FROM TOPOLOGICAL TO GEOMETRIC EQUIVALENCE IN THE CLASSIFICATION OF SINGULARITIES AT INFINITY FOR QUADRATIC VECTOR FIELDS

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Abstract. In the topological classification of phase portraits no distinctions are made between a focus and a node and neither are they made between a strong and a weak focus or between foci of different orders. These distinction are however important in the production of limit cycles close to the foci in perturbations of the systems. The distinction between the one direction node and the two directions node, which plays a role in understanding the behavior of solution curves around the singularities at infinity, is also missing in the topological classification.

In this work we introduce the notion of *geometric equivalence relation of singularities* which incorporates these important purely algebraic features. The *geometric* equivalence relation is finer than the *topological* one and also finer than the *qualitative equivalence relation* introduced in [19]. We also list all possibilities we have for singularities finite and infinite taking into consideration these finer distinctions and introduce notations for each one of them. Our long term goal is to use this finer equivalence relation to classify the quadratic family according to their different *geometric configurations of singularities*, finite and infinite.

In this work we accomplish a first step of this larger project. We give a complete global classification, using the *geometric equivalence* relation, of the whole quadratic class according to the configuration of singularities at infinity of the systems. Our classification theorem is stated in terms of invariant polynomials and hence it can be applied to any family of quadratic systems with respect to any particular normal form. The theorem we give also contains the bifurcation diagram, done in the 12-parameter space, of the *geometric configurations* of singularities at infinity, and this bifurcation set is algebraic in the parameter space. To determine the bifurcation diagram of configurations of singularities at infinity for any family of quadratic systems, given in any normal form, becomes thus a simple task using computer algebra calculations.

1. Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p,\, q \in \mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p,\, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$.

The study of the class of quadratic differential systems has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. The complete characterization of the phase portraits for real quadratic vector fields is not known and attempting to topologically classify these systems, which occur rather often in applications, is a very complex task. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time homotheties, the class ultimately depends on five parameters. This is still a large number of parameters and for the moment only subclasses depending on at most three parameters were studied globally. On the other hand we can restrict the study of this class by focusing on specific global features of the class. We may thus focus on the global study of singularities and their bifurcation diagram. The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying the differential systems on the sphere or on the Poincaré disk (see [15]).

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The global study of quadratic vector fields in the neighborhood of infinity was initiated by Nikolaev and Vulpe in [22] where they classified topologically the singularities at infinity in terms of invariant polynomials. Schlomiuk and Vulpe used geometrical concepts defined in [27], and also introduced some new geometrical concepts in [28] in order to simplify the invariant polynomials and the classification. To reduce the number of phase portraits in half, in both cases the topological equivalence relation was taken to mean the existence of a homeomorphism carrying orbits to orbits and *preserving or reversing* the orientation. In [3] the authors classified topologically (adding also the distinction between nodes and foci) the whole quadratic class according to configurations of their finite singularities.

The goal of our present work is to go deeper into these classifications by using a finer equivalence relation. In the topological classification no distinction was made among the various types of foci or saddles, strong or weak of various orders. However these distinctions, of algebraic nature, are very important in the study of perturbations of systems possessing such singularities. Indeed, the maximum number of limit cycles which can be produced close to the weak foci in perturbations depends on the orders of the foci. For these reason we shall include these distinctions in the new classification.

The distinction among weak saddles is also important since for example when a loop is formed using two separatrices of one weak saddle, the maximum number of limit cycles that can be obtained close to the loop in perturbations is the order of weak saddle.

There are also three kinds of nodes as we can see in Figure 1 below where the local phase portraits around the singularities are given.

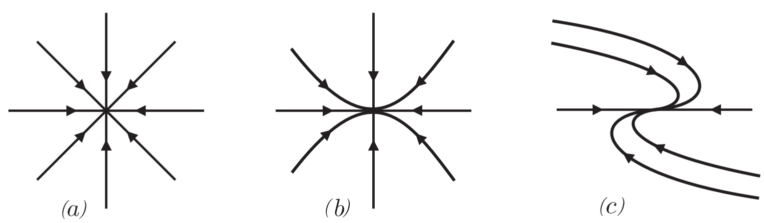


Figure 1. Different types of nodes

In the three phase portraits of Figure 1 the corresponding three singularities are stable nodes. These portraits are topologically equivalent but the solution curves do not arrive at the nodes in the same way. In the first case, any two distinct non-trivial phase curves arrive at the node with distinct slopes. Such a node is called a star node. In the second picture all non-trivial solution curves excepting two of them arrive at the node with the same slope but the two exception curves arrive at the node with a different slope. This is the generic node with two directions. In the third phase portrait all phase curves arrive at the node with the same slope.

We recall that the first and the third types of nodes could produce foci in perturbations and the first type of nodes is also involved in the existence of invariant straight lines of differential systems. For example it can be easily shown that if a quadratic differential system has two finite star nodes then necessarily the system possesses invariant straight lines of total multiplicity 6.

Furthermore, a generic node may or may not have the two exceptional curves lying on the line at infinite. This leads to two different situations for the phase portraits. For this reason we split the generic nodes at infinite in two types.

The finer equivalence relation we later introduce in this article, takes into account such distinctions.

The distinctions among the nilpotent and linearly zero singularities finite or infinite can also be refined, as it will be seen in Section 4. Such singularities are usually called *degenerate* singularities.

In this article we introduce for planar polynomial vector fields *the geometric equivalence relation* for singularities, finite or infinite. This equivalence relation is finer than the *qualitative equivalence relation* introduced by Jian and Llibre in [19] since it distinguishes among the foci of different orders and among the various types of nodes. This equivalence relation also induces a finer distinction among the more complicated degenerate singularities.

To distinguish among the foci (or saddles) of various orders we use the algebraic concept of Poincaré-Lyapunov constants. We call strong focus (or strong saddle) a focus with non–zero trace of the linearization matrix at this point. Such a focus (or saddle) will be considered to have the order zero. A focus (or saddle) with trace zero is called a weak focus (weak saddle). For details on Poincaré-Lyapunov constants and weak foci we refer to [20].

For the nodes in FIGURE 1 the distinction is also made by algebraic means: the linearization matrices at these nodes and their eigenvalues.

The finer distinctions of singularities are algebraic in nature. In fact the whole bifurcation diagram of the global configurations of singularities, finite and infinite, in quadratic vector fields and more generally in polynomial vector fields can be obtained by using only algebraic means, among them, the algebraic tool of polynomial invariants.

Algebraic information may not be significant for the local phase portrait around a singularity. For example, topologically there is no distinction between a focus and a node or between a weak and a strong focus. However, as indicated before, algebraic information plays a fundamental role in the study of perturbations of systems possessing such singularities.

In [11] Coppel wrote:

"Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients. However, attempts in this direction have met with very limited success..."

This proved to be impossible to realize. Indeed, Dumortier and Fiddelers [14] and Roussarie [25] exhibited examples of families of quadratic vector fields which have non-algebraic bifurcation sets.

Although we now sense that in trying to understand these systems, there is a limit to the power of algebraic methods, these methods have not been used far enough. In this work we go one step further in using them.

The following are legitimate questions:

How much of the behavior of quadratic (or more generally polynomial) vector fields can be described by algebraic means? How far can we go in the global theory of these vector fields by using mainly algebraic means?

For certain subclasses of quadratic vector fields the full description of the phase portraits as well as of the bifurcation diagrams can be obtained using only algebraic tools. Examples of such classes are:

- the quadratic vector fields possessing a center [36, 26, 38, 23];
- the quadratic Hamiltonian vector fields [1, 4];
- the quadratic vector fields with invariant straight lines of total multiplicity at least four [29, 30];
- the planar quadratic differential systems possessing a line of singularities at infinity [31];
- the quadratic vector fields possessing an integrable saddle [5].
- the family of Lotka-Volterra systems [32, 33], once we assume Bautin's analytic result saying that such systems have no limit cycles;

In the case of other subclasses of the quadratic class **QS**, such as the subclass of systems with a weak focus of order 3 or 2 (see [20, 2]) the bifurcation diagrams were obtained by using an interplay of algebraic, analytic and numerical methods. These subclasses were of dimensions 2 and 3 modulo the action of the affine group and time rescaling. No 4-dimensional subclasses of **QS** were studied so far and such problems are very difficult due to the number of parameters as well as the increased complexities of these classes. On the other hand we propose to study the whole class **QS** according to the configurations (see further below) of the singularities of systems in this whole class. In this paper we do this but only for singularities at infinity.

To define the notion of *configuration of singularities* at infinity we distinguish two cases:

1) If we have a finite number of infinite singular points we call *configuration of singularities at infinity* the set of all these singularities each endowed with its own multiplicity together with their local phase portraits endowed with additional geometric properties involving the concepts of tangent, order and blow–up equivalences to be defined in Section 4 and using the notations described in Section 5.

2) If the line at infinity $Z = 0$ is filled up with singularities, in each one of the charts at infinity $X \neq 0$ and $Y \neq 0$, the system is degenerate and we need to do a rescaling of an appropiate degree of the system, so that the degeneracy be removed. The resulting systems have only a finite number of singularities on the line $Z = 0$. In this case we call *configuration of singularities at infinity* the set of all points at infinity (they are all singularities) on which we single out the singularities of the "reduced" system, taken together with their local phase portraits as in the previous case.

The goal of this article is to classify the configurations of singularities at infinity of planar quadratic vector fields using the finer geometric equivalence relation which is defined Section 4. In what follows **ISP**s is a shorthand for "infinite singular points". We obtain the following

Main Theorem. (A) *The configurations of singularities at infinity of all quadratic vector fields are classified in* DIAGRAMS *1–4 according to the geometric equivalence relation. Necessary and sufficient conditions for each one of the 167 different equivalence classes can be assembled from these diagrams in terms of 27 invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 7.*

(B) *The* DIAGRAMS *1–4 actually contain the bifurcation diagram in the 12-dimensional space of parameters, of the global configurations of singularities at infinity of quadratic differential systems.*

The geometrical meaning of some of the conditions given in terms of invariant polynomials in DIAGRAMS 1–4 appear in DIAGRAMS 5–7.

This work can be extended so as to include the complete geometrical classification of all global configurations of singular points (finite and infinite) of quadratic differential systems.

The following corollary results from the proof of the Main Theorem gathering all the cases in which the polynomials defining the differential are not coprime (*degenerated systems*).

Corollary 1. *There exist exactly 30 topologically distinct phase portraits around infinity for the family of degenerate quadratic systems, given in* FIGURE *7. Moreover necessary and sufficient conditions for the realization of each one of these portraits are given in the* DIAGRAMS *1–4. These are the cases occurring for* $\mu_i = 0$ *for every* $i \in \{0, 1, 2, 3, 4\}$.

The invariants and comitants of differential equations used for proving our main results are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his discniples (see for instance [35, 37, 24, 6, 10]).

2. Some geometrical concepts

We assume that we have an isolated singularity p. Suppose that in a neighborhood U of p there is no other singularity. Consider an orbit γ in U defined by a solution $\Gamma(t) = (x(t), y(t))$ such that $\lim_{t\to\pm\infty} \Gamma(t) = p$. For a fixed t consider the unit vector $C(t) = (\overrightarrow{\Gamma(t) - p})/\|\overrightarrow{\Gamma(t) - p}\|$. Let L be a semi–line ending at p. We shall say that *the orbit* γ *is tangent to a semi–line* L *at* p if $\lim_{t\to\pm\infty} C(t)$ exists and L contains this limit point on the unit circle centered at p. In this case we may also say that the solution curve $\Gamma(t)$ *tends to* p *with a well defined angle*, which is the angle between the positive x–axis and the semi–line L measured in the counter–clockwise sense. A *characteristic orbit at a singular point* p is the orbit of a solution curve $\Gamma(t)$

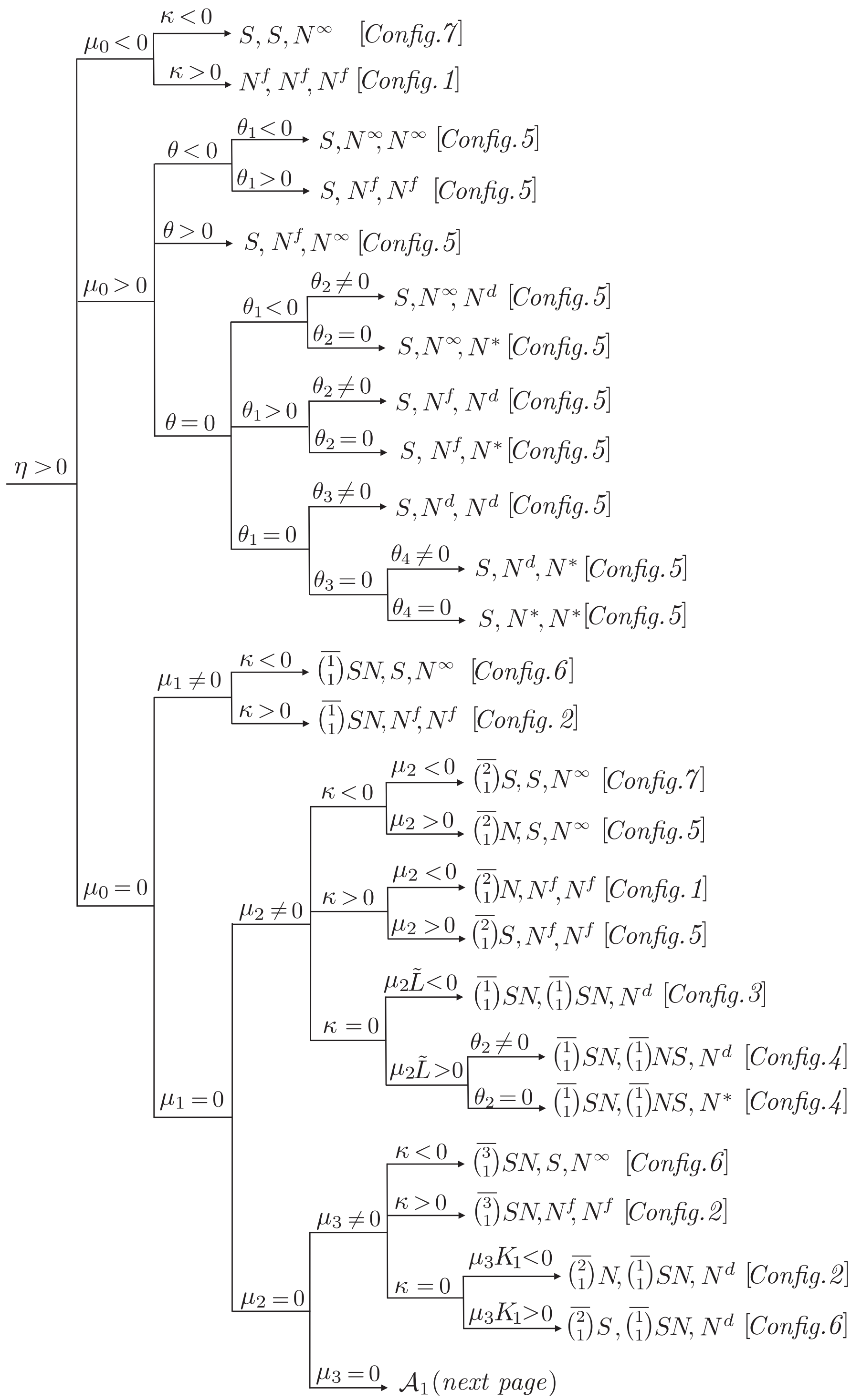


Diagram 1. Configurations of **ISP**s in the case $\boldsymbol{\eta > 0}$.

which tends to p with a well defined angle. A *characteristic angle at a singular point* p is the well defined angle in which a solution curve $\Gamma(t)$ tends to p. The line through p with this well defined angle is called a *characteristic direction*.

If a singular point has an infinite number of characteristic directions, we will call it a *star–like* point.

It is known that the neighborhood of any singular point of a polynomial vector field, which is not a focus or a center, is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic

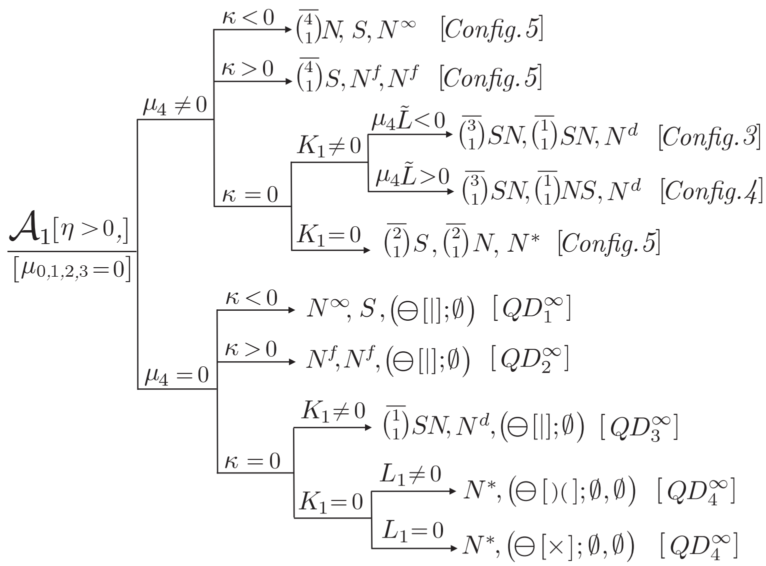


Diagram 1 (*cont.*): Configurations of **ISP**s in the case $\boldsymbol{\eta>0}$.

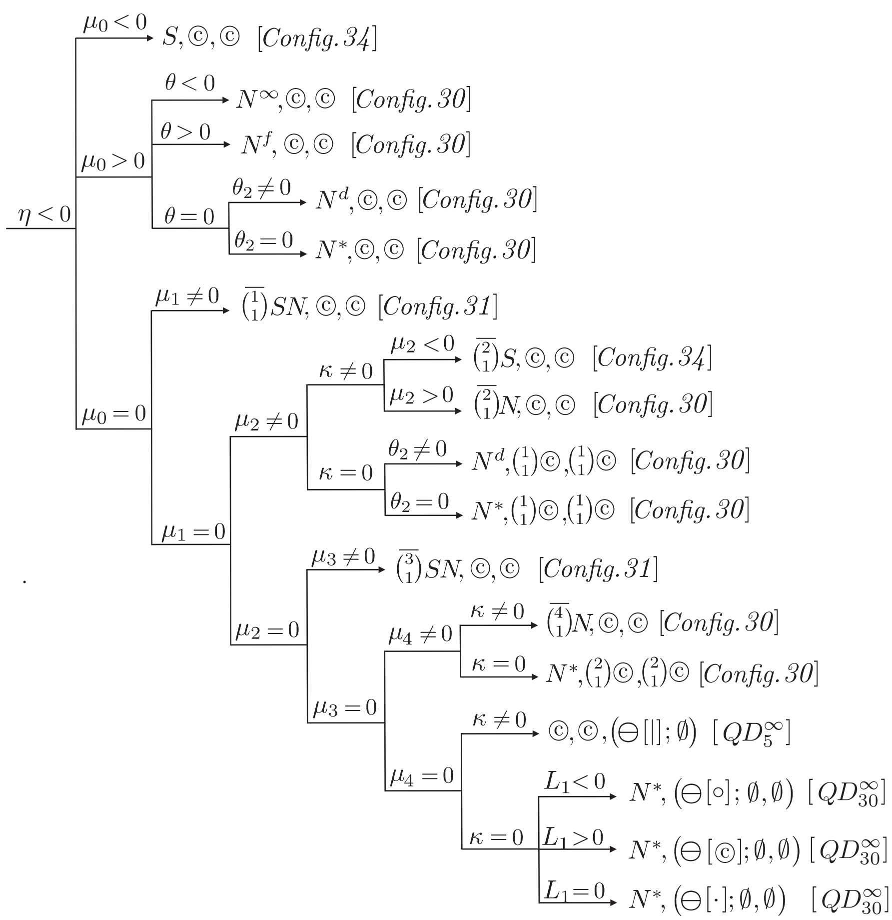


Diagram 2. Configurations of **ISP**s in the case $\boldsymbol{\eta<0}$.

and elliptic (see [15]). It is also known that any degenerate singular point can be desingularized by means of a finite number of changes of variables, called blow–up's, into elementary singular points (for more details see also [15] or Section 3).

