Resenhas IME-USP 2003, Vol. 6, No. 1, 85 - 97

STATISTICAL MEASURE OF QUADRATIC VECTOR FIELDS

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Abstract: In [2] the authors classified the 44 topological phase portraits of all the structurally stable quadratic vector fields on the Poincaré sphere $\mathbb{S}^2$ modulo limit cycles. In this topological study, no information is given about the regions in the space of all coefficients where such phase portraits take place. In this paper we use a statistical method to provide estimations of the relative frequency for such regions. We also give estimations of the relative frequencies for the regions of phase portraits having nodes, foci and limit cycles.

1. Introduction

A vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $X = (P, Q)$ where $P = \sum a_{ij}x^iy^j$ and $Q = \sum b_{ij}x^iy^j$, $0 \le i + j \le n$, is called a *planar polynomial vector field of degree* $\le n$. If $\sum_{i+j=n}(|a_{ij}| + |b_{ij}|) \neq 0$, then we say that X has degree n. In particular, the polynomial vector fields of degree 2 are called *quadratic vector fields*. The $M = (n+1)(n+2)$ real numbers a_{ij}, b_{ij} are called the *coefficients of* X. The space of these vector fields, endowed with the structure of affine $\mathbb{R}^M$–space in which X is identified with the M–tuple $(a_{00}, a_{10}, \dots, a_{0n}, b_{00}, b_{10}, \dots, b_{0n})$ of its coefficients, is denoted by $\mathcal{P}_n(\mathbb{S}^2)$.

The *Poincaré compactification* of $X \in \mathcal{P}_n(\mathbb{S}^2)$, is defined to be the unique analytic vector field $p(X)$ tangent to the sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, whose restriction to the northern hemisphere $\mathbb{S}^2_+ = \{(x, y, z) \in \mathbb{S}^2 : z > 0\}$ is given by $z^{n-1}(f_+)_*(X)$, where f_+ is the central projection from $\mathbb{R}^2$ to $\mathbb{S}^2_+$, defined by $f_+(x, y) = (x, y, 1)/(x^2 + y^2 + 1)^{1/2}$. See Section 2 of [2] for more details. The closed northern hemisphere $\{(x, y, z) \in \mathbb{S}^2 : z \ge 0\}$ is also called the *Poincaré disc*.

Let $\mathbb{S}^1 = \{(x, y, z) \in \mathbb{S}^2 : z = 0\}$ be the equator of the Poincaré sphere. Then, the vector field $X \in \mathcal{P}_n(\mathbb{S}^2)$ is said to be *topologically structurally stable* if there is a neighborhood N of X under the given topology and a continuous map $h : N \to \text{Hom}(\mathbb{S}^2, \mathbb{S}^1)$ (homeomorphisms of $\mathbb{S}^2$ which preserve $\mathbb{S}^1$) such that $h_X = \text{Id}$ and h_Y maps orbits of $p(X)$ onto orbits of $p(Y)$, for every $Y \in N$. Again see Section 2 of [2] for more details. Define by $\sum$ the set of quadratic vector fields $X \in \mathcal{P}_2(\mathbb{S}^2)$ which are topologically structurally stable.

We denote by $\varphi : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{S}^2$ the *flow generated by the vector field* $p(X)$. We call *phase portrait of the vector field* $p(X) : \mathbb{S}^2 \to \mathbb{S}^2$ the decomposition of $\mathbb{S}^2$ as union of all the orbits of $p(X)$. We consider all the orbits, different from a singular point, oriented in the sense of the integral curves of the vector field $p(X)$, i.e. if the orbit is $\varphi_x(t)$, it is oriented in the sense of the t increasing. We denote the