_1^4 + d_{111}t_3^6t_1^4 + d_{112}t_3^5t_1^4 + d_{113}t_3^4t_1^4$$
$$+ d_{114}t_3^3t_1^4 + d_{115}t_3^2t_1^4 + d_{116}t_3t_1^4 + d_{117}t_1^4 + d_{118}t_3^{12}t_1^3 + d_{119}t_3^{11}t_1^3 + d_{120}t_3^{10}t_1^3 + d_{121}t_3^9t_1^3$$
$$+ d_{122}t_3^8t_1^3 + d_{123}t_3^7t_1^3 + d_{124}t_3^6t_1^3 + d_{125}t_3^5t_1^3 + d_{126}t_3^4t_1^3 + d_{127}t_3^3t_1^3 + d_{128}t_3^2t_1^3 + d_{129}t_3t_1^3$$
$$+ d_{130}t_1^3 + d_{131}t_3^{12}t_1^2 + d_{132}t_3^{11}t_1^2 + d_{133}t_3^{10}t_1^2 + d_{134}t_3^9t_1^2 + d_{135}t_3^8t_1^2 + d_{136}t_3^7t_1^2 + d_{137}t_3^6t_1^2$$
$$+ d_{138}t_3^5t_1^2 + d_{139}t_3^4t_1^2 + d_{140}t_3^3t_1^2 + d_{141}t_3^2t_1^2 + d_{142}t_3t_1^2 + d_{143}t_1^2 + d_{144}t_3^{12}t_1 + d_{145}t_3^{11}t_1$$
$$+ d_{146}t_3^{10}t_1 + d_{147}t_3^9t_1 + d_{148}t_3^8t_1 + d_{149}t_3^7t_1 + d_{150}t_3^6t_1 + d_{151}t_3^5t_1 + d_{152}t_3^4t_1 + d_{153}t_3^3t_1$$
$$+ d_{154}t_3^2t_1 + d_{155}t_3t_1 + d_{156}t_1 + d_{157}t_3^{12} + d_{158}t_3^{11} + d_{159}t_3^{10} + d_{160}t_3^9 + d_{161}t_3^8 + d_{162}t_3^7$$
$$+ d_{163}t_3^6 + d_{164}t_3^5 + d_{165}t_3^4 + d_{166}t_3^3 + d_{167}t_3^2 + d_{168}t_3 + d_{169},$$