- $\eta=0$, $\widetilde{M}\neq 0$
  - $\mu_0<0$ → $\overline{\binom{0}{2}}SN, S$ [*Config. 22*]
  - $\mu_0>0$
    - $\theta<0$ → $\overline{\binom{0}{2}}SN, N^{\infty}$ [*Config. 18*]
    - $\theta>0$ → $\overline{\binom{0}{2}}SN, N^{f}$ [*Config. 18*]
    - $\theta=0$
      - $\theta_2\neq 0$ → $\overline{\binom{0}{2}}SN, N^{d}$ [*Config. 18*]
      - $\theta_2=0$ → $\overline{\binom{0}{2}}SN, N^{*}$ [*Config. 18*]
  - $\mu_0=0$
    - $\mu_1\neq 0$
      - $\kappa\neq 0$ → $\overline{\binom{0}{2}}SN, \overline{\binom{1}{1}}SN$ [*Config. 20*]
      - $\kappa=0$
        - $\widetilde{K}<0$ → $\widehat{\binom{1}{2}}\widehat{P}_{\lambda}E\widehat{P}_{\lambda}-H, N^{f}$ [*Config. 9*]
        - $\widetilde{K}>0$
          - $\tilde{L}<0$
            - $\widetilde{N}<0$ → $\widehat{\binom{1}{2}}\widehat{P}_{\lambda}E\widehat{P}_{\lambda}-H, S$ [*Config. 16*]
            - $\widetilde{N}>0$ → $\widehat{\binom{1}{2}}\widehat{P}_{\lambda}H\widehat{P}_{\lambda}-E, S$ [*Config. 26*]
            - $\widetilde{N}=0$ → $\widehat{\binom{1}{2}}H-E, S$ [*Config. 16*]
          - $\tilde{L}>0$ → $\widehat{\binom{1}{2}}H_{\lambda}HH_{\lambda}-H, N^{\infty}$ [*Config. 23*]
    - $\mu_1=0$
      - $\mu_2\neq 0$
        - $\kappa\neq 0$
          - $\mu_2<0$ → $\overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}S$ [*Config. 22*]
          - $\mu_2>0$ → $\overline{\binom{0}{2}}SN, \overline{\binom{2}{1}}N$ [*Config. 18*]
        - $\kappa=0$
          - $\widetilde{K}<0$
            - $\mu_2<0$ → $\binom{2}{2}\widehat{P}E\widehat{P}-\widehat{P}E\widehat{P}, N^{f}$ [*Config. 10*]
            - $\mu_2>0$
              - $K_2<0$ → $\binom{2}{2}H-H, N^{f}$ [*Config. 8*]
              - $K_2>0$ → $\binom{2}{2}\widehat{P}\widehat{P}H-\widehat{P}\widehat{P}H, N^{f}$ [*Config. 19*]
              - $K_2=0$ → $\binom{2}{2}\widehat{P}H-\widehat{P}H, N^{f}$ [*Config. 19*]
          - $\widetilde{K}>0$
            - $\tilde{L}<0$
              - $\mu_2<0$ → $\binom{2}{2}\widehat{P}H\widehat{P}-\widehat{P}H\widehat{P}, S$ [*Config. 29*]
              - $\mu_2>0$
                - $K_2<0$ → $\binom{2}{2}E-E, S$ [*Config. 17*]
                - $K_2>0$ → $\binom{2}{2}\widehat{P}\widehat{P}E-\widehat{P}\widehat{P}E, S$ [*Config. 17*]
                - $K_2=0$ → $\binom{2}{2}\widehat{P}E-\widehat{P}E, S$ [*Config. 17*]
            - $\tilde{L}>0$
              - $\mu_2<0$ → $\binom{2}{2}HHH-HHH, N^{\infty}$ [*Config. 27*]
              - $\mu_2>0$
                - $K_2<0$ → $\binom{2}{2}H-H, N^{\infty}$ [*Config. 8*]
                - $K_2>0$ → $\binom{2}{2}\widehat{P}\widehat{P}H-\widehat{P}\widehat{P}H, N^{\infty}$ [*Config. 19*]
                - $K_2=0$ → $\binom{2}{2}\widehat{P}H-\widehat{P}H, N^{\infty}$ [*Config. 19*]
          - $\widetilde{K}=0$
            - $\tilde{L}\neq 0$
              - $\kappa_1\neq 0$
                - $\theta_5\neq 0$ → $\widehat{\binom{2}{2}}\widehat{P}_{\lambda}\widehat{P}H_{\lambda}-H, N^{d}$ [*Config. 13*]
                - $\theta_5=0$ → $\widehat{\binom{2}{2}}\widehat{P}_{\lambda}\widehat{P}H_{\lambda}-H, N^{*}$ [*Config. 13*]
              - $\kappa_1=0$
                - $K_2<0$
                  - $\theta_6\neq 0$ → $\binom{2}{2}H-H, N^{d}$ [*Config. 8*]
                  - $\theta_6=0$ → $\binom{2}{2}H-H, N^{*}$ [*Config. 8*]
                - $K_2>0$
                  - $\theta_6\neq 0$ → $\binom{2}{2}\widehat{P}\widehat{P}H-\widehat{P}\widehat{P}H, N^{d}$ [*Config. 19*]
                  - $\theta_6=0$ → $\binom{2}{2}\widehat{P}\widehat{P}H-\widehat{P}\widehat{P}H, N^{*}$ [*Config. 19*]
                - $K_2=0$
                  - $\theta_6\neq 0$ → $\binom{2}{2}\widehat{P}H-\widehat{P}H, N^{d}$ [*Config. 19*]
                  - $\theta_6=0$ → $\binom{2}{2}\widehat{P}H-\widehat{P}H, N^{*}$ [*Config. 19*]
            - $\tilde{L}=0$ → $\widehat{\binom{1}{2}}\widehat{P}_{\lambda}E\widehat{P}_{\lambda}-H, \overline{\binom{1}{1}}SN$ [*Config. 11*]
      - $\mu_2=0$ → $\mathcal{A}_2$ (*next page*)

Diagram 3. Configurations of **ISP**s in the case $\boldsymbol{\eta=0,\ \widetilde{M}\neq 0}$.

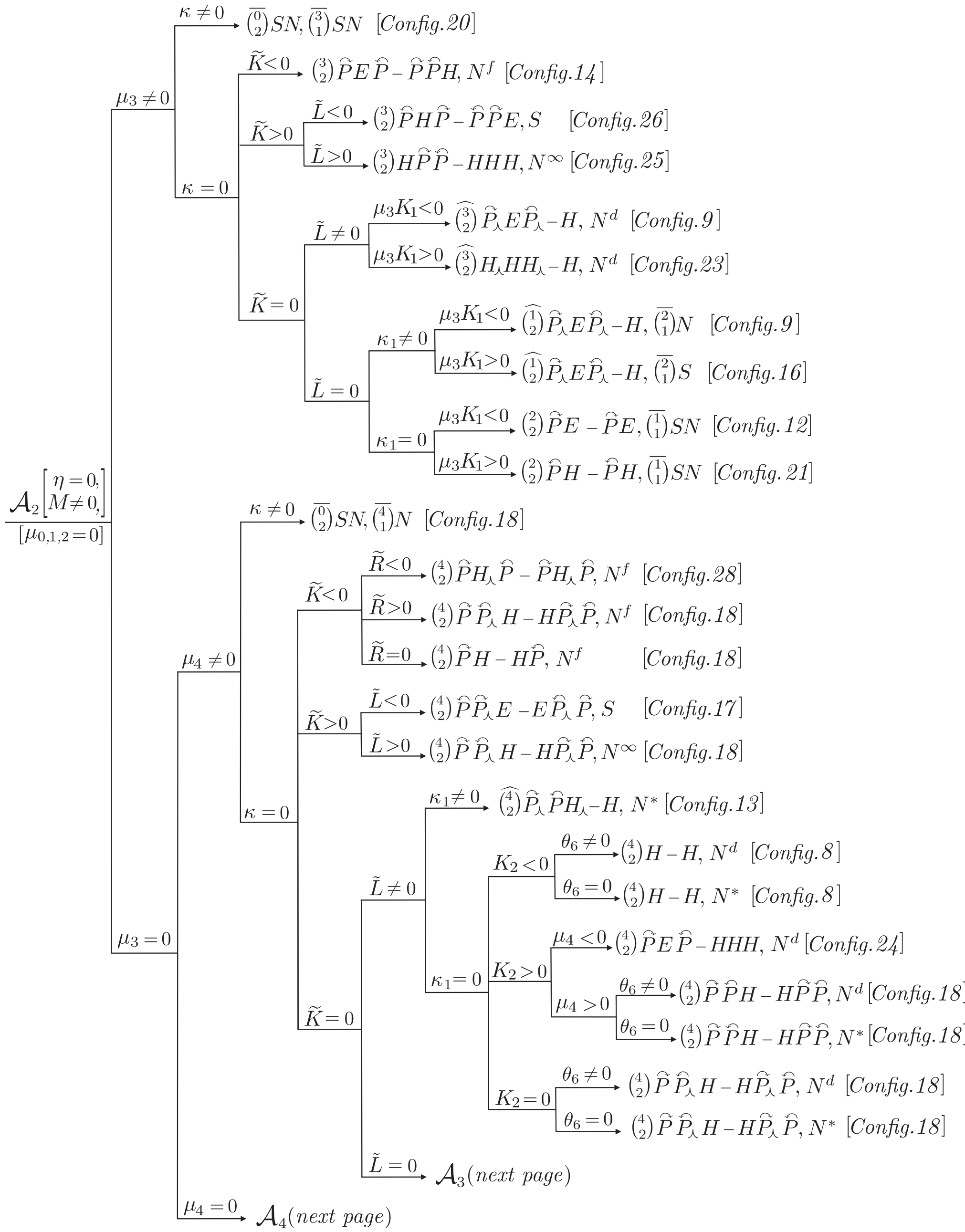


DIAGRAM 3 (*cont.*): Configurations of **ISP**s in the case $\boldsymbol{\eta=0,\ \widetilde{M}\neq 0}$.

Consider the three singular points given in FIGURE 2. All three are topologically equivalent and their neighborhoods can be described as having two elliptic sectors and two parabolic ones. But we can easily detect some geometric features that distinguish them. For example (a) and (b) have three characteristic directions and (c) has only two. Moreover in (a) the solution curves of the parabolic sectors are tangent to only one characteristic direction and in (b) they are tangent to two characteristic directions. All these properties can be determined algebraically.

The usual definition of a sector is of a topological nature and it is local with respect to a neighborhood around the singular point. We introduce a new definition of local sector which is of an algebraic nature and which distinguishes the systems of FIGURE 2.

We will call *borsec* (contraction of border and sector) any orbit of the original system which carried on through consecutive stages of the desingularization ends up as an orbit of the phase portrait in the final stage

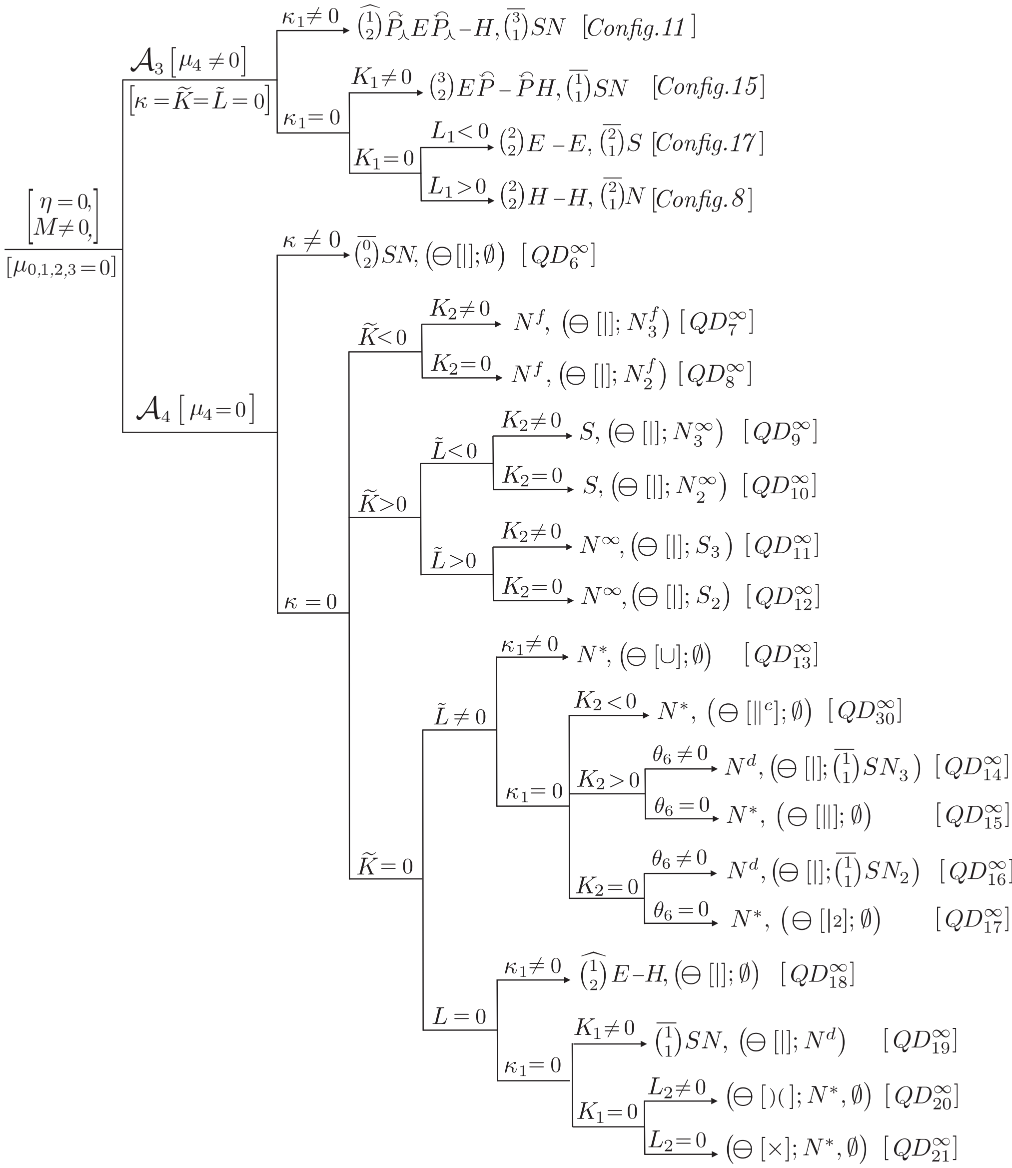


DIAGRAM 3 (*cont.*): Configurations of **ISP**s in the case $\boldsymbol{\eta=0,\ \widetilde{M}\neq 0}$.

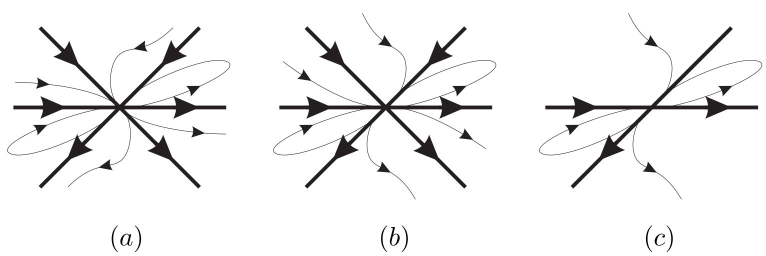


FIGURE 2. Some topologically equivalent singular points

which is either a separatrix or a representative orbit of a characteristic angle of a node or a of saddle–node in the final desingularized phase portrait.

Using this concept of borsec, we define a *geometric local sectors* with respect to a neighborhood V as a region in V delimited by two consecutive borsecs. For example, a semi–elementary saddle–node can be topologically described as a singular point having two hyperbolic sectors and a single parabolic one. But if we add the

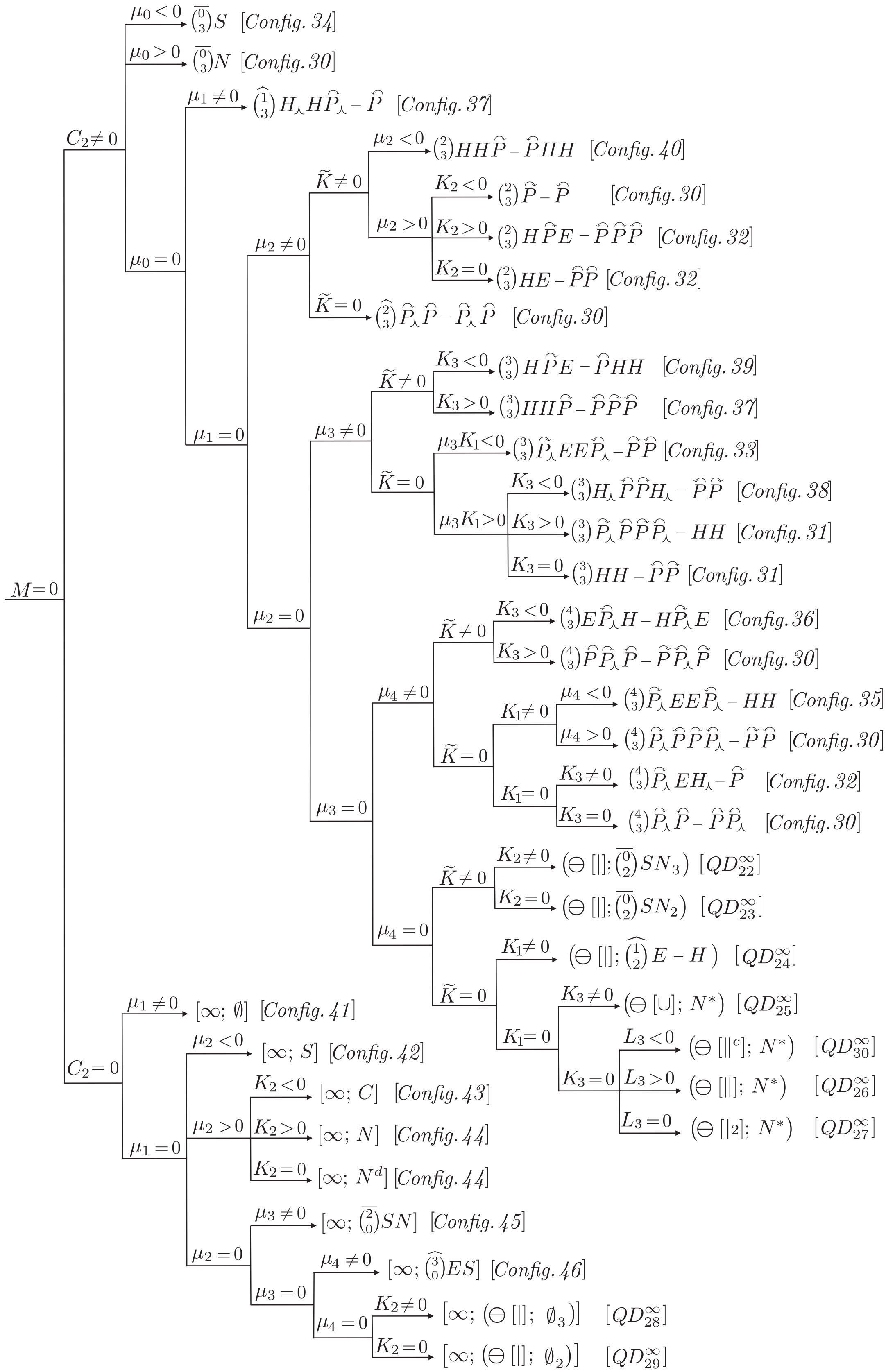


DIAGRAM 4. Configurations of **ISP**s in the case $\boldsymbol{\eta = 0}$, $\boldsymbol{\widetilde{M} = 0}$.

borsec which is any orbit of the parabolic sector, then the description would consist of two hyperbolic sectors

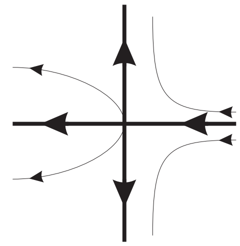


FIGURE 3. Local phase portrait of a degenerate singular point.

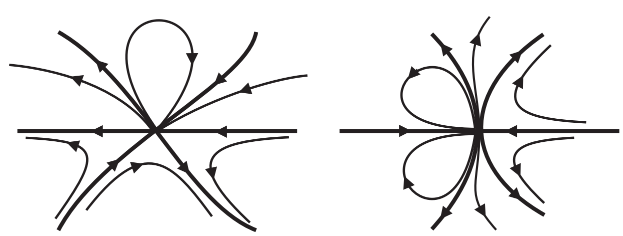
and two parabolic ones. This distinction will be critical when trying to describe a singular point like the one in FIGURE 3 which topologically is a saddle–node but qualitatively (in the sense of [20]) is different from a semi–elementary saddle–node.

Generically, a geometric local sector will be defined by two consecutive borsecs arriving at the singular point with two different well defined angles. If the sector is parabolic, then the solutions can arrive at the singular point with one of the two characteristic angles and this is a geometrical information than can be revealed with the blow–up. It may also happen that orbits arrive at the singular point in every angle inside the sector. We will call such a sector a *star–like parabolic sector* and we will be denoted by P^*.

If the sector is elliptic, then generically the solutions inside the sector will depart from and arrive at the singular point in both characteristic angles. It may also happen that orbits arrive at the singular point in every angle inside the sector. Such a sector will be called *star–like elliptic sector* and will be denoted by E^*.

There is also the possibility that two borsecs defining a geometric local sector tend to the singular point with the same well defined angle. Such a sector will be called a *cusp–like sector* which can either be hyperbolic, elliptic or parabolic respectively denoted by $H_\curlywedge$, $E_\curlywedge$ and $P_\curlywedge$.

Moreover, in the case of parabolic sectors we want to include the information as to whether the orbits arrive tangent to one or to the other borsec. We distinguish the two cases by $\overset{\curvearrowleft}{P}$ if they arrive tangent to the borsec limiting the previous sector in clock–wise sense or $\overset{\curvearrowright}{P}$ if they arrive tangent to the borsec limiting the next sector. In the case of a cusp–like parabolic sector, all orbits must arrive with only one slope, but the distinction between $\overset{\curvearrowleft}{P}$ and $\overset{\curvearrowright}{P}$ is still valid because it occurs at some stage the desingularization and this can be algebraically determined. Thus, complicated degenerate singular points like the two we see in FIGURE 4 may be described as $\overset{\curvearrowright}{P}E\overset{\curvearrowleft}{P}HHH$ (case (a)) and $E\overset{\curvearrowleft}{P}_\curlywedge HH\overset{\curvearrowright}{P}_\curlywedge E$ (case (b)), respectively.



(a) *(b)*

FIGURE 4. Two phase portraits of degenerate singular points.

A star–like point can either be a node or something much more complicated with elliptic and hyperbolic sectors included. In case there are hyperbolic sectors, they must be cusp–like. Elliptic sectors can either be cusp–like or star–like. So, some special angles will be relevant. We will call *special characteristic angle* any well defined angle in which not a unique solution curve tends to p (that is, either none or more than one solution curve tends to p within this well defined angle). We will call *special characteristic direction* any line such that at least one of the two angles defining it, is a special characteristic angle.

3. The blow–up technique

To draw the phase portrait around an elementary hyperbolic singularity of a smooth planar vector field we just need to use the Hartman-Grobman theorem. For an elementary non-hyperbolic singularity the system can be brought by an affine change of coordinates and time rescaling to the form $dx/dt = -y + ...$, $dy/dt = x + ...$ and it is well known that in this case the singularity is either a center or a focus. One way to see this is by the Poincaré-Lyapounov theory. In the quadratic case we can actually determine using the Poincaré-Lyapounov constants if it is a focus or a center so the local phase portrait is known. For higher order systems we have the center-focus problem: we can only say that the phase portrait around the singularity is of a center or of a focus but we cannot determine with certainty which one of the two it is.

In case of a more complicated singularity, such as a degenerate one, we need to use of the *blow–up technique*. This is a well known technique but since it plays such a crucial role in this work and also in order to make this article as self-contained as possible, we shall briefly describe it here. Another reason why we need to insist on describing this technique here is because we are going to use it in a slightly modified (actually simplified) way so as to lighten the calculations. For this modified way to be perfectly clear, we show below that it is in complete agreement with the usual blow–up procedure.

