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ON THE NUMBER OF INVARIANT STRAIGHT LINES FOR POLYNOMIAL DIFFERENTIAL SYSTEMS

Joan C. Artés, Branko Grünbaum and Jaume Llibre

If P and Q are two real polynomials in the real variables x and y such that the degree of $P^2 + Q^2$ is $2n$, then we say that the polynomial differential system $x' = P(x, y)$, $y' = Q(x, y)$ has degree n. Let $\alpha(n)$ be the maximum number of invariant straight lines possible in a polynomial differential systems of degree $n > 1$ having finitely many invariant straight lines. In the 1980's the following conjecture circulated among mathematicians working in polynomial differential systems. Conjecture: $\alpha(n)$ is $2n+1$ if n is even, and $\alpha(n)$ is $2n+2$ if n is odd. The conjecture was established for $n = 2, 3, 4$. In this paper we prove that, in general, the conjecture is not true for $n > 4$. Specifically, we prove that $\alpha(5) = 14$. Moreover, we present counterexamples to the conjecture for $n \in \{6, 7, \dots, 20\}$. We also show that $2n + 1 \leq \alpha(n) \leq 3n - 1$ if n is even, and that $2n + 2 \leq \alpha(n) \leq 3n - 1$ if n is odd.

1. Introduction and statement of the main results.

Let P and Q be two real polynomials in the real variables x and y. We say that the polynomial differential system

$$x' = P(x, y)\,, \;\; y' = Q(x, y), \tag{1}$$

has *degree* n if the degree of the polynomial $P^2 + Q^2$ is $2n$.

Studies of polynomial differential systems were carried out by Poincaré in [**P1**], [**P2**] and [**P3**]. The algebraic feature of polynomial differential systems renders natural certain questions and problems of an algebraic or an algebro–geometric nature, such as to recognize when system (1) has invariant algebraic curves, or is algebraically integrable. See the interesting survey of Schlomiuk [**Sc**] on these questions. This paper deals with the former aspect.

The straight line $ax + by + c = 0$ is invariant for the flow of system (1), and we call it an *invariant straight line* of system (1) if $ax' + by' = aP(x, y) + bQ(x, y) = (ax + by + c)R(x, y)$ for some real polynomial R.

Suppose that the polynomial differential system (1) of degree n has finitely many invariant straight lines; then we denote by $\alpha(n, P, Q)$ the number