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THE GEOMETRY OF QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS WITH A WEAK FOCUS AND AN INVARIANT STRAIGHT LINE

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Planar quadratic differential systems occur in many areas of applied mathematics. Although more than a thousand papers were written on these systems, a complete understanding of this family is still missing. Classical problems, and in particular, Hilbert's 16th problem [Hilbert, 1900, 1902], are still open for this family. In this article, we conduct a global study of the class QW^{I} of all real quadratic differential systems which have a weak focus and invariant straight lines of total multiplicity of at least two. This family modulo the action of the affine group and time homotheties is three-dimensional and we give its bifurcation diagram with respect to a normal form, in the three-dimensional real projective space of the parameters of this form. The bifurcation diagram yields 73 phase portraits for systems in QW^{I} plus 26 additional phase portraits with the center at its border points. Algebraic invariants are used to construct the bifurcation set. We show that all systems in QW^{I} necessarily have their weak focus of order one and invariant straight lines of total multiplicity exactly two. The phase portraits are represented on the Poincaré disk. The bifurcation set is algebraic and all points in this set are points of bifurcation of singularities. We prove that there is no phase portrait with limit cycles in this class but that there is a total of five phase portraits with graphics, four having the invariant line as a regular orbit and one phase portrait with an infinity of graphics which are all homoclinic loops inside a heteroclinic graphic with two singularities, both at infinity.

Keywords: Quadratic differential systems; weak focus; limit cycle; phase portraits; bifurcation diagram; invariant straight line.

1. Introduction, Brief Review of the Literature and Statement of Results

In this paper, we call *quadratic differential systems* or simply *quadratic systems*, differential systems of the form

$$\dot{x} = p(x, y),\qquad \dot{y} = q(x, y), \tag{1}$$