The idea behind the blow–up technique is to replace a singular point p by a line or by a circle on which the "composite" degenerate singularity decomposes (ideally) into a finite number of simpler singularities p_i. For this idea to work we need to construct a new surface on which we have a diffeomorpic copy of our vector field on $\mathbb{R}^2\backslash\{p\}$ or at least on the complement of a line passing through p, and whose associated foliation with singularities extends also to the circle (or to a line) which replaces the point p on the new surface.

One way to do this is to use polar coordinates. Clearly we may assume that the singularity is placed at the origin. Consider the map $\phi : \mathbb{S}^1 \times \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by $\phi(\theta, r) \mapsto (r\cos\theta, r\sin\theta)$. This map is a diffeomorphism for $r \in (0, \infty)$ and for $r \in (-\infty, 0)$ onto $\mathbb{R}^2\backslash\{(0,0)\}$ but $\phi^{-1}(0,0)$ is the circle $\mathbb{S}^1 \times \{0\}$. This application defines a diffeomorphic vector field on the upper part of the cylinder $\mathbb{S}^1 \times \mathbb{R}$. In fact this is the passing to polar coordinates. The resulting smooth vector field extends to the whole cylinder just by allowing r to be negative or zero. This full vector field on the cylinder has either a finite number of singularities on the circle (this occurs when the initial singular point is nilpotent) or the circle is filled up with singularities (when we start with a linearly zero point). In this latter case we need to make a time rescaling $T = r^s t$ of the vector field with an adequate s to obtain a finite number of singularities. The map ϕ collapses the circle on the cylinder (and hence the singularities located on this circle) to the origin of coordinates in the plane. In case the phase portraits around the singularities on the circle can be drawn then the inverse process of *blowing down* the upper side of the cylinder completed with the circle allows us to draw the portrait around the origin of $\mathbb{R}^2$. In case the singularities on the circle are still degenerate, we need to repeat the process a finite number of times. This is guaranteed by the theorem of desingularization of singularities (see [7] and [12])

The blow–up by polar coordinates is simple, leading to a simple surface (the cylinder), on which a diffeomorphic copy of our vector field on $\mathbb{R}^2\backslash\{(0,0)\}$ extends to a vector field on the full cylinder. The origin of the plane "blows-up" to the circle $\phi^{-1}(0,0)$ on which the singularity splits into several simpler singularities. The visualization of this blow–up is easy. But this process has the disadvantage of using the transcendental functions: *cos* and *sin* and in case several such blow–ups are needed this is computationally very inconvenient.

It would be more advantageous to use a construction involving rational functions. More difficult to visualize, this algebraic blow–up is computationally simpler, using only rational transformations. To blow–up a point of the plane means to replace the point with a line (directional blow–up) viewed as the space of directions of $\mathbb{R}^2$ at this point and to construct a manifold playing the role of the cylinder in the preceding case. The point is replaced by a line with the change $(x, y) \to (x, zx)$, then the surface will not be a cylinder but an algebraic surface.

We start with a polynomial differential system (1) with a degenerate singular point at the origin $(0,0)$, and we want to do a blow–up in the direction of the y–axis so as to split the singularity at the origin into several singularities on the axis $x = 0$. In order to do this correctly we must be sure that $x = 0$ is not a characteristic direction. In this case we have $p(x,y) = p_1(x,y) + \ldots + p_n(x,y)$ and $q(x,y) = q_1(x,y) + \ldots + q_n(x,y)$ where $p_i(x,y)$ and $q_i(x,y)$ (for $i = 1\ldots,n$) are the homogeneous terms involving $x^r y^l$ with $r + l = i$ of p and q. We call the *starting degree of* (1) the positive integer m such that $p_m(x,y)^2 + q_m(x,y)^2 \neq 0$ but $p_i(x,y)^2 + q_i(x,y)^2 = 0$ for $i = 0,1,\ldots,m-1$.

Then, we define the *Polynomial of Characteristic Directions* as $PCD(x,y) = yp_m(x,y) - xq_m(x,y)$ where m is the starting degree of (1). In case $PCD(x,y) \not\equiv 0$ the factorization of $PCD(x,y)$ gives the characteristic directions at the origin. So, in order to be sure that the y–axis is not a characteristic direction we only need to show that x is not a factor of $PCD(x,y)$. In case it is, we need to do a linear change of variables which moves this direction out of the vertical axis and does not move any other characteristic direction into it. If all the directions are characteristic, i.e. $PCD(x,y) \equiv 0$, then the degenerate point will be star–like and at least two blow–ups must be done to obtain the desingularization. Anyway there are no degenerate star–like singular points in quadratic systems. So, the number of characteristic directions is finite and there exists the possibility to make such a linear change. We will use changes of the type $(x,y) \to (x + ky, y)$ where k is some number (usually 1). It seems natural to call this linear change a *k–twist* as the y–axis gets twisted with some angle depending on k. It is obvious that the phase portrait of the degenerate point which is studied cannot depend on the set of k's used in the desingularization process.

Once we are sure that we have no characteristic direction on the y–axis we do the directional blow–up $(x,y) = (X, XY)$. This change preserves invariant the axis $y = 0$ ($Y = 0$ after the change) and replaces the singular point $(0,0)$ with a whole vertical axis. The old orbits which arrived at $(0,0)$ with a well defined slope s now arrive at the singular point $(0,s)$ of the new system. Studying these new singular points, one can determine the local behavior around them and their separatrices which after the blow–down describe the behavior of the orbits around the original singular point up to geometrical equivalence (for definition see next section). Often one needs to do a tree of blow–up's (combined with some translation and/or twists) if some of the singular points which appear on $X = 0$ after the first blow–up are also degenerate.

4. Equivalence relations for singularities of planar polynomial vector fields

We first recall the *topological equivalence relation* as it is used in most of the literature. Two singularities p_1 and p_2 are *topologically equivalent* if there exist open neighborhoods N_1 and N_2 of these points and a homeomorphism $\Psi : N_1 \to N_2$ carrying orbits to orbits and preserving their orientations. To reduce the number of cases, by *topological equivalence* we shall mean here that the homeomorphism Ψ *preserves or reverses* the orientation. This second notion is also used sometimes elsewhere in the literature (see [19, 2]).

In [19] Jiang and Llibre introduced another equivalence relation for singularities which is finer than the topological equivalence:

We say that p_1 and p_2 are *qualitatively equivalent* if i) they are topologically equivalent through a local homeomorphism Ψ; and ii) two orbits are tangent to the same straight line at p_1 if and only if the corresponding two orbits are tangent to the same straight line at p_2.

We say that two simple finite nodes, with the respective eigenvalues λ_1, λ_2 and σ_1, σ_2, of a planar polynomial vector field are *tangent equivalent* if and only if they satisfy one of the following three conditions: a) $(\lambda_1 - \lambda_2)(\sigma_1 - \sigma_2) \neq 0$; b) $\lambda_1 - \lambda_2 = 0 = \sigma_1 - \sigma_2$ and both linearization matrices at the two singularities are diagonal; c) $\lambda_1 - \lambda_2 = 0 = \sigma_1 - \sigma_2$ and the corresponding linearization matrices are not diagonal.

We say that two infinite simple nodes P_1 and P_2 are *tangent equivalent* if and only if their corresponding singularities on the sphere are tangent equivalent and in addition, in case they are generic nodes, we have

$(|\lambda_1| - |\lambda_2|)(|\sigma_1| - |\sigma_2|) > 0$ where λ_1 and σ_1 are the eigenvalues of the eigenvectors tangent to the line at infinity.

Finite and infinite singular points may either be real of complex. In case we have a complex singular point we will specify this with the symbols ⓒ and Ⓒ for finite and infinite points respectively. We point out that the sum of the multiplicities of all singular points of a quadratic system (with a finite number of singular points) is always 7. (Here of course we refer to the compactification on the complex projective space $P_2(\mathbb{R})$ of the foliation with singularities associated to the complexification of the vector field.) The sum of the multiplicities of the infinite singular points is always at least 3, more precisely it is always 3 plus the sum of the multiplicities of the finite points which have gone to infinity.

We use here the following terminology for singularities:

> We call *elemental* a singular point with its both eigenvalues not zero;
> We call *semi–elemental* a singular point with exactly one of its eigenvalues equal to zero;
> We call *nilpotent* a singular point with both its eigenvalues zero but with its Jacobian matrix at that point not identically zero;
> We call *intricate* a singular point with its Jacobian matrix identically zero.

The *intricate* singularities are usually called in the literature *linearly zero*. We use here the term *intricate* to indicate the rather complicated behavior of phase curves around such a singularity.

Roughly speaking a singular point p of an analytic differential system χ is a *multiple singularity of multiplicity m* if p produces m singularities, as closed to p as we wish, in analytic perturbations χ_ε of this system and m is the maximal such number. In polynomial differential systems of fixed degree n we have several possibilities for obtaining multiple singularities. i) A finite singular point splits into several finite singularities in n-degree polynomial perturbations. ii) An infinite singular point splits into some finite and some infinite singularities in n-degree polynomial perturbations. iii) An infinite singularity splits only in infinite singular points of the systems in n-degree perturbations. To all these cases we can give a precise mathematical meaning using the notion of intersection multiplicity at a point p of two algebraic curves.

We will say that two foci (or saddles) are *order equivalent* if their corresponding orders coincide.

Semi–elemental saddle–nodes are always topologically equivalent.

To define the notion of *geometric equivalence relation* of singularities we first define the notion of *blow–up equivalence*, necessary for nilpotent and intricate singular points. We start by having a degenerate singular point p_1 at the origin of the plane (x_0, y_0) with a finite number of characteristic directions. We define an ε-twist as a k-twist with k small enough so that no characteristic direction (or special characteristic direction in case of a star point) with negative slope is moved to positive slope. Then if $x_0 = 0$ is a characteristic direction, we do an ε-twist. After the blow–up $(x_0, y_0) = (x_1, y_1x_1)$ the singular point is replaced by the straight line $x_1 = 0$ in the plane (x_1, y_1). The neighborhood of the straight line $x_1 = 0$ in the projective plane obtained identifying the opposite infinite points of the Poincaré disk is a Möebius band M_1.

The straight line $x_1 = 0$ will be invariant and may be formed by a continuous of singular points. In that case, with a time change, this degeneracy may be removed and the y_1–axis will remain invariant.

Now we have a number k_1 of singularities located on the axis $x_1 = 0$. We do not include the infinite singular point at the origin of the local chart U_2 at infinity ($Y \neq 0$) because we already know that it does not play any role in understanding the local phase portrait of the singularity p_1. We can then list the k_1 singularities as $p_{1,1}, p_{1,2}, ..., p_{1,k_1}$ with decreasing order of the y_1 coordinate. The $p_{1,i}$ is adjacent to $p_{1,i+1}$ in the usual sense and p_{1,k_1} is also adjacent to $p_{1,1}$ on the Möebius band.

Assume now we have a degenerate singular point p_1 at the origin of the plane (x_0, y_0) with an infinite number of characteristic directions. Then if $x_0 = 0$ is a special characteristic direction, we do an ε-twist. After the blow–up $(x_0, y_0) = (x_1, y_1x_0)$ the singular point is replaced by the straight line $x_1 = 0$ in the plane

(x_1, y_1). The neighborhood of the straight line $x_1 = 0$ in the projective plane obtained identifying the opposite infinite points of the Poincaré disk is a Möebius band M_1.

The straight line $x_1 = 0$ will be invariant and formed by a continuous of singular points. In that case, with a time change, this degeneracy may be removed and the y_1–axis will not anymore be invariant.

Now we have a set of cardinality k_1 formed by singularities located on the axis $x_1 = 0$ plus contact points of the flow with the axis $x_1 = 0$. Again we do not include the infinite singular point at the origin of the local chart U_2 at infinity ($Y \neq 0$) because we already know that it does not play any role in understanding the local phase portrait of the singularity p_1. We list again the k_1 points as $p_{1,1}, p_{1,2}, ..., p_{1,k_1}$ with decreasing order of the y_1 coordinate. The $p_{1,i}$ is adjacent to $p_{1,i+1}$ in the usual sense and p_{1,k_1} is also adjacent to $p_{1,1}$ by the Möebius band.

Let p_2 be another degenerate singularity located at the origin of another plane $(\bar{x}_0, \bar{y}_0)$.

The next definition works whether the singular points are star–like or not.

We say that p_1 and p_2 are *one step blow–up equivalent* if modulus a rotation with center p_2 (before the blow–up) and a reflection (if needed) we have:

(i) the cardinality k_1 from p_1 equals the cardinality k_2 from p_2;
(ii) we can construct a homeomorphism $\phi^1_{p_1} : M_1 \to M_2$ such that $\phi^1_{p_1}(\{x_1 = 0\}) = \{\bar{x}_1 = 0\}$, $\phi^1_{p_1}$ sends the points $p_{1,i}$ to $p_{2,i}$ and the phase portrait in a neighborhood U of the axis $x_1 = 0$ is topologically equivalent to the phase portrait on $\phi^1_{p_1}(U)$;
(iii) $\phi^1_{p_1}$ sends an elemental (respectively semi–elemental, nilpotent or intricate) singular point to an elemental (respectively semi–elemental, nilpotent or intricate) singular point;
(iv) $\phi^1_{p_1}$ sends a contact point to a contact point.

Assuming $p_{1,j}$ and $\phi^1_{p_1}(p_{1,j}) = p_{2,j}$ are both intricate or both nilpotent, then the process of desingularization (blow–up) must be continued.

We do exactly the same study we did before for p_1 and p_2 now for $p_{1,j}$ and $p_{2,j}$. We move them to the respective origins of the planes (x_1, y_1) and $(\bar{x}_1, \bar{y}_1)$ and we determine whether they are one step blow–up equivalent or not.

If successive degenerate singular points appear from desingularization of p_1 we do the same kind of changes that we did for $p_{1,j}$ and apply the corresponding definition of one step blow–up equivalence. This is repeated until after a finite number of blow–up's all the singular points that appear are elemental or semi–elemental.

We say that two singularities p_1 and p_2, both nilpotent or both intricate, of two polynomial vector fields χ_1 and χ_2, are *blow–up equivalent* if and only if

(i) they are one step blow–up equivalent;
(ii) at each level j in the process of desingularization of p_1 and of p_2, two singularities which are related via the corresponding homeomorphism are one step blow–up equivalent.

Definition 1. *Two singularities p_1 and p_2 of two polynomial vector fields are locally geometrically equivalent if and only if they are topologically equivalent, they have the same multiplicity and one of the following conditions is satisfied:*

- *p_1 and p_2 are order equivalent foci (or saddles);*
- *p_1 and p_2 are tangent equivalent simple nodes;*
- *p_1 and p_2 are both centers;*
- *p_1 and p_2 are both semi–elemental singularities;*
- *p_1 and p_2 are blow–up equivalent nilpotent or intricate singularities.*

We say that two infinite singularities P_1 and P_2 of two polynomial vector fields are *blow–up equivalent* if they are blow–up equivalent finite singularities in the corresponding infinite local charts and the number, type and ordering of sectors on each side of the line at infinity of P_1 coincide with those of P_2.

Definition 2. *Let χ_1 and χ_2 be two polynomial vector fields each having a finite number of singularities. We say that χ_1 and χ_2 have geometric equivalent configurations of singularities if and only if we have a bijection ϑ carrying the singularities of χ_1 to singularities of χ_2 and for every singularity p of χ_1, $\vartheta(p)$ is geometric equivalent with p.*

5. Notations for singularities of polynomial differential systems

In this work we encounter all the possibilities we have for the geometric features of the infinite singularities in the whole quadratic class as well as the way they assemble in systems of this class. Since we want to describe precisely these geometric features and in order to facilitate understanding, it is important to have a clear, compact and congenial notation which conveys easily the information. Of course this notation must be compatible with the one used to describe finite singularities, so we start with the finite ones. The notation we use, even though it is used here to describe finite and infinite singular points of quadratic systems, can easily be extended to general polynomial systems.

We describe the finite and infinite singularities, denoting the first ones with lower case letters and the second with capital letters. When describing in a sequence both finite and infinite singular points, we will always place first the finite ones and only later the infinite ones, separating them by a semicolon';'.

Elemental points: We use the letters 's','S' for "saddles"; 'n', 'N' for "nodes"; 'f' for "foci" and 'c' for "centers". In order to augment the level of precision we will distinguish the finite nodes as follows:

- 'n' for a node with two distinct eigenvalues (generic node);
- 'n^d' (a one–direction node) for a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal;
- 'n^*' (a star–node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

Moreover, in the case of an elemental infinite generic node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is, in absolute value, greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. This is relevant because this determines if all the orbits except one on the Poincaré disk arrive at infinity tangent to the line at infinity or transversal to this line. We will denote them as 'N^∞' and 'N^f' respectively.

Finite elemental foci and saddles are classified as strong or weak foci, respectively strong or weak saddles. When the trace of the Jacobian matrix evaluated at those singular points is not zero, we call them strong saddles and strong foci and we maintain the standard notations 's' and 'f.' But when the trace is zero, except for centers and saddles of infinite order (i.e. saddles with all their Poincaré-Lyapounov constants equal to zero), it is known that the foci and saddles, in the quadratic case, may have up to 3 orders. We denote them by '$s^{(i)}$' and '$f^{(i)}$' where $i = 1, 2, 3$ is the order. In addition we have the centers which we denote by 'c' and saddles of infinite order (integrable saddles) which we denote by '$\$$'.

Foci and centers cannot appear as singular points at infinity and hence there is no need to introduce their order in this case. In case of saddles, we can have weak saddles at infinity but the maximum order of weak singularities in cubic systems is not yet known. For this reason, a complete study of weak saddles at infinity cannot be done at this stage. Due to this, in this work we shall not even distinguish between a saddle and a weak saddle at infinity.

All non–elemental singular points are multiple points, in the sense that there are perturbations which have at least two elemental singular points as close as we wish to the multiple point. For finite singular points we denote with a subindex their multiplicity as in '$\overline{s}_{(5)}$' or in '$\widehat{es}_{(3)}$' (the notation '$^-$' indicates that the saddle is semi–elemental and '$\widehat{es}_{(3)}$' indicates that the singular point is nilpotent). In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [28]. Thus we denote by '$\binom{a}{b}$...' the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example '$\overline{\binom{1}{1}}SN$' means a saddle–node at infinity produced by the collision of one finite singularity with an infinite one; '$\overline{\binom{0}{3}}S$' means a saddle produced by the collision of 3 infinite singularities.

Semi–elemental points: They can either be nodes, saddles or saddle–nodes, finite or infinite. We will denote the semi–elemental ones always with an overline, for example '$\overline{sn}$', '$\overline{s}$' and '$\overline{n}$' with the corresponding multiplicity. In the case of infinite points we will put '$^{-}$' on top of the parenthesis with multiplicities.

Moreover, in cases that will be explained later, an infinite saddle–node may be denoted by '$\overline{\binom{1}{1}}NS$' instead of '$\overline{\binom{1}{1}}SN$'. Semi–elemental nodes could never be 'n^d' or 'n^*' since their eigenvalues are always different. In case of an infinite semi–elemental node, the type of collision determines whether the point is denoted by 'N^f' or by 'N^∞' where '$\overline{\binom{2}{1}}N$' is an 'N^f' and '$\overline{\binom{0}{3}}N$' is an 'N^∞'.

Nilpotent points: They can either be saddles, nodes, saddle–nodes, elliptic–saddles, cusps, foci or centers. The first four of these could be at infinity. We denote the nilpotent singular points with a hat '$\widehat{\ }$' as in $\widehat{es}_{(3)}$ for a finite nilpotent elliptic–saddle of multiplicity 3 and $\widehat{cp}_{(2)}$ for a finite nilpotent cusp point of multiplicity 2. In the case of nilpotent infinite points, we will put the '$\widehat{\ }$' on top of the parenthesis with multiplicity, for example $\widehat{\binom{1}{2}}PEP-H$ (the meaning of $PEP-H$ will be explained in next paragraph). The relative position of the sectors of an infinite nilpotent point, with respect to the line at infinity, can produce topologically different phase portraits. This forces us to use a notation for these points similar to the notation which we will use for the intricate points.

Intricate points: It is known that the neighborhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [15]). Then, a reasonable way to describe intricate and nilpotent points at infinity is to use a sequence formed by the types of their sectors. The description we give is the one which appears in the clock–wise direction (starting anywhere) once the blow–down of the desingularization is done. Thus in non degenerate quadratic systems, we have just seven possibilities for finite intricate singular points of multiplicity four (see [3]) which are the following ones:

- a) $phpphp_{(4)}$;
- b) $phph_{(4)}$;
- c) $hh_{(4)}$;
- d) $hhhhhh_{(4)}$;
- e) $peppep_{(4)}$;
- f) $pepe_{(4)}$;
- g) $ee_{(4)}$.

We use lower case because of the finite nature of the singularities and add the subindex (4) since they are all of multiplicity 4.

For infinite intricate and nilpotent singular points, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\binom{2}{2}PHP-PHP$ and $\binom{2}{2}PPH-PPH$.

Whenever we have an infinite nilpotent or intricate singular point, we will always start with a sector bordering the infinity (to avoid using two dashes). When one needs to describe a configuration of singular points at infinity, then the relative positions of the points, is relevant in some cases. In this paper this situation only occurs once for systems having two semi–elemental saddle–nodes at infinity and a third singular point which is elemental. In this case we need to write NS instead of SN for one of the semi–elemental points in order to have coherence of the positions of the parabolic (nodal) sector of one point with respect to the hyperbolic (saddle) of the other semi–elemental point. More concretely, *Figure* 3 from [28] (which corresponds to *Config. 3* in FIGURE 4) must be described as $\overline{\binom{1}{1}}SN$, $\overline{\binom{1}{1}}SN$, N since the elemental node lies always between the hyperbolic sectors of one saddle–node and the parabolic ones of the other. However, *Figure* 4 from [28] (which corresponds to *Config. 4* in FIGURE 4) must be described as $\overline{\binom{1}{1}}SN$, $\overline{\binom{1}{1}}NS$, N since the hyperbolic sectors of each saddle–node lie between the elemental node and the parabolic sectors of the other saddle–node. These two configurations have exactly the same description of singular points but their relative position produces topologically (and geometrically) different portraits.