where the coefficients d_i are

$d_1 = 9658413369501255163276163798755600773$	$d_2 = 35317295585910647321645419281661863456$
$d_3 = 15882166700089440555049128988946135766$	$d_4 = -105388392832058172214709783809655818464$
$d_5 = -350325349443127055311977332321980548393$	$d_6 = -428381522780610943179781647065840260416$
$d_7 = -15809160534680777659466556344143159372$	$d_8 = 147223860351680840789833153696442108096$
$d_9 = -6615219861428775459196868998657919693$	$d_{10} = -130457607152598424884258075670839593696$
$d_{11} = -66028157665812914294761894112469629034$	$d_{12} = -18827813456539996093060500369404280288$
$d_{13} = -4438557146599625422697512301083785327$	$d_{14} = -18862342571408613262814141417644415424$
$d_{15} = -27098581623582545267174529043905218880$	$d_{16} = -164348132527117286631607204863806542344$
$d_{17} = -108963336738218450613804747575390174400$	$d_{18} = -302829116835585602363956236127401975960$
$d_{19} = -206842844152946337723563587261549468800$	$d_{20} = -273865734699193772482856811909138308880$
$d_{21} = -231398698055413101271840354030223548800$	$d_{22} = -214120289749485949651766944724108262960$
$d_{23} = -142629302486948265492336291357833294400$	$d_{24} = -109393235536598237729637222417243283944$
$d_{25} = -36208693469845596597429306057674258880$	$d_{26} = -11795353606429131565563916921033480824$
$d_{27} = 57729284157066353866155204954743858226$	$d_{28} = 234656987160644447922639514794924595296$
$d_{29} = 121820635321241028646077092132716546224$	$d_{30} = -559506549687102693511196488061990037984$
$d_{31} = -2118567476935918029212416222051124573838$	$d_{32} = -2477495357417913260398615281171412696896$
$d_{33} = -203625537106801792577927445360122094672$	$d_{34} = 947931986191336347426324901057572154176$
$d_{35} = -102183554730617865953018442800748720138$	$d_{36} = -754420171555262600826706436279072729376$
$d_{37} = -395268568494608310922992670905533033376$	$d_{38} = -106507047854476707871067577816742947168$
$d_{39} = -28645092258199231344060934367609146074$	$d_{40} = -73970283591967734228551806698844431876$
$d_{41} = -134189170000435600917229454708240589120$	$d_{42} = -676324943209845484012609069384477401456$
$d_{43} = -513753920087134271811361522644923025600$	$d_{44} = -1166568377609732222391110648329955513540$
$d_{45} = -912386100447926698984152962989285251200$	$d_{46} = -943207740013608747433032966943709157120$
$d_{47} = -966450443162048947134195117857859491200$	$d_{48} = -766862279563437264907714826444795926540$
$d_{49} = -579812415326676879975230163786352385600$	$d_{50} = -432571835027556162389038838648966019856$
$d_{51} = -146183322525855960931055940981095709120$	$d_{52} = -44703577485873172307303593503556706476$
$d_{53} = 191315042643624893059513334646493610352$	$d_{54} = 679843111749526518876465654522124399680$
$d_{55} = 740250951786465306163790653404374475432$	$d_{56} = -1051495204182701241337520378923832357760$
$d_{57} = -4567298913320431082807614362880426101740$	$d_{58} = -5601443897393791581320665217099419427840$
$d_{59} = 49378570712444040552799773994988051760$	$d_{60} = 2974683728903811939898123617369472019840$
$d_{61} = 210689382944802130009482394572683194760$	$d_{62} = -1530144929589544094017213291483429106240$
$d_{63} = -711268829965776415000812749116172887768$	$d_{64} = -176991913799198956655158497920363796480$
$d_{65} = -42828243453612253016847113385061063948$	$d_{66} = -71403228819803075647235037594292987164$
$d_{67} = -276340325428861805181413653270972231680$	$d_{68} = -826830748066137855145509801288955145784$
$d_{69} = -958698623616697982429997693878013158400$	$d_{70} = -1028876778950424268929788905061289143060$
$d_{71} = -1448686586234176482148961046779916876800$	$d_{72} = -230538758872980450035127216361479577680$
$d_{73} = -1300323422737240100069148907901297996800$	$d_{74} = -357915239475244484172345572739976525060$
$d_{75} = -707679954094433657504337184620036838400$	$d_{76} = -433986950721373872114326335761653083384$
$d_{77} = -173684819403533862335565282891614791680$	$d_{78} = -33161210415020348545593838016529151564$
$d_{79} = 350871651626256743176411109116152088858$	$d_{80} = 1089714202226686546548282214306454394880$
$d_{81} = 1859003338875912157376044120131439040308$	$d_{82} = -725364496690152082851904149676874420480$
$d_{83} = -4615250140408917891828870452214289122690$	$d_{84} = -6330834902933215056529571936973326750720$
$d_{85} = 1225792967970035594750060525069677451320$	$d_{86} = 5112490037263997374490230331701725099520$
$d_{87} = 1247826542758169314920002701108756600310$	$d_{88} = -1370741153366541356303913048202579205120$
$d_{89} = -381782134472456115772808601958238312492$	$d_{90} = -68397626439287293434385493446302240000$
$d_{91} = 4254886772984477457428118126258612658$	$d_{92} = 12133046104963011972623401053615565724$
$d_{93} = -291844218340594690554353431215560197120$	$d_{94} = -244846173890652927950500933729293270856$
$d_{95} = -885120217451118186476551063313792665600$	$d_{96} = 431235518431186969261268616512628765460$
$d_{97} = -978105138303039807026880710070743731200$	$d_{98} = 1627071619724896597197103977159274604880$
$d_{99} = -488292671827481256785879223370613171200$	$d_{100} = 944651435641691915646022513671241027460$

$d_{101} = -123529706847342385625553413706364825600$	$d_{102} = 30059231916484628334052636883333050744$
$d_{103} = -20066174212377439944357268308262917120$	$d_{104} = 24264677572886400700741149075829026124$
$d_{105} = 298947609281456947471060011296941304352$	$d_{106} = 1015571184479274887945888415901154623680$
$d_{107} = 1870011388733431021645249868715687489432$	$d_{108} = 113151275426306775770268615262777482240$
$d_{109} = -3004180338976207791453575875424774221740$	$d_{110} = -3841787582529920346766663371050306307840$
$d_{111} = 747067120340221896655180543948731311760$	$d_{112} = 4553872354179739072488325653565854579840$
$d_{113} = 958780374827108496447812202232516424760$	$d_{114} = -670791068842013309414772587386715106240$
$d_{115} = -58684982615440335953157510489007593768$	$d_{116} = 33911229657016281872715783221501147520$
$d_{117} = 18362457607105685410733138741227700052$	$d_{118} = 23269440792038131824508691360632637556$
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