For the description of the topological phase portraits around the isolated singular points the information described above is sufficient. However we are interested in additional geometrical features such as the number of characteristic directions which figure in the final global picture of the desingularization. In order to add this information we need to introduce more notation. If two borsecs (the limiting orbits of a sector) arrive at the singular point with the same slope and direction, then the sector will be denoted by $H_{\curlywedge}$, $E_{\curlywedge}$ or $P_{\curlywedge}$. The index in this notation refers to the cusp–like form of limiting trajectories of the sectors. Moreover, in the case of parabolic sectors we want to make precise whether the orbits arrive tangent to one borsec or to the other. We distinguish the two cases by $\overset{\curvearrowleft}{P}$ if they arrive tangent to the borsec limiting the previous sector in clock–wise sense or $\overset{\curvearrowright}{P}$ if they arrive tangent to the borsec limiting the next sector. Clearly, a parabolic sector denoted by P^* would correspond to a sector in which orbits arrive with all possible slopes between the borsecs. In the case of a cusp–like parabolic sector, all orbits must arrive with only one slope, but the distinction between $\overset{\curvearrowleft}{P}$ and $\overset{\curvearrowright}{P}$ is still valid if we consider the different desingularizations we obtain from them. Thus, complicated intricate singular points like the two we see in FIGURE 4 may be described as $\binom{4}{2}\ \overset{\curvearrowright}{P}E\,\overset{\curvearrowleft}{P}-HHH$ (case (a)) and $\binom{4}{3}\ E\overset{\curvearrowleft}{P}_{\curlywedge}H-H\overset{\curvearrowright}{P}_{\curlywedge}E$ (case (b)), respectively.

The lack of finite singular points will be encapsulated in the notation $\scriptstyle\emptyset$. In the cases we need to point out the lack of an infinite singular point, we will use the symbol $\emptyset$.

Finally there is also the possibility that we have an infinite number of finite or of infinite singular points. In the first case, this means that the polynomials defining the differential system are not coprime. Their common factor may produce a line or conic with real coefficients filled up with singular points.

Line at infinity filled up with singularities: It is known that any such system has in a sufficiently small neighborhood of infinity one of 6 topological distinct phase portraits (see [31]). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. In case a singular point still remains on the line at infinity we study such a point. In [31] the tangential behavior of the solution curves was not considered in the case of a node. If after the removal of the degeneracy in the local charts at infinity a node remains, this could either be of the type N^d, N and $N^{\star}$ (this last case does not occur in quadratic systems as we will see in this paper). Since no eigenvector of such a node N (for quadratic systems) will have the direction of the line at infinity we do not need to distinguish N^f and N^{∞}. Other types of singular points at infinity of quadratic systems, after removal of the degeneracy, can be saddles, centers, semi–elemental saddle–nodes or nilpotent elliptic–saddles. We also have the possibility of no singularities after the removal of the degeneracy. To convey the way these singularities were obtained as well as their nature, we use the notation $[\infty;\ \emptyset]$, $[\infty;\ N]$, $[\infty;\ N^d]$, $[\infty;\ S]$, $[\infty;\ C]$, $[\infty;\ \overline{\binom{1}{0}}SN]$ or $[\infty;\ \widehat{\binom{3}{0}}ES]$.

Degenerate systems: We will denote with the symbol $\ominus$ the case when the polynomials defining the system have a common factor. This symbol stands for the most generic of these cases which corresponds to a real line filled up with singular points. The degeneracy can also be produced by a common quadratic factor which defines a conic. It is well known that by an affine transformation any conic over $\mathbb{R}$ can be brought to one of the following forms: $x^2+y^2-1=0$ (real ellipse), $x^2+y^2+1=0$ (complex ellipse), $x^2-y^2=1$ (hyperbola), $y-x^2=0$ (parabola), $x^2-y^2=0$ (pair of intersecting real lines), $x^2+y^2=0$ (pair of intersecting complex lines), $x^2-1=0$ (pair of parallel real lines), $x^2+1=0$ (pair of parallel complex lines), $x^2=0$ (double line).

We will indicate each case by the following symbols:

- $\ominus[|]$ for a real straight line;
- $\ominus[\circ]$ for a real ellipse;
- $\ominus[©]$ for a complex ellipse;
- $\ominus[\,)(\,]$ for an hyperbola;
- $\ominus[\cup]$ for a parabola;
- $\ominus[\times]$ for two real straight lines intersecting at a finite point;
- $\ominus[\cdot\,]$ for two complex straight lines which intersect at a real finite point.
- $\ominus[||]$ for two real parallel lines;
- $\ominus[||^c]$ for two complex parallel lines;
- $\ominus[|_2]$ for a double real straight line.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If the reduced system has no finite singularity on this curve, we will use the symbol $\emptyset$ to describe this situation. If some singular points remain we will use the corresponding notation of their types. As an example we complete the notation above as follows:

- $(\ominus\,[|];\emptyset)$ denotes the presence of a real straight line filled up with singular points such that the reduced system has no singularity on this line;
- $(\ominus\,[|];f)$ denotes the presence of the same straight line such that the reduced system has a strong focus on this line;
- $(\ominus\,[\cup];\emptyset)$ denotes the presence of a parabola filled up with singularities such that no singular point of the reduced system is situated on this parabola.

Degenerate systems with non–isolated singular points at infinity, which are however isolated on the line at infinity: The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non–isolated singularity.

In order to describe correctly the singularities at infinity, we must mention also this kind of phenomena and describe what happens to such points at infinity after the removal of the common factor. To show the existence of the common factor we will use the same symbol $\ominus$ as before, and for the type of degeneracy we use the symbols introduced above. We will use the symbol $\emptyset$ to denote the non–existence of real infinite singular points after the removal of the degeneracy. We will use the corresponding capital letters to describe the singularities which remain there. Let us take note that a simple straight line, two parallel lines (real or complex), one double line or one parabola defined by the common factor (all taken over the reals) imply the existence of one real non–isolated singular point at infinity in the original degenerate system. However a hyperbola and two real straight lines intersecting at a finite point imply the presence of two real non–isolated singular points at infinity in the original degenerate system. Finally, a complex ellipse and two complex straight lines which intersect at a real finite point imply the presence of two complex non–isolated singular points at infinity in the original degenerate system. Thus, in the reduced system these points may disappear as singularities and in case they remain, they must be described. For the first five cases mentioned above we will give the description of the corresponding infinite point. In the next five cases we will give the description of the corresponding two singular points. As agreed, we will use capital letters to denote them since they are on the line at infinity. We give below some examples:

- $N^f, S, (\ominus\,[|];\emptyset)$ means that the system has a node at infinity such that an infinite number of orbits arrive tangent to the eigenvector in the affine part, a saddle, and one non–isolated singular point which belongs to a real affine straight line filled up with singularities, and that the reduced linear system has no infinite singular points in that position;

- $S, \left(\ominus\, [|]; N^*\right)$ means that the system has a saddle at infinity, and one non–isolated singular point which belongs to a real affine straight line filled up with singularities, and that the reduced linear system has a star node in that position;
- $S, \left(\ominus\, [\,)(\,]; \emptyset, \emptyset\right)$ means that the system has a saddle at infinity, and two non–isolated singular points which belong to a hyperbola filled up with singularities, and that the reduced constant system has no singularities in those positions;
- $\left(\ominus\, [\times]; N^*, \emptyset\right)$ means that the system has two non–isolated singular points at infinity which belong to two real intersecting straight lines filled up with singularities, and that the reduced constant system has a star node in one of those positions and no singularities in the other;
- $S, \left(\ominus\, [\circ]; \emptyset, \emptyset\right)$ means that the system has a saddle at infinity, and two non–isolated (complex) singular points which are located on the complexification of a real conic which has no real points at infinity, and the reduced constant system has no singularities in those positions.

When there is a non–isolated infinite singular point such that the reduced system has a singularity at that position, it may happen that one or several characteristic directions at this point, directed towards the affine plane, could coincide with a tangent line to the curve of singularities at this point. This situation could produce many different geometrical (or even topological) combinations but in the quadratic case we only have a few of them for which we introduce a coherent notation. This notation can be further developed for higher degree systems. In quadratic systems we only need to distinguish among some situations in which, after the removal of the degeneracy, a characteristic direction of the infinite singular point may coincide or may not coincide with a tangent line to the curve of singularities at this point. We show in Figure 5 two cases that need to be distinguished (case (a) and (b)). Here we will use a numerical subscript which denotes the cardinal number $\mathcal{K}$ of the union of the set characteristic directions, together with the set of tangent lines to the curve of singularities at this point, all of them considered in a neighborhood of the point at infinity on the Poincaré sphere. The singularities at infinity of the examples of Figure 5 would then be denoted by $S, \left(\ominus\, [|]; N_3^\infty\right)$ (case (a)) and $S, \left(\ominus\, [|]; N_2^\infty\right)$ (case (b)).

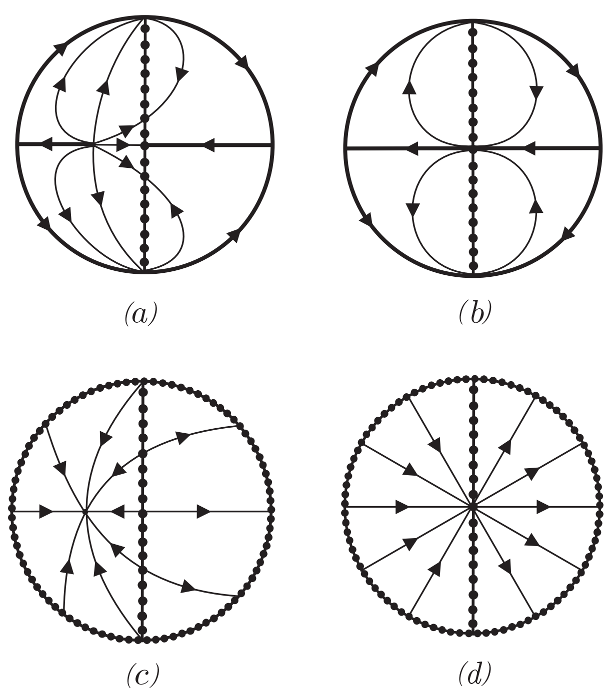


Figure 5

Degenerate systems with the line at infinity filled up with singularities: For a quadratic system this implies that the polynomials must have a common linear factor and there are only two possible phase portraits, which can be seen in FIGURE 5 (the portraits (c) and (d)). In order to be consistent with our notation and considering generalization to higher degree systems, we describe the two cases in a way coherent with what we have done up to now.

The case (c) is denoted by $[\infty; \big(\ominus\, [|]; \emptyset_3\big)]$ which means:

- the line at infinity is filled up with singular points;
- the reduced quadratic system has on one of the infinite local charts a non–isolated singular point on the line at infinity due to the affine line of degeneracy;
- once the original system at infinity is reduced to a linear one by removing the common factor, the infinity continues to be filled up with singular points;
- once the system on a local chart at infinity around the singularity which is common to both lines filled up with singular points, is reduced by completely removing the degeneracy, there is no singular point on that intersection;
- the cardinal number $\mathcal{K}$ is 3. This means that apart from the line of singularities and the line at infinity, we have another characteristic direction pointing towards the affine plane.

The second case is denoted by $[\infty; \big(\ominus\, [|]; \emptyset_2\big)]$, which means exactly the same items as above with the exception that cardinal number $\mathcal{K}$ is 2. That is, beyond the line of singularities and the line at infinity, we have no other characteristic direction.

6. ASSEMBLING MULTIPLICITIES FOR GLOBAL CONFIGURATIONS OF SINGULARITIES AT INFINITY USING DIVISORS

The singular points at infinity belong to compactifications of planar polynomial differential systems, defined on the affine plane. We begin this section by briefly recalling these compactifications.

6.1. **Compactifications associated to planar polynomial differential systems.**

6.1.1. *Compactification on the sphere and on the Poincaré disk.* Planar polynomial differential systems (1) can be compactified on the sphere. For this we consider the affine plane of coordinates (x, y) as being the plane $Z = 1$ in $\mathbb{R}^3$ with the origin located at $(0, 0, 1)$, the x–axis parallel with the X–axis in $\mathbb{R}^3$, and the y–axis parallel to the Y–axis. We use central projection to project this plane on the sphere as follows: for each point $(x, y, 1)$ we consider the line joining the origin with $(x, y, 1)$. This line intersects the sphere in two points $P_1 = (X, Y, Z)$ and $P_2 = (-X, -Y, -Z)$ where $(X, Y, Z) = (1/\sqrt{x^2 + y^2 + 1})(x, y, 1)$. The applications $(x, y) \mapsto P_1$ and $(x, y) \mapsto P_2$ are bianalytic and associate to a vector field on the plane (x, y) an analytic vector field Ψ on the upper hemisphere and also an analytic vector field Ψ'on the lower hemisphere. A theorem stated by Poincaré and proved in [16] says that there exists an analytic vector field Θ on the whole sphere which simultaneously extends the vector fields on the two hemispheres. By the *Poincaré compactification on the sphere* of a planar polynomial vector field we mean the restriction $\bar{\Psi}$ of the vector field Θ to the union of the upper hemisphere with the equator. For more details we refer to [20]. The vertical projection of $\bar{\Psi}$ on the plane $Z = 0$ gives rise to an analytic vector field Φ on the unit disk of this plane. By the *compactification on the Poincaré disk* of a planar polynomial vector field we understand the vector field Φ. By a *singular point at infinity* of a planar polynomial vector field we mean a singular point of the vector field $\bar{\Psi}$ which is located on the equator of the sphere, respectively a singular point of the vector field Φ located on the circumference of the Poincaré disk.

6.1.2. *Compactification on the projective plane.* To a polynomial system (1) we can associate a differential equation $\omega_1 = q(x, y)dx - p(x, y)dy = 0$. Assuming the differential system (1) is with real coefficients, we may associate to it a foliation with singularities on the real, respectively complex, projective plane as indicated below. The equation $\omega_1 = 0$ defines a foliation with singularities on the real or complex plane depending if we consider the equation as being defined over the real or complex affine plane. It is known that we can compactify these foliations with singularities on the real respectively complex projective plane. In the study of real planar polynomial vector fields, their associated complex vector fields and their singularities play an important role. In particular such a vector field could have complex, non-real singularities, by this meaning singularities of the associated complex vector field. We briefly recall below how these foliations with singularities are defined.

The application $\Upsilon : K^2 \longrightarrow P_2(K)$ defined by $(x, y) \mapsto [x : y : 1]$ is an injection of the plane K^2 over the field K into the projective plane $P_2(K)$ whose image is the set of $[X : Y : Z]$ with $Z \neq 0$. If K is $\mathbb{R}$ or $\mathbb{C}$ this application is an analytic injection. If $Z \neq 0$ then $(\Upsilon)^{-1}([X : Y : Z]) = (x, y)$ where $(x, y) = (X/Z, Y/Z)$. We obtain a map $i : K^3 - \{Z = 0\} \longrightarrow K^2$ defined by $[X : Y : Z] \mapsto (X/Z, Y/Z)$.

Considering that $dx = d(X/Z) = (ZdX - XdZ)/Z^2$ and $dy = (ZdY - YdZ)/Z^2$, the pull-back of the form ω_1 via the map i yields the form $i * (\omega_1) = q(X/Z, Y/Z)(ZdX - XdZ)/Z^2 - p(X/Z, Y/Z)(ZdY - YdZ)/Z^2$ which has poles on $Z = 0$. Then the form $\omega = Z^{m+2} i * (\omega_1)$ on $K^3 - \{Z = 0\}$, K being $\mathbb{R}$ or $\mathbb{C}$ and m being the degree of systems (1) yields the equation $\omega = 0$:

$$A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ = 0$$

on $K^3 - \{Z = 0\}$ where A, B, C are homogeneous polynomials over K whith $A(X, Y, Z) = ZQ(X, Y, Z)$, $Q(X, Y, Z) = Z^m q(X/Z, Y/Z)$, $B(X, Y, Z) = ZP(X, Y, Z)$, $P(X, Y, Z) = Z^m p(X/Z, Y/Z)$ and $C(X, Y, Z) = YP(X, Y, Z) - XQ(X, Y, Z)$.

The equation $AdX + BdY + CdZ = 0$ defines a foliation F with singularities on the projective plane over K with K either $\mathbb{R}$ or $\mathbb{C}$. The *points at infinity* of the foliation defined by $\omega_1 = 0$ on the affine plane are the points $[X : Y : 0]$ and the line $Z = 0$ is called the *line at infinity* of the foliation with singularities generated by $\omega_1 = 0$.

The singular points of the foliation F are the solutions of the three equations $A = 0$, $B = 0$, $C = 0$. In view of the definitions of A, B, C it is clear that the singular points at infinity are the points of intersection of $Z = 0$ with $C = 0$.

6.2. **Assembling data on infinite singularities in divisors of the line at infinity.** In the previous sections we have seen that there are two types of multiplicities for a singular point p at infinity: one expresses the maximum number m of infinite singularities which can split from p, in small perturbations of the system and the other expresses the maximum number m' of finite singularities which can split from p, in small perturbations of the system. In Section 2 we mentioned that we shall use a column $(m, m')^t$ to indicate this situation.

We are interested in the global picture which includes *all* singularities at infinity. Therefore we need to assemble the data for individual singularities in a convenient, precise way. To do this we use for this situation the notion of *cycle* on an algebraic variety as indicated in [23] and which was used in [20] as well as in [28].

We briefly recall here the definition of this notion. Let V be an irreducible algebraic variety over a field K. A *cycle* of dimension r or $r - cycle$ on V is a formal sum $\sum_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of the coefficients n_W are non-zero. The degree $\deg(J)$ of a cycle J is defined by $\sum_W n_W$. An $(n-1)$-cycle is called a divisor on V. These notions were used for classification purposes of planar quadratic differential systems in [23, 20, 28].

To a system (1) we can associate two divisors on the line at infinity $Z = 0$ of the complex projective plane: $D_S(P, Q; Z) = \sum_w I_w(P, Q)w$ and $D_S(C, Z) = \sum_w I_w(C, Z)w$ where $w \in \{Z = 0\}$ and where by $I_w(F, G)$ we mean the intersection multiplicity at w of the curves $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$, with F and G homogeneous polynomials in X, Y, Z over $\mathbb{C}$. For more details see [20].

Following [28] we assemble the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

$$D_S = \sum_{\omega\in\{Z=0\}} \begin{pmatrix} I_w(P,Q) \\ I_w(C,Z) \end{pmatrix} w.$$

This divisor encodes for us the total number of singularities at infinity of a system (1) as well as the two kinds of multiplicities which each singularity has. The meaning of these two kinds of multiplicities are described in the definition of the two divisors $D_S(P,Q;Z)$ and $D_S(C,Z)$ on the line at infinity.

7. Invariant polynomials and preliminary results

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{2}$$

with homogeneous polynomials p_i and q_i $(i = 0,1,2)$ of degree i in x, y:

$$\begin{aligned} p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of systems (2) and denote $\mathbb{R}[\tilde{a}, x, y] = \mathbb{R}[a_{00}, \dots, b_{02}, x, y]$.

7.1. **Affine invariant polynomials associated to infinite singularities.** It is known that on the set **QS** of all quadratic differential systems (2) acts the group $\mathit{Aff}(2,\mathbb{R})$ of the affine transformation on the plane (cf. [28]). For every subgroup $G \subseteq \mathit{Aff}(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (2) with a subset of $\mathbb{R}^{12}$ via the map **QS**$\longrightarrow \mathbb{R}^{12}$ which associates to each system (2) the 12–tuple $(a_{00}, \dots, b_{02})$ of its coefficients.

For the definitions of a GL–comitant and invariant as well as for the definitions of a T–comitant and a CT–comitant we refer the reader to the paper [28] (see also [35]). Here we shall only construct the necessary T–comitants and CT–comitants associated to configurations of infinite singularities (including multiplicities) of quadratic systems (2).

Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P^* + \beta Q^* \in \mathbb{R}[\tilde{a}, X, Y, Z, \alpha, \beta]$ where $P^* = Z^2 P(X/Z, Y/Z)$, $Q^* = Z^2 Q(X/Z, Y/Z)$, $P,\, Q \in \mathbb{R}[\tilde{a}, x, y]$ and $\max(\deg_{(x,y)} P, \deg_{(x,y)} Q) = 2$. Then

$$\Phi_{\alpha,\beta} = s_{11}(\tilde{a},\alpha,\beta)X^2 + 2s_{12}(\tilde{a},\alpha,\beta)XY + s_{22}(\tilde{a},\alpha,\beta)Y^2 + 2s_{13}(\tilde{a},\alpha,\beta)XZ + 2s_{23}(\tilde{a},\alpha,\beta)YZ + s_{33}(\tilde{a},\alpha,\beta)Z^2$$

and we denote

$$\begin{aligned} \widetilde{D}(\tilde{a},x,y) =& 4\det\|s_{ij}(\tilde{a},y,-x)\|_{i,j\in\{1,2,3\}}\,, \\ \widetilde{H}(\tilde{a},x,y) =& 4\det\|s_{ij}(\tilde{a},y,-x)\|_{i,j\in\{1,2\}}\,. \end{aligned}$$

We consider the polynomials

$$\begin{aligned} C_i(\tilde{a},x,y) &= y p_i(\tilde{a},x,y) - x q_i(\tilde{a},x,y), \\ D_i(\tilde{a},x,y) &= \frac{\partial}{\partial x} p_i(\tilde{a},x,y) + \frac{\partial}{\partial y} q_i(\tilde{a},x,y), \end{aligned} \tag{3}$$

in $\mathbb{R}[\tilde{a}, x, y]$ for $i = 0,1,2$ and $i = 1,2$ respectively. Using the so–called *transvectant of order* k (see [17], [21]) of two polynomials $f,\, g \in \mathbb{R}[\tilde{a}, x, y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\,\frac{\partial^k g}{\partial x^h \partial y^{k-h}},$$

we construct the following GL—comitants of the second degree with the coefficients of the initial system

$$\begin{aligned}
T_1 &= (C_0, C_1)^{(1)}, & T_2 &= (C_0, C_2)^{(1)}, & T_3 &= (C_0, D_2)^{(1)},\\
T_4 &= (C_1, C_1)^{(2)}, & T_5 &= (C_1, C_2)^{(1)}, & T_6 &= (C_1, C_2)^{(2)},\\
T_7 &= (C_1, D_2)^{(1)}, & T_8 &= (C_2, C_2)^{(2)}, & T_9 &= (C_2, D_2)^{(1)}.
\end{aligned}$$

Using these GL—comitants as well as the polynomials (3) we construct the additional invariant polynomials (see also [28])

$$\begin{aligned}
\widetilde{M}(\tilde{a}, x, y) =&(C_2, C_2)^{(2)} \equiv 2\text{Hess}\big(C_2(\tilde{a}, x, y)\big);\\
\eta(\tilde{a}) =&(\widetilde{M}, \widetilde{M})^{(2)}/384 \equiv \text{Discrim}\big(C_2(\tilde{a}, x, y)\big);\\
\widetilde{K}(\tilde{a}, x, y) =&\text{Jacob}\big(p_2(\tilde{a}, x, y), q_2(\tilde{a}, x, y)\big);\\
K_1(\tilde{a}, x, y) =&p_1(\tilde{a}, x, y)q_2(\tilde{a}, x, y) - p_2(\tilde{a}, x, y)q_1(\tilde{a}, x, y);\\
K_2(\tilde{a}, x, y) =&4(T_2, \widetilde{M} - 2\widetilde{K})^{(1)} + 3D_1(C_1, \widetilde{M} - 2\widetilde{K})^{(1)} - (\widetilde{M} - 2\widetilde{K})\big(16T_3 - 3T_4/2 + 3D_1^2\big);\\
K_3(\tilde{a}, x, y) =&C_2^2(4T_3 + 3T_4) + C_2(3C_0\widetilde{K} - 2C_1T_7) + 2K_1(3K_1 - C_1D_2);\\
\tilde{L}(\tilde{a}, x, y) =&4\widetilde{K} + 8\widetilde{H} - \widetilde{M};\\
L_1(\tilde{a}, x, y) =&(C_2, \widetilde{D})^{(2)};\\
L_2(\tilde{a}, x, y) =&(C_2, \widetilde{D})^{(1)};\\
L_3(\tilde{a}, x, y) =&C_1^2 - 4C_0C_2;\\
\widetilde{R}(\tilde{a}, x, y) =&\tilde{L} + 8\widetilde{K};\\
\kappa(\tilde{a}) =&(\widetilde{M}, \widetilde{K})^{(2)}/4;\\
\kappa_1(\tilde{a}) =&(\widetilde{M}, C_1)^{(2)};\\
\kappa_2(\tilde{a}) =&(D_2, C_0)^{(1)};\\
\widetilde{N}(\tilde{a}, x, y) =&\widetilde{K}(\tilde{a}, x, y) + \widetilde{H}(\tilde{a}, x, y);\\
\theta(\tilde{a}) =&-(\widetilde{N}, \widetilde{N})^{(2)}/2 \equiv \text{Discrim}\big(\widetilde{N}(\tilde{a}, x, y)\big);\\
\theta_1(\tilde{a}) =&16\eta(\tilde{a}) + \kappa(\tilde{a});\\
\theta_2(\tilde{a}) =&\big(C_1, \widetilde{N}\big)^{(2)}/16;\\
\theta_3(\tilde{a}) =&\Big(2\big(\widetilde{F}, \widetilde{N}\big)^{(2)} - \big((\widetilde{D}, \widetilde{H})^{(2)}, D_2\big)^{(1)}\Big)/32;\\
\theta_4(\tilde{a}) =&\big((C_2, \widetilde{E})^{(2)}, D_2\big)^{(1)};\\
\theta_5(\tilde{a}, x, y) =&2C_2(T_6, T_7)^{(1)} - (T_5 + 2D_2C_1)(C_1, D_2^2)^{(2)};\\
\theta_6(\tilde{a}, x, y) =&C_1T_8 - 2C_2T_6,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{E} &= \Big[D_1(2T_9 - T_8) - 3\,(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)\Big]/72,\\
\widetilde{F} &= \Big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widetilde{E} -\\
&\quad -24\,\Big(C_2, \widetilde{D}\Big)^{(2)} + 120\,\Big(D_2, \widetilde{D}\Big)^{(1)} - 36C_1\,(D_2, T_7)^{(1)} + 8D_1\,(D_2, T_5)^{(1)}\Big]/144.
\end{aligned}$$

The geometrical meaning of the invariant polynomials C_2, $\widetilde{M}$ and η is revealed in the next lemma (see [28]).

Lemma 1. *The form of the divisor $D_S(C, Z)$ for systems* (2) *is determined by the corresponding conditions indicated in Table 1, where we write $w_1^c + w_2^c + w_3$ if two of the points, i.e. w_1^c, w_2^c, are complex but not real. Moreover, for each form of the divisor $D_S(C, Z)$ given in Table 1 the quadratic systems* (2) *can be brought via* *a linear transformation to one of the following canonical systems* $(\mathbf{S}_I)-(\mathbf{S}_V)$ *corresponding to their behavior at infinity.*

Table 1

Case	Form of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants
1	$w_1+w_2+w_3$	$\eta>0$
2	$w_1^c+w_2^c+w_3$	$\eta<0$
3	$2w_1+w_2$	$\eta=0,\quad \widetilde{M}\neq 0$
4	$3w$	$\widetilde{M}=0,\quad C_2\neq 0$
5	$D_S(C,Z)$ undefined	$C_2=0$

$$\begin{cases}\dot{x} &= a+cx+dy+gx^2+(h-1)xy,\\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2;\end{cases}\qquad (\mathbf{S}_I)$$

$$\begin{cases}\dot{x} &= a+cx+dy+gx^2+(h+1)xy,\\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2;\end{cases}\qquad (\mathbf{S}_{II})$$

$$\begin{cases}\dot{x} &= a+cx+dy+gx^2+hxy,\\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2;\end{cases}\qquad (\mathbf{S}_{III})$$

$$\begin{cases}\dot{x} &= a+cx+dy+gx^2+hxy,\\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2,\end{cases}\qquad (\mathbf{S}_{IV})$$

$$\begin{cases}\dot{x} &= a+cx+dy+x^2,\\ \dot{y} &= b+ex+fy+xy.\end{cases}\qquad (\mathbf{S}_V)$$

Consider the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ acting on $\mathbb{R}[a,x,y]$ constructed in [9], where

$$\begin{aligned}\mathbf{L}_1 = {}& 2a_{00}\tfrac{\partial}{\partial a_{10}}+a_{10}\tfrac{\partial}{\partial a_{20}}+\tfrac{1}{2}a_{01}\tfrac{\partial}{\partial a_{11}}+2b_{00}\tfrac{\partial}{\partial b_{10}}+b_{10}\tfrac{\partial}{\partial b_{20}}+\tfrac{1}{2}b_{01}\tfrac{\partial}{\partial b_{11}},\\ \mathbf{L}_2 = {}& 2a_{00}\tfrac{\partial}{\partial a_{01}}+a_{01}\tfrac{\partial}{\partial a_{02}}+\tfrac{1}{2}a_{10}\tfrac{\partial}{\partial a_{11}}+2b_{00}\tfrac{\partial}{\partial b_{01}}+b_{01}\tfrac{\partial}{\partial b_{02}}+\tfrac{1}{2}b_{10}\tfrac{\partial}{\partial b_{11}}.\end{aligned}$$

Using this operator and the affine invariant $\mu_0=\operatorname{Res}_x\big(p_2(\tilde{a},x,y),q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y)=\frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ \ i=1,..,4,$$

where $\mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$.

These polynomials are in fact comitants of systems (2) with respect to the group $GL(2,\mathbb{R})$ (see [9]). Their geometrical meaning is revealed in Lemmas 2 and 3 below.

Lemma 2. ([8]) *The total multiplicity of all finite singularities of a quadratic system* (2) *equals* k *if and only if for every* $i\in\{0,1,\dots,k-1\}$ *we have* $\mu_i(\tilde{a},x,y)=0$ *in* $\mathbb{R}[x,y]$ *and* $\mu_k(\tilde{a},x,y)\neq 0$. *Moreover a system* (2) *is degenerate (i.e.* $\gcd(P,Q)\neq$ *constant) if and only if* $\mu_i(\tilde{a},x,y)=0$ *in* $\mathbb{R}[x,y]$ *for every* $i=0,1,2,3,4$.

Lemma 3. ([9]) The point $M_0(0,0)$ is a singular point of multiplicity k $(1\leq k\leq 4)$ for a quadratic system (2) if and only if for every $i\in\{0,1,\dots,k-1\}$ we have $\mu_{4-i}(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_{4-k}(\tilde{a},x,y)\neq 0$.

We base our work here on results obtained in [28] and [31] where integer valued invariants and invariant polynomials were used to classify globally singularities in the neighborhood of infinity. We integrate here this information, using invariant polynomials and types of divisors on the line at infinity, in a unified theorem where we replace *Figure j* to *Config. j* from $j=1,\dots,46$. This theorem is stated as follows:

Theorem 1. *Consider the family of planar quadratic differential systems. The bifurcation diagram of the phase portraits around infinity in the 12-dimensional parameter space of coefficients is given by using invariant polynomials in* Diagrams *5-7 and their local phase portraits are given in* Figure *6.*

8. The proof of the Main Theorem

As we have to examine the infinite singularities we shall consider step by step each one of the five canonical systems $(\mathbf{S}_I)-(\mathbf{S}_V)$ (see Lemma 1) which are associated to infinite singularities.

8.1. **The family of systems $(\mathbf{S}_I)$.** For these systems we have $C_2 = yp_2(x,y) - xq_2(x,y) = xy(x-y)$ and $\eta > 0$. Therefore at infinity we have three real distinct singularities: $R_1(1,0,0)$, $R_2(0,1,0)$ and $R_3(1,1,0)$. Constructing the corresponding systems at infinity (possessing the point R_i $(i = 1,2,3)$ at the origin of coordinates) we get respectively:

$$
\begin{aligned}
&R_1 \rightarrow \begin{cases} \dot{u} = u - ez - u^2 + (c-f)uz - bz^2 + du^2z + auz^2, \\ \dot{z} = gz + (h-1)uz + cz^2 + duz^2 + az^3; \end{cases}\\
&R_2 \rightarrow \begin{cases} \dot{v} = v - dz - v^2 + (-c+f)vz - az^2 + ev^2z + bvz^2, \\ \dot{z} = hz + (g-1)vz + fz^2 + evz^2 + bz^3; \end{cases}\\
&R_3 \rightarrow \begin{cases} \dot{u} = u - (c+d-e-f)z + u^2 - (c+2d-f)uz - (a-b)z^2 - du^2z - auz^2, \\ \dot{z} = (1-g-h)z - (h-1)uz - (c+d)z^2 - duz^2 - az^3. \end{cases}
\end{aligned} \tag{4}
$$

So the corresponding matrices for these singularities are as follows:

$$
\begin{gathered}
R_1 \Rightarrow \begin{pmatrix} 1 & -e \\ 0 & g \end{pmatrix}; \qquad R_2 \Rightarrow \begin{pmatrix} 1 & -d \\ 0 & h \end{pmatrix};\\
R_3 \Rightarrow \begin{pmatrix} 1 & -c-d+e+f \\ 0 & 1-g-h \end{pmatrix}.
\end{gathered} \tag{5}
$$

Remark 1. The eigenvalues of R_1 (respectively R_2; R_3) are $\lambda_1 = 1$ and $\lambda_2 = g$ (respectively $\lambda_2 = h$; $\lambda_2 = 1-g-h$). We also denote $\xi = -e$ (respectively $\xi = -d$; $\xi = -c-d+e+f$) for R_1 (respectively R_2; R_3). The eigenvalue λ_1 is associated to the eigenvector tangent to the line at infinity whereas λ_2 is associated to the eigenvector directed towards the affine plane. Thus the point R_i for $i = 1,2,3$ is a node if $\lambda_2 > 0$ and according to the notation introduced in Section 5, when $\lambda_2 > 1$ the singular point R_i is N^∞ and if $\lambda_2 < 1$ it is N^f. Moreover, when $\lambda_2 = 1$ the singular point R_i is a star node (i.e. N^*) if $\xi = 0$ and it is a one direction node (i.e. N^d) if $\xi \neq 0$.

Following Theorem 1 (see the Diagram 1) we calculate for systems $(\mathbf{S}_I)$ the value of the corresponding invariant polynomials:

$$
\mu_0 = gh(g+h-1), \quad \kappa = 16(g+h-g^2-gh-h^2). \tag{6}
$$

8.1.1. **The case $\mu_0 < 0$.** According to Theorem 1 all three infinite singularities are elemental. Moreover by [28] we have a node and two saddles if $\kappa < 0$ and three nodes if $\kappa > 0$. We claim that in the first case we have a node N^∞, whereas in the second case all three nodes are of the type N^f.

Indeed, assume first $\kappa < 0$, i.e. we have a node and two saddles. These means that two of the values g, h and $1-g-h$ are negative and one positive. Without loss of generality we may assume $g > 0$ (i.e. R_1 is a node), $h < 0$ and $1-g-h < 0$. Then $g > 1-h > 1$ and according to Remark 1 R_1 is a node N^∞.

Suppose now $\kappa > 0$, i.e. we have three nodes. Therefore according to (4) the relations $g > 0$, $h > 0$ and $g+h < 1$ must hold. Hence $g < 1$, $h < 1$ and by Remark 1 all three nodes are N^f. Thus our claim is proved.

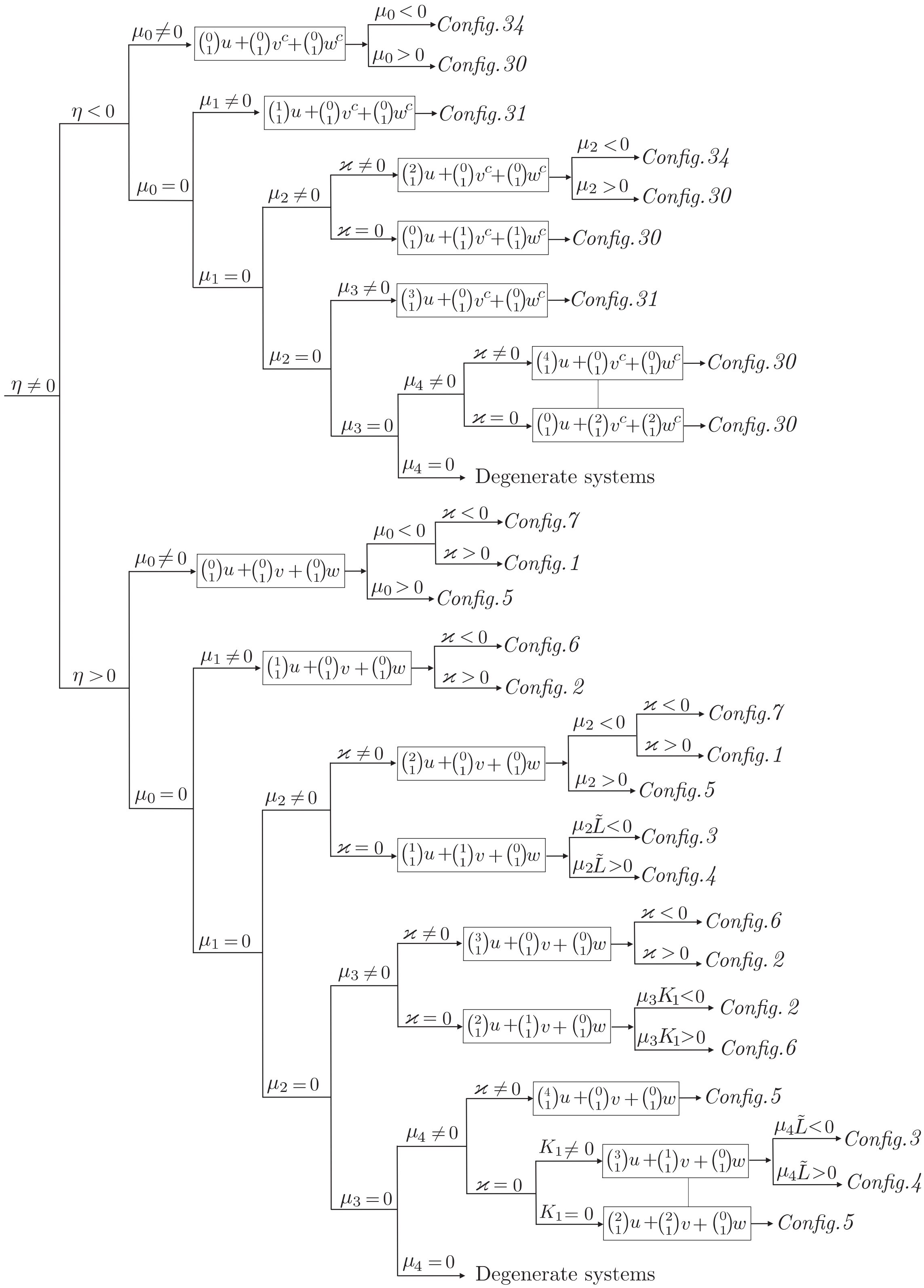


DIAGRAM 5. Topological configuration for the case $\eta \neq 0$.

8.1.2. **The case $\mu_0 > 0$.** By Theorem 1 and [28] systems $(\mathbf{S}_I)$ possess at infinity one saddle and two nodes. According to Remark 1 the types of the nodes depend on the three values $\lambda_2 - 1$ with $\lambda_2 \in \{g, h, 1-g-h\}$. Moreover, if one of these values vanishes (for example, $h - 1 = 0$) then in order to distinguish between a star

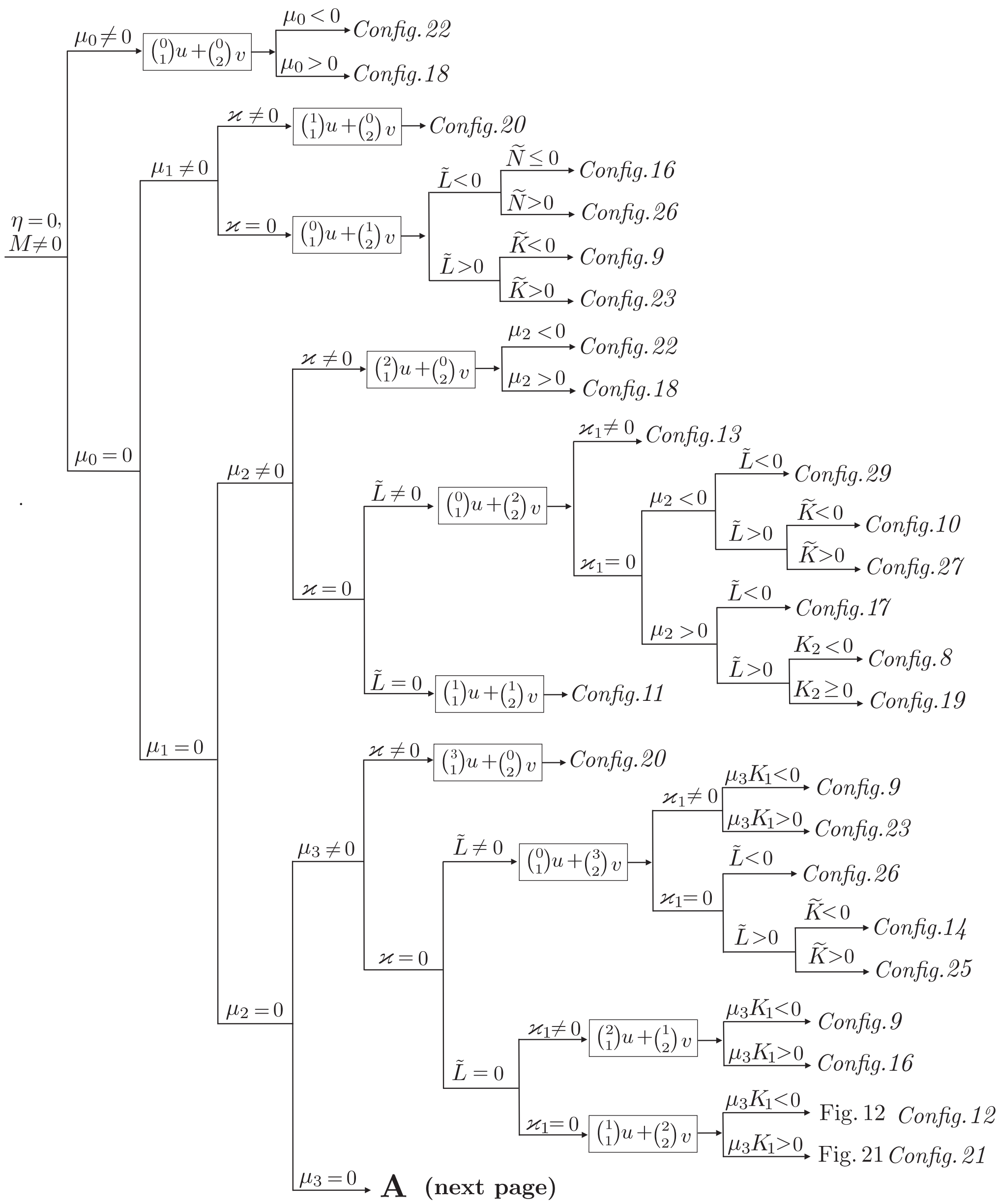


Diagram 6. Topological configuration for the case $\eta=0$, $\widetilde{M}\neq 0$.

node and a one direction node we need to distinguish if either the value ξ (which in this case is $-d$) vanishes or not. So it is convenient to introduce for the singular points R_i $(i=1,2,3)$ the following additional notations:

$$\begin{array}{lll} \tau_1=g-1, & \tau_2=h-1, & \tau_3=(1-g-h)-1=-(g+h); \\ \xi_1=-e, & \xi_2=-d, & \xi_3=-c-d+e+f. \end{array} \tag{7}$$

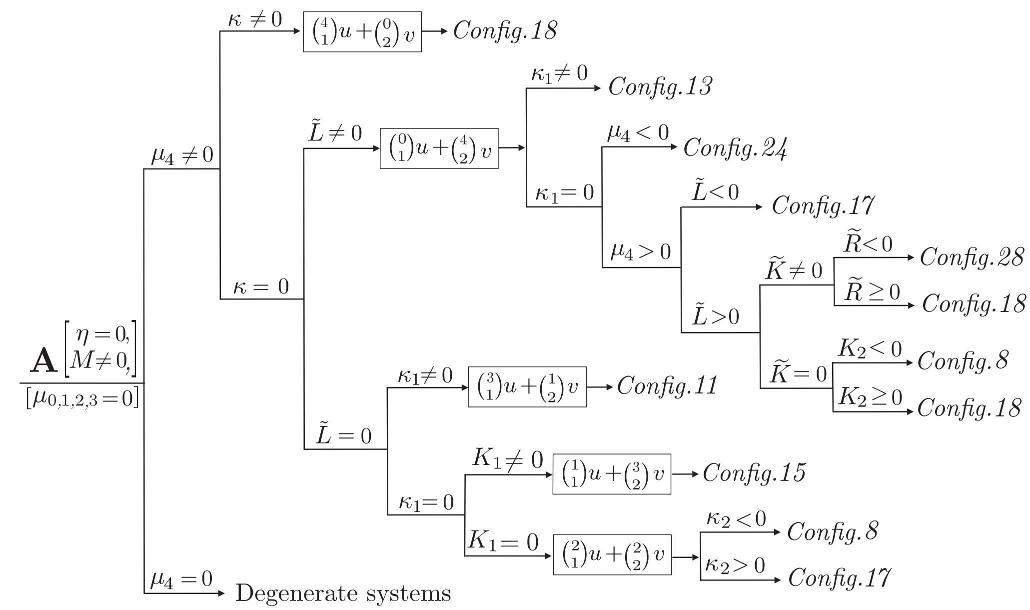


Diagram 6 (*cont.*). Topological configuration for the case $\eta=0$, $\widetilde{M}\neq 0$.

Then for systems $(\mathbf{S}_I)$ we calculate

$$
\begin{aligned}
\theta &= 8\tau_1\tau_2\tau_3,\\
\theta_1 &= 16(\tau_1\tau_2+\tau_1\tau_3+\tau_2\tau_3),\\
4\theta_2 &= (\tau_1\tau_2\xi_3+\tau_1\tau_3\xi_2-\tau_2\tau_3\xi_1),\\
\theta_3\big|_{\{\tau_1=\tau_2=0\}} &= -2\xi_1\xi_2, \quad \theta_4\big|_{\{\tau_1=\tau_2=0\}} = \xi_1+\xi_2,\\
\theta_3\big|_{\{\tau_1=\tau_3=0\}} &= -2\xi_1\xi_3, \quad \theta_4\big|_{\{\tau_1=\tau_3=0\}} = -(\xi_1+\xi_3),\\
\theta_3\big|_{\{\tau_2=\tau_3=0\}} &= 2\xi_2\xi_3, \quad \theta_4\big|_{\{\tau_2=\tau_3=0\}} = -(\xi_2-\xi_3),
\end{aligned}
\tag{8}
$$

In order to distinguish the signs of the values τ_1, τ_2 and τ_3, using the Viète's theorem we construct the equation of degree three possessing these quantities as the roots:

$$z^3-(\tau_1+\tau_2+\tau_3)z^2+(\tau_1\tau_2+\tau_1\tau_3+\tau_2\tau_3)z-\tau_1\tau_2\tau_3=0.$$

Considering (7) the above equation is equivalent to

$$F(z)\equiv z^3+2z^2+\frac{\theta_1}{16}z-\frac{\theta}{8}=0. \tag{9}$$

We note that the existence of one saddle among the singular points R_i $(i=1,2,3)$ implies that one of the roots of the equation (9) is negative.

8.1.2.1. *The subcase* $\theta<0$. Then the remaining two roots are both of the same sign. Moreover, considering the zeros of the function $F'(z)=3z^2+4z+\theta_1/16$ we conclude, that besides the negative zero of (9) we have two negative roots if $\theta_1>0$ and two positive ones if $\theta_1<0$. We note that the conditions $\mu_0>0$, $\theta<0$ and $\theta_1=0$ are incompatible as it can be easily seen using the respective graphic.

Thus besides the saddle we have at infinity two nodes N^∞, N^∞ if $\theta_1<0$ and N^f, N^f if $\theta_1>0$.

8.1.2.2. *The subcase* $\theta>0$. Then the remaining two zeros are of opposite signs and hence, beside the saddle we have at infinity the nodes N^f, N^∞.

8.1.2.3. *The subcase* $\theta=0$. In this case one of the roots of (9) vanishes and hence at infinity we have a node with two coinciding eigenvalues.

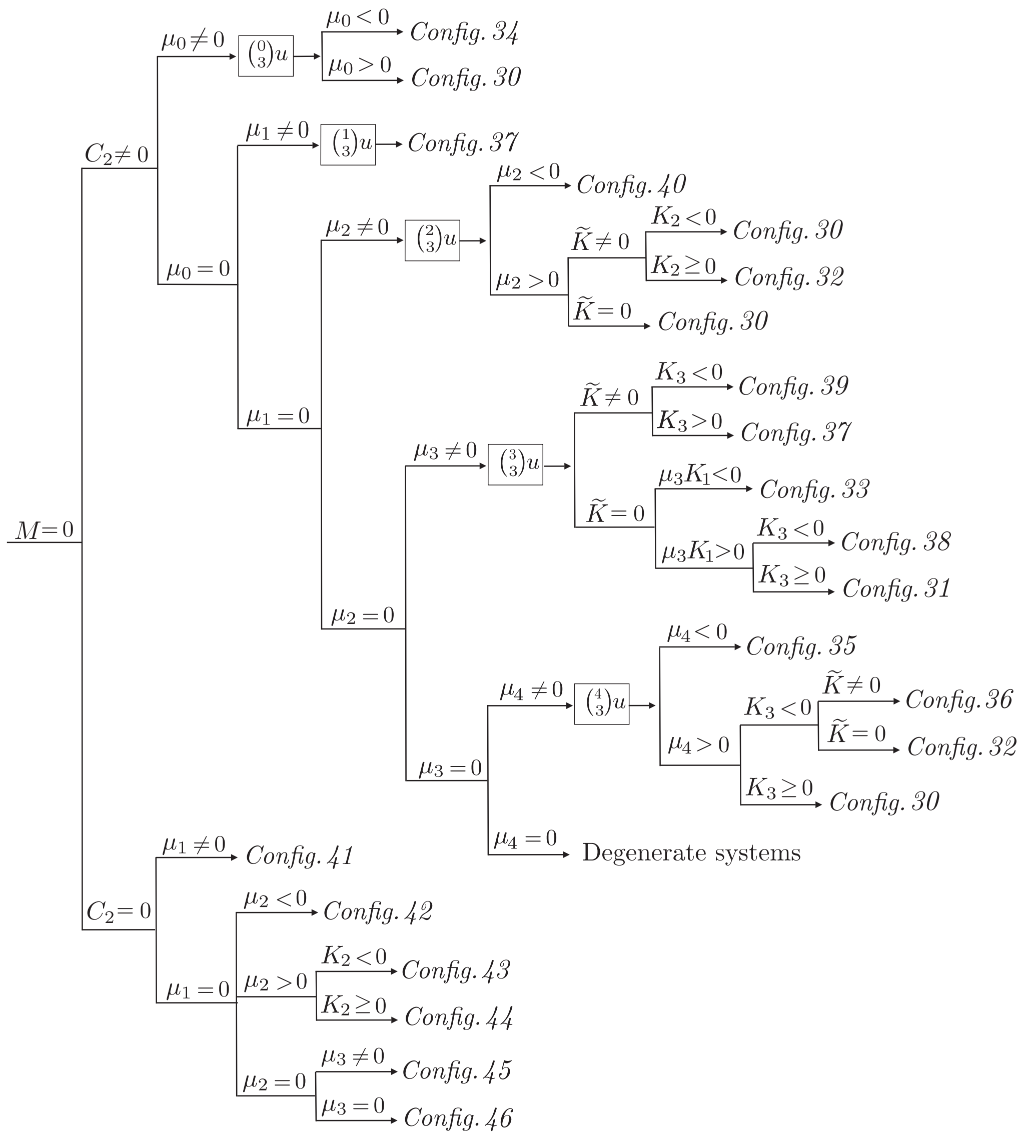


Diagram 7. Topological configuration for the case $\widetilde{M}=0$.

8.1.2.3.1. Assume first $\theta_1\neq 0$, i.e. other two zeros do not vanish. More exactly, as one of the zeros of F is negative, the second one is positive if $\theta_1<0$, and it is negative if $\theta_1>0$.

It remains to distinguish whether the node with two coinciding eigenvalues is a N^d or it is a N^*. We may assume that such a node is R_1 (otherwise we can apply a linear transformation). Therefore the condition $g=1$ (i.e. $\tau_1=0$) holds and considering (8) we obtain $\theta_1=16\tau_2\tau_3\neq 0$ and $\theta_2=-\tau_2\tau_3\xi_1/4$. So due to $\theta_1\neq 0$ the condition $\xi_1=0$ is equivalent to $\theta_2=0$.

Thus in the case $\theta=0$ and $\theta_1\neq 0$ we arrive at the following configurations, respectively:

$$\begin{array}{lllll} \theta_1<0, & \theta_2\neq 0 & \Rightarrow & S,\, N^\infty,\, N^d; \\ \theta_1<0, & \theta_2=0 & \Rightarrow & S,\, N^\infty,\, N^*; \\ \theta_1>0, & \theta_2\neq 0 & \Rightarrow & S,\, N^f,\, N^d; \\ \theta_1>0, & \theta_2=0 & \Rightarrow & S,\, N^f,\, N^*. \end{array} \tag{10}$$

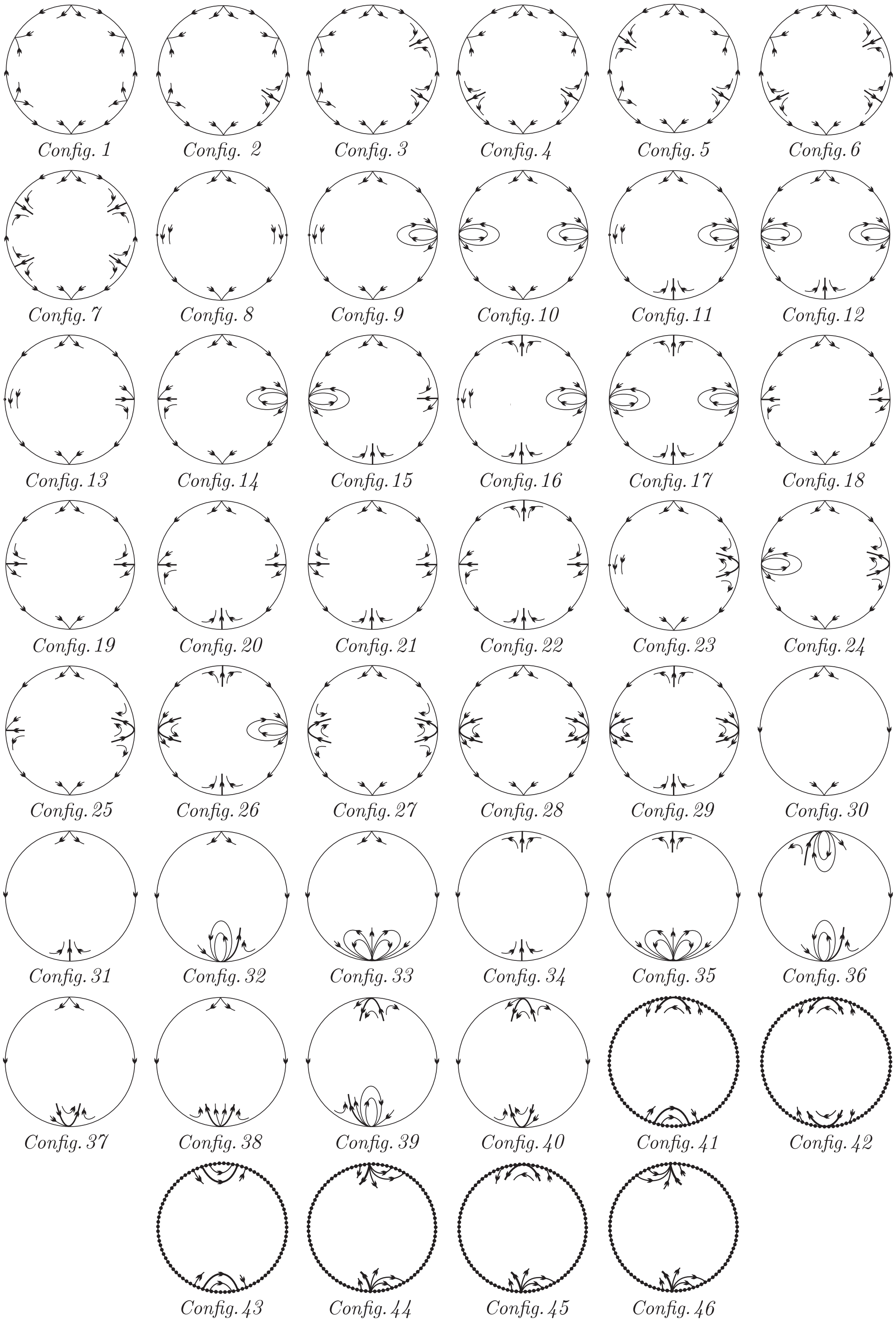


FIGURE 6. Topologically distinct local configurations of **ISP**s ([28],[31])

8.1.2.3.2. Suppose now that $\theta_1 = 0$. Then two of the roots of (9) vanish and therefore at infinity we have two nodes of the type either N^d or N^*. We may consider that the infinite singular point N_3 is a saddle (then $\tau_3 < 0$) and in this case the condition $\tau_1 = \tau_2 = 0$ (i.e. $g = h = 1$) must be satisfied. According to (8) in this case we obtain $\theta_3 = -2\xi_1\xi_2$ and $\theta_4 = \xi_1 + \xi_2$. So evidently we obtain N^d, N^d (respectively N^d, N^*; N^*, N^*) if $\theta_3 \neq 0$ (respectively $\theta_3 = 0, \theta_4 \neq 0$; $\theta_3 = \theta_4 = 0$).

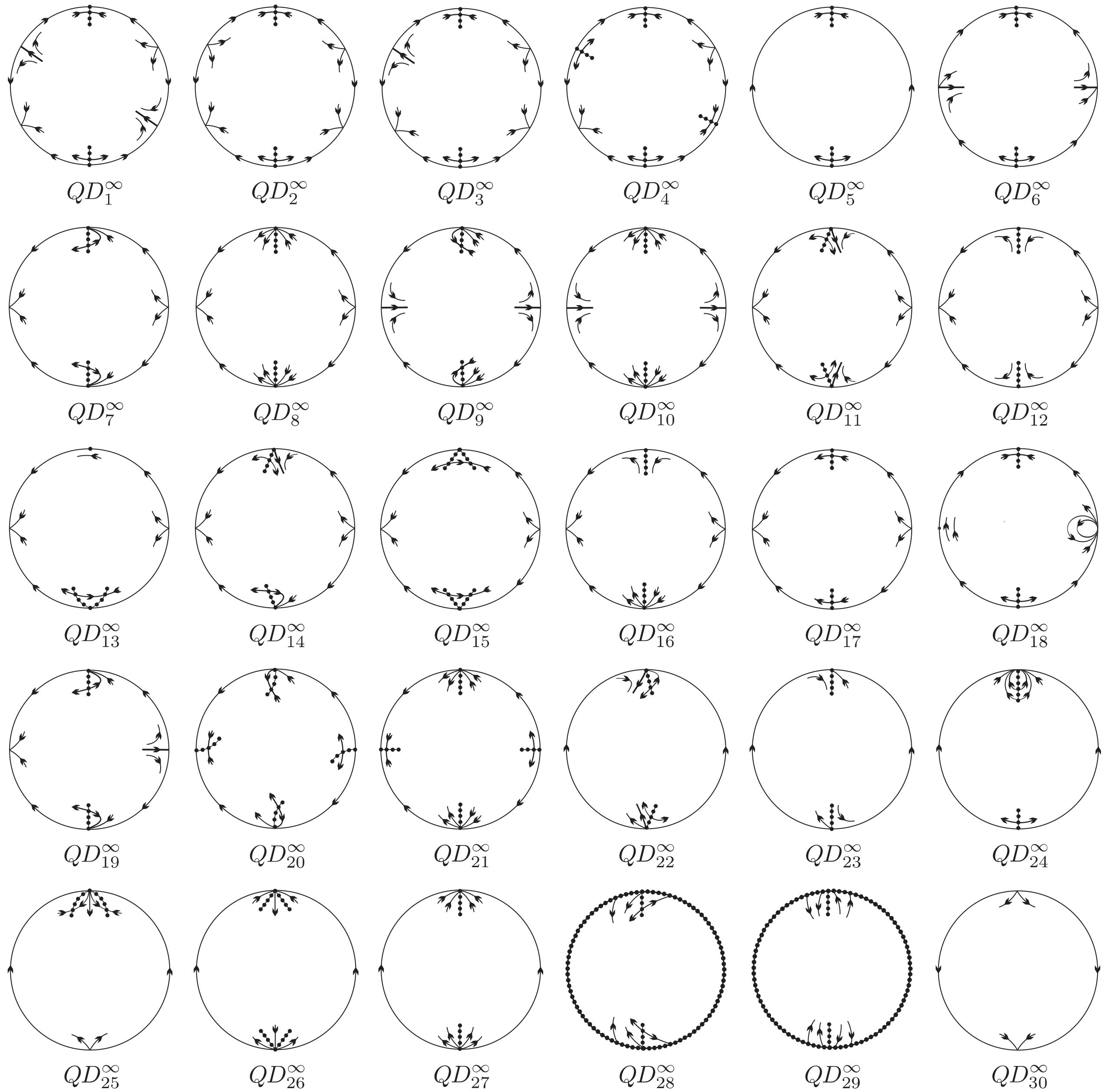


FIGURE 7. Topologically distinct local configurations of **ISP**s for degenerate quadratic systems

8.1.3. **The case $\mu_0 = 0,\ \mu_1 \neq 0$.** In this case exactly one finite point has gone to infinity. Considering (6) we have $gh(g + h - 1) = 0$ and we may assume $g = 0$ due to a linear transformation (which replaces the corresponding lines defined by the factors of $C_2 = xy(x - y)$). So the singular point R_1 becomes a semi–elemental saddle–node and for systems $(\mathbf{S}_I)$ we calculate

$$\mu_0 = 0, \quad \mu_1 = (1 - h)h(c - e + eh)y \neq 0, \quad \kappa = 16h(1 - h). \tag{11}$$

Remark 2. *If $\kappa \neq 0$ (i.e. $h(h - 1) \neq 0$) then considering* (5) *and $g = 0$ we conclude that R_2 and R_3 are elemental infinite singularities. Moreover, we have a saddle and a node N^∞ if $\kappa < 0$ and there are two nodes N^f, N^f if $\kappa > 0$.*

The condition $\mu_1 \neq 0$ implies $\kappa \neq 0$ and by the above remark besides the saddle–node at infinity we have the singular points S, N^∞ if $\kappa < 0$ and N^f, N^f if $\kappa > 0$.

8.1.4. **The case $\boldsymbol{\mu_0 = \mu_1 = 0}$.** We shall consider two geometrically distinct situations (see Theorem 1): $\kappa \neq 0$ (when systems $(\mathbf{S}_I)$ have at infinity only one multiple singularity) and $\kappa = 0$ (when at infinity there are two multiple singularities).

8.1.4.1. *The subcase* $\kappa \neq 0$. Then non–degenerate systems $(\mathbf{S}_I)$ (i.e. systems with $\sum_{i=0}^{4} \mu_i^2 \neq 0$) possess at infinity only one multiple singularity (in this case it is the point R_1). Moreover its multiplicity is three (respectively four; five) if $\mu_2 \neq 0$ (respectively $\mu_2 = 0$, $\mu_3 \neq 0$; $\mu_2 = \mu_3 = 0$, $\mu_4 \neq 0$). It is clear that this point is a semi–elemental saddle–node in the case of even multiplicity and it is either a saddle or a node if its multiplicity is an odd number. Considering Theorem 1, Remark 2 and [28] for non–degenerate systems $(\mathbf{S}_I)$ in the case $\mu_0 = \mu_1 = 0$ and $\kappa \neq 0$ we obtain the following configurations of infinite singularities:

$$
\begin{array}{lll}
\mu_2 < 0, & \kappa < 0 & \Rightarrow\ \overline{\binom{2}{1}} S,\ S,\ N^\infty; \\
\mu_2 < 0, & \kappa > 0 & \Rightarrow\ \overline{\binom{2}{1}} N,\ N^f,\ N^f; \\
\mu_2 > 0, & \kappa < 0 & \Rightarrow\ \overline{\binom{2}{1}} N,\ S,\ N^\infty; \\
\mu_2 > 0, & \kappa > 0 & \Rightarrow\ \overline{\binom{2}{1}} S,\ N^f,\ N^f; \\
\mu_2 = 0 \neq \mu_3, & \kappa < 0 & \Rightarrow\ \overline{\binom{3}{1}} SN,\ S,\ N^\infty; \\
\mu_2 = 0 \neq \mu_3, & \kappa > 0 & \Rightarrow\ \overline{\binom{3}{1}} SN,\ N^f,\ N^f; \\
\mu_2 = \mu_3 = 0 \neq \mu_4, & \kappa < 0 & \Rightarrow\ \overline{\binom{4}{1}} N,\ S,\ N^\infty; \\
\mu_2 = \mu_3 = 0 \neq \mu_4, & \kappa > 0 & \Rightarrow\ \overline{\binom{4}{1}} S,\ N^f,\ N^f.
\end{array}
$$

It remains to examine the case of degenerate systems $(\mathbf{S}_I)$, i.e. when the conditions $\mu_i = 0$ for each $i = 0, 1, \ldots, 4$ hold. We shall construct the canonical form of such systems in the case $\kappa \neq 0$. By (11) the condition $\mu_1 = 0$ implies $c - e + eh = 0$. Moreover, as $g = 0$ via a translation we may assume $e = f = 0$. Then $c = 0$ and systems $(\mathbf{S}_I)$ become

$$\dot{x} = a + dy + (h-1)xy, \quad \dot{y} = b - xy + hy^2,$$

and for these systems we calculate

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = h(h-1)(a - b + bh)y^2, \quad \kappa = 16h(1-h).$$

So since $\kappa \neq 0$ the condition $\mu_2 = 0$ gives $a = b(1-h)$ and then we calculate

$$
\begin{aligned}
\mu_3 &= bd(1-h)hy^3, \\
\mu_4 &= -by^3(d^2x - d^2hy - bh^2y + 2bh^3y - bh^4y).
\end{aligned}
$$

Clearly the condition $\mu_3 = \mu_4 = 0$ is equivalent to $b = 0$. Therefore we arrive at the degenerate systems

$$\dot{x} = y(d - x + hx), \quad \dot{y} = -y(x - hy), \tag{12}$$

possessing the invariant singular line $y = 0$ and the corresponding linear systems have the matrix

$$\begin{pmatrix} h-1 & 0 \\ -1 & h \end{pmatrix}.$$

As $\kappa \neq 0$ then considering the notation of singularities (see Section 5) we obtain the following configurations of infinite singularities of quadratic systems (12): N^∞, S, $(\ominus\, [|]; \emptyset)$ if $\kappa < 0$ and N^f, N^f, $(\ominus\, [|]; \emptyset)$ if $\kappa > 0$.

On the other hand we observe that the behavior of the trajectories at infinity in this case is topologically equivalent to the portraits QD_1^∞ if $\kappa < 0$ and QD_2^∞ if $\kappa > 0$ (see Figure 7).

8.1.4.2. *The subcase* $\kappa = 0$. Then $h(h-1) = 0$ and without loss of generality we may consider $h = 0$ in the systems $(\mathbf{S}_I)$ with $g = 0$ (due to a linear transformation which keeps the line $y = 0$ and replaces the line $y = x$ with $x = 0$). Moreover since $g = h = 0$ (doing a translation) we may assume $d = e = 0$ and systems $(\mathbf{S}_I)$ become

$$\dot{x} = a + cx - xy, \quad \dot{y} = b + fy - xy, \tag{13}$$

which possess at infinity two semi–elemental singular points $R_1(1,0,0)$ and $R_2(0,1,0)$ and the elemental singular point $R_3(1,1,0)$. For the last point we have the corresponding linear matrix (see (5)) $\begin{pmatrix} 1 & -c+f \\ 0 & 1 \end{pmatrix}$. Therefore $R_3(1,1,0)$ is a node of the type either N^d if $f-c\neq 0$, or N^* if $f-c=0$.

On the other hand for systems (13) we calculate:

$$\mu_0=\mu_1=0,\quad \mu_2=cfxy,\quad \tilde{L}=8xy,\quad \theta_2=(f-c)/4,\quad K_1=-xy(cx-fy),$$

and therefore we arrive at the next result.

Remark 3. *The elemental singular point $R_3(1,1,0)$ is a node of the type N^d, if $\theta_2\neq 0$, and N^* if $\theta_2=0$.*

8.1.4.2.1. Assume $\mu_2\neq 0$. Then by Theorem 1 the singularities $R_1(1,0,0)$ and $R_2(0,1,0)$ are both of multiplicity 2 and hence they are semi–elemental saddle–nodes. Considering [28] we conclude that in the case $\mu_2\neq 0$ we have the following configurations of infinite singularities:

$$\begin{array}{lll} \mu_2\tilde{L}<0 & & \Rightarrow\ \overline{\binom{1}{1}}SN,\, \overline{\binom{1}{1}}SN,\, N^d; \\ \mu_2\tilde{L}>0, & \theta_2\neq 0 & \Rightarrow\ \overline{\binom{1}{1}}SN,\, \overline{\binom{1}{1}}NS,\, N^d; \\ \mu_2\tilde{L}>0, & \theta_2=0 & \Rightarrow\ \overline{\binom{1}{1}}SN,\, \overline{\binom{1}{1}}NS,\, N^*. \end{array}$$

We notice that the condition $\mu_2\tilde{L}<0$ (i.e. $cf<0$) implies $\theta_2\neq 0$.

8.1.4.2.2. Admit now $\mu_2=0$. Then $cf=0$ and we may assume $f=0$ since the change $(x,y,a,b,c,f)\mapsto(y,x,b,a,f,c)$ conserves the systems (13). Then the semi–elemental singular point $R_2(0,1,0)$ becomes of the multiplicity ≥ 3. Moreover, according to Theorem 1 the multiplicities of the semi–elemental singularities are governed in the case $\kappa=0$ by the invariant polynomials μ_3, μ_4 and K_1.

1) Assume first $K_1\neq 0$. For systems (13) in the case $f=0$ we have

$$\mu_3=(b-a)cx^2y,\quad \mu_4=-bc^2x^3y+(a-b)^2x^2y^2,\quad K_1=-cx^2y,\quad \theta_2=-c/4. \tag{14}$$

Therefore the condition $K_1\neq 0$ implies $\theta_2\neq 0$.

a) If $\mu_3\neq 0$ by Theorem 1, Remark 3 and [28] we get the configurations $\overline{\binom{2}{1}}N$, $\overline{\binom{1}{1}}SN$, N^d if $\mu_3K_1<0$ and $\overline{\binom{2}{1}}S$, $\overline{\binom{1}{1}}SN$, N^d if $\mu_3K_1>0$.

b) Assume $\mu_3=0$. Since $K_1\neq 0$ (i.e. $c\neq 0$) we obtain $b=a$ and then $\mu_4=-ac^2x^3y$ and $\tilde{L}=xy$.

So if $\mu_4\neq 0$ (i.e. systems (13) are non–degenerate) then at infinity we have an elemental singularity (which is N^d by Remark 3), and two semi–elemental saddle–nodes: $R_1(1,0,0)$ of multiplicity two and $R_2(0,1,0)$ of multiplicity four. Considering [28] we obtain the configurations $\overline{\binom{3}{1}}SN$, $\overline{\binom{1}{1}}SN$, N^d if $\mu_4\tilde{L}<0$ and $\overline{\binom{3}{1}}SN$, $\overline{\binom{1}{1}}NS$, N^d if $\mu_4\tilde{L}>0$.

Assuming $\mu_4=0$ (i.e. $a=0$) since $c\neq 0$ we may take $c=1$ due to a rescaling and hence, we get the degenerate system $\dot{x}=x(1-y)$, $\dot{y}=-xy$, possessing the invariant line $x=0$ filled with singularities. Clearly the singular point $R_2(0,1,0)$ at infinity becomes a non-isolated singularity for the above system. So applying our notations (see Section 5) in the case of degenerate systems and $\kappa=0$ and $K_1\neq 0$ we get the configuration $\overline{\binom{1}{1}}SN$, N^d, $(\ominus\,[||];\emptyset)$. On the other hand we observe that the phase portrait around infinity in this case is topologically equivalent to the portrait QD_3^{∞} (see FIGURE 7).

2) Suppose now $K_1=0$. Then for systems (13) with $f=0$ considering (14) we obtain $c=0$ and then

$$\mu_3=0,\quad \mu_4=(a-b)^2x^2y^2,\quad \theta_2=0.$$

If $\mu_4\neq 0$ by Theorem 1 we have at infinity an elemental singularity (which is a star node by Remark 3) and two semi–elemental singular points both of multiplicity three. Considering [28] one of them is a node and another one is a saddle. Thus we get $\overline{\binom{2}{1}}S$, $\overline{\binom{2}{1}}N$, N^*.

Assume that $\mu_4 = 0$. Then for systems (13) the condition $\mu_2 = \mu_3 = \mu_4 = K_1 = 0$ gives us $f = c = 0$ and $b = a$, i.e. we get the degenerate systems $\dot{x} = a - xy, \quad \dot{y} = a - xy$. These systems possess an invariant hyperbola $xy - a = 0$ filled with singularities, which splits in two lines if $a = 0$. As for the above systems we have $L_1 = 12a(x - y)^2$, considering the notations in Section 5 we obtain N^*, $\big(\ominus\,[\,)(\,]; \emptyset, \emptyset\big)$ if $L_1 \neq 0$ and N^*, $\big(\ominus\,[\times]; \emptyset, \emptyset\big)$ if $L_1 = 0$. On the other hand we observe that the phase portrait around infinity in both the cases is topologically equivalent to the portrait QD_4^∞ (see FIGURE 7).

As all the cases are examined, the Main Theorem is proved for the family of systems $(\mathbf{S}_I)$.

8.2. **The family of systems $(\mathbf{S}_{II})$.** For these systems we have $C_2 = yp_2(x,y) - xq_2(x,y) = x(x^2+y^2)$. Therefore clearly at infinity we have one real singular point $R_2(0,1,0)$ and two complex singularities $R_{1,3}(1, \pm i, 0)$. Constructing the corresponding systems at infinity (possessing the real point at the origin of coordinates) we get the family of systems:

$$R_2 : \begin{cases} \dot{v} = -v - dz + (-c + f)vz - az^2 - v^3 + ev^2z + bvz^2, \\ \dot{z} = hz + gvz + fz^2 - v^2z + evz^2 + bz^3 \end{cases}$$

with the corresponding linear matrix $\begin{pmatrix} -1 & -d \\ 0 & h \end{pmatrix}$. Considering the Remark 1 we arrive at the next result.

Remark 4. *If $R_2(0,1,0)$ is an elemental singular point (i.e. $h \neq 0$) then it is a saddle if $h > 0$; a node N^f if $-1 < h < 0$; a node N^d if $h = -1$ and $d \neq 0$; a node N^* if $h = -1$ and $d = 0$; and it is a node N^∞ if $h < -1$.*

On the other hand for systems $(\mathbf{S}_{II})$ we have:

$$\begin{aligned} &\mu_0 = -h\big[(h+1)^2 + g^2\big], \ \kappa = -16\big[g^2 + (h+1)(1-3h)\big], \\ &\theta = 8(h+1)\big[(h-1)^2 + g^2\big], \ \theta_2\big|_{\{h=-1\}} = d(4+g^2)/4. \end{aligned} \tag{15}$$

8.2.1. *The case* $\boldsymbol{\mu_0 \neq 0}$. In this case we obtain $\text{sign}\,(\mu_0) = -\text{sign}\,(h)$ and by Theorem 1 together with [28] and taking into account the above remark we get the following configurations of infinite singularities:

$$\begin{array}{rcl} \mu_0 < 0 & \Rightarrow & S,\ ©, ©; \\ \mu_0 > 0,\ \theta < 0 & \Rightarrow & N^\infty,\ ©, ©; \\ \mu_0 > 0,\ \theta > 0 & \Rightarrow & N^f,\ ©, ©; \\ \mu_0 > 0,\ \theta = 0,\ \theta_2 \neq 0 & \Rightarrow & N^d,\ ©, ©; \\ \mu_0 > 0,\ \theta = 0,\ \theta_2 = 0 & \Rightarrow & N^*,\ ©, ©. \end{array}$$

We notice that in the case $\mu_0 > 0$ we get $h < 0$ and then by (15) the condition $\theta = 0$ is equivalent to $h = -1$.

8.2.2. *The case* $\boldsymbol{\mu_0 = 0,\ \mu_1 \neq 0}$. According to Lemma 2 in this case only one finite point has gone to infinity and clearly it must be a real one. So R_2 becomes a semi–elemental double singular point and clearly we get the configuration $\overline{\binom{1}{1}}SN$, ©, ©.

8.2.3. **The case $\boldsymbol{\mu_0 = \mu_1 = 0}$.** Considering Theorem 1 we shall distinguish again two geometrically different situations: when only the real infinite singular point increases its multiplicity (then $\kappa \neq 0$) and when the complex points become multiple (then $\kappa = 0$).

8.2.3.1. *The subcase* $\kappa \neq 0$. Then for non–degenerate systems $(\mathbf{S}_{II})$ the conditions $\mu_0 = 0$ and $\kappa \neq 0$ imply $h = 0$. In this case we may assume $c = d = 0$ (doing a translation) and then the condition $\mu_1 = -f(1+g^2)x = 0$ gives $f = 0$. Therefore we get the systems

$$\dot{x} = a + gx^2 + xy, \quad \dot{y} = b + ex - x^2 + gxy \tag{16}$$

for which calculations yield

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = (ag - b)(1 + g^2)x^2, \quad \kappa = -16(1 + g^2). \tag{17}$$

By Theorem 1 only the real infinite singularity is a multiple singularity for these systems which is R_2. Moreover its multiplicity is three (respectively four; five) if $\mu_2 \neq 0$ (respectively $\mu_2 = 0$, $\mu_3 \neq 0$; $\mu_2 = \mu_3 = 0$, $\mu_4 \neq 0$). Clearly, this point is a semi–elemental saddle–node in the case of even multiplicity and it is either a saddle or a node if its multiplicity is an odd number. Considering Theorem 1, Remark 2 and [28] for non–degenerate systems $(\mathbf{S}_{II})$ in the case $\mu_0 = \mu_1 = 0$ and $\kappa \neq 0$ we obtain the following configurations of infinite singularities:

$$\begin{array}{rcl}
\mu_2 < 0 & \Rightarrow & \overline{\binom{2}{1}}S,\ ©, ©; \\
\mu_2 > 0 & \Rightarrow & \overline{\binom{2}{1}}N,\ ©, ©; \\
\mu_2 = 0 \neq \mu_3 & \Rightarrow & \overline{\binom{3}{1}}SN,\ ©, ©; \\
\mu_2 = \mu_3 = 0 \neq \mu_4 & \Rightarrow & \overline{\binom{4}{1}}N,\ ©, ©.
\end{array}$$

Consider now the case of degenerate systems $(\mathbf{S}_{II})$ when $\kappa \neq 0$. Therefore we have to impose the conditions $\mu_2 = \mu_3 = \mu_4 = 0$ for systems (16). By (17) the condition $\mu_2 = 0$ yields $b = ag$ and then we obtain

$$\mu_3 = ae(1 + g^2)x^3, \quad \mu_4 = a\big[a(1 + g^2)^2 + e^2 g\big]x^4 + ae^2x^3y.$$

Clearly the condition $\mu_3 = \mu_4 = 0$ is equivalent to $a = 0$ and we arrive at the systems

$$\dot{x} = x(gx + y), \quad \dot{y} = x(e - x + gy)$$

possessing the invariant line $x = 0$ filled with singularities. The corresponding linear systems have the complex infinite points and hence, according to our notations (see Section 5) for degenerate systems (16) we get the configuration ©, ©, $\big(\ominus\,[|];\emptyset\big)$. In this case the phase portrait around infinity is topologically equivalent to the portrait QD_5^∞ (see FIGURE 7).

8.2.3.2. *The subcase* $\kappa = 0$. As $\mu_0 = 0$ then considering (15) we have $h + 1 = g = 0$ and then we may assume $e = f = 0$ doing a translation. So we get the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b - x^2 - y^2 \tag{18}$$

for which we calculate

$$\kappa = \theta = \mu_0 = \mu_1 = 0, \quad \mu_2 = (c^2 + d^2)(x^2 + y^2), \quad \theta_2 = d.$$

If $\mu_2 \neq 0$ then by Theorem 1, the complex singularities $R_{1,3}(1, \pm i, 0)$ are both of multiplicity 2. As the condition $d = 0$ is equivalent to $\theta_2 = 0$, considering Remark 4 we obtain the configurations N^d, $\binom{1}{1}$ ©, $\binom{1}{1}$ © if $\theta_2 \neq 0$ and N^*, $\binom{1}{1}$ ©, $\binom{1}{1}$ © if $\theta_2 = 0$.

Assuming $\mu_2 = 0$ we have $c = d = 0$ and then for systems (18) we obtain

$$\mu_2 = \mu_3 = \theta = \theta_2 = 0, \quad \mu_4 = a^2(x^2 + y^2)^2.$$

For non–degenerate systems we have $\mu_4 \neq 0$ (i.e. $a \neq 0$) and by Theorem 1 and Remark 4 we obtain the configuration N^*, $\binom{2}{1}$ ©, $\binom{2}{1}$ ©.

It remains to examine the case $\mu_4 = 0$, i.e. when $\kappa = 0$ and systems $(\mathbf{S}_{II})$ are degenerate. So setting $\mu_4 = 0$ (i.e. $a = 0$) in systems (18) with $c = d = 0$ we get the systems $\dot{x} = 0, \quad \dot{y} = b - x^2 - y^2$ possessing the invariant conics $x^2 + y^2 = b$ filled with singularities. For these systems we have $L_1 = -48bx^2$, i.e. $\operatorname{sign}(b) = -\operatorname{sign}(L_1)$ if $b \neq 0$. Therefore considering the notations in Section 5 and Remark 4 we obtain the following configurations:

$$\begin{array}{lcl} L_1 < 0 & \Rightarrow & N^*,\ \big(\ominus\,[\circ];\emptyset,\emptyset\big);\\ L_1 > 0 & \Rightarrow & N^*,\ \big(\ominus\,[\copyright];\emptyset,\emptyset\big);\\ L_1 = 0 & \Rightarrow & N^*,\ \big(\ominus\,[\,\cdot\,];\emptyset,\emptyset\big). \end{array}$$

On the other hand we observe that the phase portrait around infinity in all three cases is topologically equivalent to the portrait QD_{30}^{∞} (see FIGURE 7). This completes the proof of the Main Theorem in the case of the family of systems $(\mathbf{S}_{II})$.

8.3. **The family of systems** $(\mathbf{S}_{III})$**.** For these systems we have $\eta = 0$, $\widetilde{M} \neq 0$ and according to Lemma 1, at infinity we have two distinct real singularities. As $C_2 = yp_2(x,y) - xq_2(x,y) = x^2y$ these singularities are $R_1(1,0,0)$ and $R_2(0,1,0)$. We note that by Theorem 1 the divisor encoding the multiplicities of infinite singular points has the form $\binom{i}{1}u + \binom{j}{2}v$ with $i+j \in \{0,1,\ldots,4\}$. Constructing the corresponding systems at infinity (possessing the points R_i $(i=1,2)$ each one at the origin of coordinates of the corresponding local chart) we get respectively:

$$(19)\qquad \begin{array}{l} R_1 \longrightarrow \begin{cases} \dot{u} = u - ez + (c-f)uz - bz^2 + du^2z + auz^2,\\ \dot{z} = gz + huz + cz^2 + duz^2 + az^3; \end{cases}\\[2ex] R_2 \longrightarrow \begin{cases} \dot{v} = -dz - v^2 - (c-f)vz - az^2 + ev^2z + bvz^2,\\ \dot{z} = hz + (g-1)vz + fz^2 + evz^2 + bz^3. \end{cases} \end{array}$$

So the corresponding matrices for these singularities are as follows:

$$(20)\qquad R_1 \Rightarrow \begin{pmatrix} 1 & -e\\ 0 & g \end{pmatrix}; \qquad R_2 \Rightarrow \begin{pmatrix} 0 & -d\\ 0 & h \end{pmatrix}$$

and therefore R_1 is an elemental singular point if $g \neq 0$ and R_2 is a semi–elemental singularity if $h \neq 0$.

Remark 5. *If $R_1(1,0,0)$ is an elemental singular point (i.e. $g \neq 0$) then it is a saddle if $g < 0$; a node N^f if $0 < g < 1$; a node N^d if $g = 1$ and $e \neq 0$; a node N^* if $g = 1$ and $e = 0$; and it is a node N^∞ if $g > 1$.*

On the other hand for systems $(\mathbf{S}_{III})$ calculations yield

$$(21)\qquad \mu_0 = gh^2, \quad \kappa = -16h^2, \quad \theta = -8h^2(g-1), \quad \theta_2\big|_{\{g=1\}} = -eh^2/4.$$

8.3.1. *The case* $\boldsymbol{\mu_0 \neq 0}$. In this case we obtain $\operatorname{sign}(\mu_0) = \operatorname{sign}(g)$ and $\operatorname{sign}(\theta) = -\operatorname{sign}(g-1)$. Therefore by Theorem 1 and [28] and taking into account the above remark we get the following configurations of infinite singularities:

$$\begin{array}{lcl} \mu_0 < 0 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\ S;\\ \mu_0 > 0,\ \theta < 0 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\ N^\infty;\\ \mu_0 > 0,\ \theta > 0 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\ N^f;\\ \mu_0 > 0,\ \theta = 0 \neq \theta_2 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\ N^d;\\ \mu_0 > 0,\ \theta = \theta_2 = 0 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\ N^*. \end{array}$$

8.3.2. *The case* $\boldsymbol{\mu_0 = 0}$. According to Theorem 1 at least one finite singular point has gone to infinity. Moreover this point has coalesced either with $R_1(1,0,0)$ or with $R_2(0,1,0)$ and these two possibilities are governed by the invariant polynomial κ.

8.3.2.1. *The subcase* $\kappa \neq 0$. Then for systems $(\mathbf{S}_{III})$ the conditions $\mu_0 = 0$ and $\kappa \neq 0$ imply $g = 0$, i.e. the finite singular point has coalesced with $R_1(1,0,0)$. We note that by Theorem 1 if $\kappa \neq 0$ then all the finite singularities which have gone to infinity, have coalesced only with the point R_1 whose multiplicity is $\binom{i}{1}$. Moreover this point remains a semi–elemental singularity whose multiplicity $i+1$ depends of the number of the vanishing invariant polynomials μ_j $(j \in \{0,1,\ldots,3\}$ (see Lemma 2). It is clear that this point is a semi–elemental saddle–node in the case of even multiplicity and it is either a saddle or a node if its multiplicity is odd.

Therefore considering Theorem 1 and [28], in the case $\mu_0 = 0$ and $\kappa \neq 0$, for non–degenerate systems $(\mathbf{S}_{III})$ we obtain the following configurations of infinite singularities:

$$
\begin{array}{rcl}
\mu_1 \neq 0 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\, \overline{\tbinom{1}{1}}SN; \\
\mu_1 = 0,\, \mu_2 < 0 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\, \overline{\tbinom{2}{1}}S; \\
\mu_1 = 0,\, \mu_2 > 0 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\, \overline{\tbinom{2}{1}}N; \\
\mu_1 = \mu_2 = 0 \neq \mu_3 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\, \overline{\tbinom{3}{1}}SN; \\
\mu_1 = \mu_2 = \mu_3 = 0 \neq \mu_4 & \Rightarrow & \overline{\tbinom{0}{2}}SN,\, \overline{\tbinom{4}{1}}N.
\end{array}
$$

.

In order to finish the case $\kappa \neq 0$ we consider the degenerate systems $(\mathbf{S}_{III})$, i.e. by Lemma 2 the conditions $\mu_i = 0$ must hold for each $i = 0,1,\ldots,4$.

As $\kappa \neq 0$ we have $h \neq 0$ and we may assume $h = 1$ and $c = d = 0$ due to the affine transformation $x_1 = x + d/h$, $y_1 = hy + (ch - 2dg)/h$. It was mentioned above that the conditions $\mu_0 = 0$ and $\kappa \neq 0$ yield $g = 0$ and then $\mu_1 = -ey = 0$ implies $e = 0$. Thus we obtain the systems

$$\dot{x} = a + xy, \quad \dot{y} = b + fy - xy + y^2$$

for which we have $\mu_2 = (a+b)y^2$. So $\mu_2 = 0$ gives $b = -a$ and then we calculate

$$\mu_3 = afy^3, \quad \mu_4 = ay^3(f^2x + ay).$$

Clearly the condition $\mu_3 = \mu_4 = 0$ is equivalent to $a = 0$ and then we obtain the degenerate systems

$$\dot{x} = xy, \quad \dot{y} = y(f - x + y)$$

with $f \in \{0,1\}$ by doing a rescaling. These systems possess the invariant line $y = 0$ filled with singularities and the corresponding linear systems possess a double point at infinity which corresponds to the point $R_2(0,1,0)$ of quadratic systems. So using the notations given in Section 5 we arrive at the configuration $\overline{\binom{0}{2}}SN$, $(\ominus[|];\emptyset)$. On the other hand we observe that the phase portrait around infinity is, in this case, topologically equivalent to the portrait QD_6^{∞} (see Figure 7).

8.3.2.2. *The subcase* $\kappa = 0$. Then by (21) we get $h = 0$ and this implies $\mu_0 = 0$. We observe that in this case the singular point $R_2(0,1,0)$ becomes either a nilpotent or intricate point and for systems $(\mathbf{S}_{III})$ we calculate

$$\mu_1 = dg(g-1)^2x, \quad \widetilde{K} = 2g(g-1)x^2, \quad \tilde{L} = 8gx^2, \quad \kappa_1 = -32d, \quad \widetilde{N} = (g^2-1)x^2. \tag{22}$$

If $\mu_1 \neq 0$ then $\tilde{L}\widetilde{K} \neq 0$, $\operatorname{sign}(g) = \operatorname{sign}(\tilde{L})$ and if $\tilde{L} > 0$ then $\operatorname{sign}(g-1) = \operatorname{sign}(\widetilde{K})$.

The condition $\mu_1 \neq 0$ implies $d \neq 0$ and hence the second singular point $R_2(0,1,0)$ is nilpotent of multiplicity three. As $d(g-1) \neq 0$ then for systems $(\mathbf{S}_{III})$ with $h = 0$ we may assume $e = f = 0$ and $d = 1$ (doing a translation and a rescaling). So considering (19) we have the following systems

$$\dot{v} = -z - v^2 - cvz - az^2 + bvz^2, \quad \dot{z} = (g-1)vz + bz^3 \tag{23}$$

which have R_2 at the origin of coordinates. The phase portrait in a neighborhood of this point depends on the parameter g. More exactly, applying a blow–up and using our notation from Section 5 we obtain the following types of the singularity R_2 (depending on g):

$$\begin{array}{lcl} g<-1 & \Rightarrow & \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge}-E; \\ g=-1 & \Rightarrow & \widehat{\binom{1}{2}}\ H-E; \\ -1<g<0 & \Rightarrow & \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H; \\ 0<g<1 & \Rightarrow & \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H; \\ g>1 & \Rightarrow & \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge}-H. \end{array}$$

We observe that the above intervals for the parameter g are completely defined by the invariant polynomials $\widetilde{K}$, $\tilde{L}$ and $\widetilde{N}$ given in (22). So considering Remark 5 in the case $\kappa=0$ and $\mu_1\neq 0$ we obtain the following configurations of infinite singularities:

$$\begin{array}{lcl} \widetilde{K}<0 & \Rightarrow & \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H,\ N^f; \\ \widetilde{K}>0,\ \tilde{L}<0,\ \widetilde{N}<0 & \Rightarrow & \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H,\ S; \\ \widetilde{K}>0,\ \tilde{L}<0,\ \widetilde{N}=0 & \Rightarrow & \widehat{\binom{1}{2}}\ H-E,\ S; \\ \widetilde{K}>0,\ \tilde{L}<0,\ \widetilde{N}>0 & \Rightarrow & \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge}-E,\ S; \\ \widetilde{K}>0,\ \tilde{L}>0 & \Rightarrow & \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge}-H,\ N^{\infty}. \end{array}$$

In what follows we assume $\mu_1=0$ and we consider two cases: $\widetilde{K}\neq 0$ and $\widetilde{K}=0$.

8.3.2.2.1. Assume first $\widetilde{K}\neq 0$. Considering (22) we have $g(g-1)\neq 0$ and therefore the condition $\mu_1=0$ gives $d=0$. In this case we may assume $e=f=0$ (due to a translation) and we get the systems

$$\dot{x}=a+cx+gx^2,\quad \dot{y}=b+(g-1)xy, \tag{24}$$

for which we have

$$\mu_0=\mu_1=\kappa_1=0,\quad \mu_2=ag(g-1)^2x^2,\ \widetilde{K}=2g(g-1)x^2,\quad \tilde{L}=8gx^2,\ K_2=48(c^2-4ag)(2-g+g^2)x^2. \tag{25}$$

The condition $\widetilde{K}\neq 0$ implies $\tilde{L}\neq 0$ and the infinite singular point $R_1(1,0,0)$ of the above systems is elemental. Its type is described by Remark 5.

On the other hand since $h=d=0$, by (20) the second infinite singularity $R_2(0,1,0)$ becomes an intricate singular point whose multiplicity, by Lemma 2, depends of the number of the vanishing invariant polynomials μ_i $(i=2,3)$.

1) Assume $\mu_2\neq 0$. Then $R_2(0,1,0)$ has multiplicity four: two infinite and two finite singularities have coalesced all together. The corresponding systems (19) in this case are the systems

$$\dot{v}=-v^2-cvz-az^2+bvz^2,\quad \dot{z}=(g-1)vz+bz^3,$$

where $ag(g-1)\neq 0$, having R_2 at the origin of coordinates. Applying a blow–up we determine that the behavior of the trajectories in the neighborhood of this point depends on the parameters a, c and g. More exactly, using our notation from Section 5 we obtain the following types of the singularity R_2:

$$\begin{array}{lcl} ag<0,\ g<0 & \Rightarrow & \binom{2}{2}\ \widehat{P} H \widehat{P}-\widehat{P} H \widehat{P}; \\ ag<0,\ 0<g<1 & \Rightarrow & \binom{2}{2}\ \widehat{P} E \widehat{P}-\widehat{P} E \widehat{P}; \\ ag<0,\ g>1 & \Rightarrow & \binom{2}{2}\ HHH-HHH; \\ ag>0,\ g<0,\ \Delta<0 & \Rightarrow & \binom{2}{2}\ E-E; \\ ag>0,\ g<0,\ \Delta=0 & \Rightarrow & \binom{2}{2}\ \widehat{P} E-\widehat{P} E; \\ ag>0,\ g<0,\ \Delta>0 & \Rightarrow & \binom{2}{2}\ \widehat{P}\,\widehat{P} E-\widehat{P}\,\widehat{P} E; \\ ag>0,\ g>0,\ \Delta<0 & \Rightarrow & \binom{2}{2}\ H-H; \\ ag>0,\ g>0,\ \Delta=0 & \Rightarrow & \binom{2}{2}\ \widehat{P} H-\widehat{P} H; \\ ag>0,\ g>0,\ \Delta>0 & \Rightarrow & \binom{2}{2}\ \widehat{P}\,\widehat{P} H-\widehat{P}\,\widehat{P} H, \end{array}$$

where $\Delta = c^2 - 4ag$. According to (25) if $\mu_2 K_2 \widetilde{K} \neq 0$ then we have

$$\operatorname{sign}(ag) = \operatorname{sign}(\mu_2), \quad \operatorname{sign}(c^2 - 4ag) = \operatorname{sign}(K_2), \quad \operatorname{sign}(g) = \operatorname{sign}(\tilde{L}).$$

Moreover, as by Remark 5 the type of the elemental node R_1 depends on the sign $(g-1)$ we notice that sign $(g-1) = \operatorname{sign}(\tilde{L}\widetilde{K})$.

Thus considering the types of the intricate singular point R_2 (described above) and Remark 5 in the case $\kappa = \mu_1 = 0$ and $\widetilde{K}\mu_2 \neq 0$ we obtain the following configurations of infinite singularities:

$$\begin{array}{rcl}
\mu_2 < 0,\ \widetilde{K} < 0 & \Rightarrow & \binom{2}{2}\ \widehat{P}E\widehat{P} - \widehat{P}E\widehat{P},\ N^f;\\
\mu_2 < 0,\ \widetilde{K} > 0,\ \tilde{L} < 0 & \Rightarrow & \binom{2}{2}\ \widehat{P}H\widehat{P} - \widehat{P}H\widehat{P},\ S;\\
\mu_2 < 0,\ \widetilde{K} > 0,\ \tilde{L} > 0 & \Rightarrow & \binom{2}{2}\ HHH - HHH,\ N^\infty;\\
\mu_2 > 0,\ \widetilde{K} < 0,\ K_2 < 0 & \Rightarrow & \binom{2}{2}\ H - H,\ N^f;\\
\mu_2 > 0,\ \widetilde{K} < 0,\ K_2 > 0 & \Rightarrow & \binom{2}{2}\ \widehat{P}\widehat{P}H - \widehat{P}\widehat{P}H,\ N^f;\\
\mu_2 > 0,\ \widetilde{K} < 0,\ K_2 = 0 & \Rightarrow & \binom{2}{2}\ \widehat{P}H - \widehat{P}H,\ N^f;\\
\mu_2 = 0,\ \widetilde{K} > 0,\ \tilde{L} < 0,\ K_2 < 0 & \Rightarrow & \binom{2}{2}\ E - E,\ S;\\
\mu_2 = 0,\ \widetilde{K} > 0,\ \tilde{L} < 0,\ K_2 > 0 & \Rightarrow & \binom{2}{2}\ \widehat{P}\widehat{P}E - \widehat{P}\widehat{P}E,\ S;\\
\mu_2 = 0,\ \widetilde{K} > 0,\ \tilde{L} < 0,\ K_2 = 0 & \Rightarrow & \binom{2}{2}\ \widehat{P}E - \widehat{P}E,\ S;\\
\mu_2 = 0,\ \widetilde{K} > 0,\ \tilde{L} > 0,\ K_2 < 0 & \Rightarrow & \binom{2}{2}\ H - H,\ N^\infty;\\
\mu_2 = 0,\ \widetilde{K} > 0,\ \tilde{L} > 0,\ K_2 > 0 & \Rightarrow & \binom{2}{2}\ \widehat{P}\widehat{P}H - \widehat{P}\widehat{P}H,\ N^\infty;\\
\mu_2 = 0,\ \widetilde{K} > 0,\ \tilde{L} > 0,\ K_2 = 0 & \Rightarrow & \binom{2}{2}\ \widehat{P}H - \widehat{P}H,\ N^\infty.
\end{array}$$

2) Suppose now $\mu_2 = 0$ and $\mu_3 \neq 0$. Considering (25) we have $a = 0$ and then

$$\mu_3 = -bcg(g-1)x^3, \quad \mu_4 = bx^3\left[bg^2x + c^2(g-1)y\right], \quad \widetilde{K} = 2g(g-1)x^2. \tag{26}$$

As $\mu_3 \neq 0$ the intricate singular point $R_2(0,1,0)$ has multiplicity five. The corresponding systems (19) in this case are of the form

$$\dot{v} = -v^2 - cvz + bvz^2, \quad \dot{z} = (g-1)vz + bz^3,$$

where $bcg(g-1) \neq 0$ having R_2 at the origin of coordinates. Applying again a blow–up we determine that the behavior of the trajectories in the neighborhood of this point depends only on the parameter g. More exactly, using our notation from Section 5 we obtain the following types of the singularity R_2:

$$\begin{array}{rcl}
g < 0 & \Rightarrow & \binom{3}{2}\ \widehat{P}H\widehat{P} - \widehat{P}\widehat{P}E;\\
0 < g < 1 & \Rightarrow & \binom{3}{2}\ \widehat{P}E\widehat{P} - \widehat{P}\widehat{P}H;\\
g > 1 & \Rightarrow & \binom{3}{2}\ H\widehat{P}\widehat{P} - HHH.
\end{array}$$

Therefore considering Remark 5 in the case $\kappa = \mu_1 = \mu_2 = 0$ and $\widetilde{K}\mu_3 \neq 0$ we obtain the following configurations of infinite singularities:

$$\begin{array}{rcl}
\widetilde{K} < 0 & \Rightarrow & \binom{3}{2}\ \widehat{P}E\widehat{P} - \widehat{P}\widehat{P}H,\ N^f;\\
\widetilde{K} > 0,\ \tilde{L} < 0 & \Rightarrow & \binom{3}{2}\ \widehat{P}H\widehat{P} - \widehat{P}\widehat{P}E,\ S;\\
\widetilde{K} > 0,\ \tilde{L} > 0 & \Rightarrow & \binom{3}{2}\ H\widehat{P}\widehat{P} - HHH,\ N^\infty.
\end{array}$$

3) If $\mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$ considering (26) we get $c = 0$ and $bg(g-1) \neq 0$. In this case the intricate singular point $R_2(0,1,0)$ becomes a singularity of multiplicity six. The corresponding systems (19) in this case are of the form

$$\dot{v} = -v^2 + bvz^2, \quad \dot{z} = (g-1)vz + bz^3, \tag{27}$$

where $bg(g-1) \neq 0$ and we need to examine the point $(0,0)$ of these systems. Similarly as before, applying a blow–up we determine that the behavior of the trajectories in the neighborhood of this point depends only on the parameter g. More exactly, using our notation from Section 5 we obtain the following types of the singularity of R_2:

$$\begin{array}{lcl} g<0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,\widehat{P}_{\curlywedge}E-E\,\widehat{P}_{\curlywedge}\,\widehat{P};\\ 0<g<1/2 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,H_{\curlywedge}\widehat{P}-\widehat{P}\,H_{\curlywedge}\widehat{P};\\ g=1/2 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,H-H\,\widehat{P};\\ 1/2<g<1 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,\widehat{P}_{\curlywedge}H-H\,\widehat{P}_{\curlywedge}\widehat{P};\\ g>1 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,\widehat{P}_{\curlywedge}H-H\,\widehat{P}_{\curlywedge}\widehat{P}. \end{array}$$

We note that for systems $(\mathbf{S}_{III})$ in this case we have $\widetilde{R}=8g(2g-1)x^2$ and hence if $g>0$ we have $\operatorname{sign}(2g-1)=\operatorname{sign}(\widetilde{R})$.

Thus considering the types of the intricate singular point R_2 (described above) and Remark 5 in the case $\kappa=\mu_1=\mu_2=\mu_3=0$ and $\widetilde{K}\mu_4\neq 0$ we obtain the following configurations of infinite singularities:

$$\begin{array}{lcl} \widetilde{K}<0,\ \widetilde{R}<0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,H_{\curlywedge}\widehat{P}-\widehat{P}\,H_{\curlywedge}\widehat{P},\ N^f;\\ \widetilde{K}<0,\ \widetilde{R}>0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,\widehat{P}_{\curlywedge}H-H\,\widehat{P}_{\curlywedge}\widehat{P},\ N^f;\\ \widetilde{K}<0,\ \widetilde{R}=0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,H-H\,\widehat{P},\ N^f;\\ \widetilde{K}>0,\ \tilde{L}<0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,\widehat{P}_{\curlywedge}E-E\,\widehat{P}_{\curlywedge}\,\widehat{P},\ S;\\ \widetilde{K}>0,\ \tilde{L}>0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\,\widehat{P}_{\curlywedge}H-H\,\widehat{P}_{\curlywedge}\widehat{P},\ N^{\infty}. \end{array}$$

4) Assuming $\mu_2=\mu_3=\mu_4=0$, the systems $(\mathbf{S}_{III})$ become degenerate.

Considering (26) we observe that the condition $\mu_3=\mu_4=0$ is equivalent to $b=0$. Therefore systems (24) with $a=0$ are of the form

$$\dot{x}=x(c+gx),\quad \dot{y}=(g-1)xy \tag{28}$$

possessing the invariant line $x=0$ filled with singularities. The corresponding linear systems possess two infinite singularities $R_1(1,0,0)$ and $R_2(0,1,0)$. The corresponding matrices for these singularities are the following:

$$R_1\ \Rightarrow\ \begin{pmatrix}1&0\\0&g\end{pmatrix};\qquad R_2\ \Rightarrow\ \begin{pmatrix}1&0\\0&1-g\end{pmatrix}.$$

As $\widetilde{K}=2g(g-1)x^2\neq 0$ we conclude that both singularities are elemental. Moreover their types are governed by the parameter g as follows:

$$\begin{array}{ccl} g<0 & \Rightarrow & R_1\ \rightarrow\ S,\quad R_2\ \rightarrow\ N^{\infty};\\ 0<g<1 & \Rightarrow & R_1\ \rightarrow\ N^f,\quad R_2\ \rightarrow\ N^f;\\ g>1 & \Rightarrow & R_1\ \rightarrow\ N^{\infty},\quad R_2\ \rightarrow\ S. \end{array}$$

We observe that the invariant line $x=0$ of systems (28) coincides with an invariant line of the corresponding linear systems if and only if $c=0$. Therefore as $\tilde{L}=8gx^2$ and $K_2=48c^2(2-g-g^2)x^2$ then considering our notations (see Section 5) for the degenerate systems $(\mathbf{S}_{III})$ in the case $\kappa=0$ and $\widetilde{K}\neq 0$ we obtain the following configurations of infinite singularities and the corresponding topological behavior at infinity (see FIGURE 7):

$$\begin{array}{lcll} \widetilde{K}<0,\ K_2\neq 0 & \Rightarrow & N^f,\ \big(\ominus\,[|];N_3^f\big) & QD_7^{\infty};\\ \widetilde{K}<0,\ K_2=0 & \Rightarrow & N^f,\ \big(\ominus\,[|];N_2^f\big) & QD_8^{\infty};\\ \widetilde{K}>0,\ \tilde{L}<0\neq K_2 & \Rightarrow & S,\ \big(\ominus\,[|];N_3^{\infty}\big) & QD_9^{\infty};\\ \widetilde{K}>0,\ \tilde{L}<0=K_2 & \Rightarrow & S,\ \big(\ominus\,[|];N_2^{\infty}\big) & QD_{10}^{\infty};\\ \widetilde{K}>0,\ \tilde{L}>0\neq K_2 & \Rightarrow & N^{\infty},\ \big(\ominus\,[|];S_3\big) & QD_{11}^{\infty};\\ \widetilde{K}>0,\ \tilde{L}>0=K_2 & \Rightarrow & N^{\infty},\ \big(\ominus\,[|];S_2\big) & QD_{12}^{\infty}. \end{array}$$

8.3.2.2.2. Suppose now $\widetilde{K} = 0$. Then by (22) we get $g(g-1) = 0$ and we shall consider two subcases: $\tilde{L} \neq 0$ and $\tilde{L} = 0$.

1) Assume first $\tilde{L} \neq 0$. Then $g = 1$ and we may assume $c = 0$ due to a translation. So we get the systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + fy \tag{29}$$

for which we have

$$\begin{aligned} &\mu_0 = \mu_1 = \widetilde{K} = 0, \quad \mu_2 = f^2x^2, \quad \kappa_1 = -d, \\ &\tilde{L} = 8x^2, \quad \theta_5 = 96dex^3, \quad \theta_6\big|_{\{d=0\}} = 8ex^4, \quad K_2\big|_{\{d=0\}} = -384ax^2. \end{aligned} \tag{30}$$

a) Suppose $\mu_2 \neq 0$, i.e. $f \neq 0$. Then via a rescaling we may assume $f = 1$. Since $g = 1$, by Remark 5, the singular point $R_1(1,0,0)$ is a node N^d if $e \neq 0$ and it is a star node if $e = 0$. The singularity $R_2(0,1,0)$ has multiplicity four: two infinite and two finite singularities have coalesced all together. Moreover R_2 is a nilpotent singularity if $d \neq 0$ and it is an intricate singular point if $d = 0$. The corresponding systems (19) in this case are the systems

$$\dot{v} = -dz - v^2 + vz + ev^2z - az^2 + bvz^2, \quad \dot{z} = z^2 + evz^2 + bz^3,$$

having R_2 at the origin of coordinates. Applying a blow–up we show that the behavior of the trajectories in the neighborhood of this point depends on the parameters a and d. Furthermore, using our notation from Section 5 we obtain the following types of the singularity R_2:

$$\begin{array}{lcl}
d \neq 0 & \Rightarrow & \widehat{\binom{2}{2}}\ \overset{\curvearrowright}{P}_{\curlywedge}\ \overset{\curvearrowleft}{P} H_{\curlywedge} - H; \\
d = 0,\ a < 0 & \Rightarrow & \binom{2}{2}\ \overset{\curvearrowright}{P}\ \overset{\curvearrowleft}{P} H - \overset{\curvearrowright}{P}\ \overset{\curvearrowleft}{P} H; \\
d = 0,\ a = 0 & \Rightarrow & \binom{2}{2}\ \overset{\curvearrowright}{P} H - \overset{\curvearrowright}{P} H; \\
d = 0,\ a > 0 & \Rightarrow & \binom{2}{2}\ H - H.
\end{array}$$

Therefore considering the types of the elemental singular point R_1 and (30), in the case $\kappa = \widetilde{K} = 0$ and $\mu_2\tilde{L} \neq 0$ we obtain the following configurations of infinite singularities:

$$\begin{array}{lcl}
\kappa_1 \neq 0 \neq \theta_5 & \Rightarrow & \widehat{\binom{2}{2}}\ \overset{\curvearrowright}{P}_{\curlywedge}\ \overset{\curvearrowleft}{P} H_{\curlywedge} - H,\ N^d; \\
\kappa_1 \neq 0 = \theta_5 & \Rightarrow & \widehat{\binom{2}{2}}\ \overset{\curvearrowright}{P}_{\curlywedge}\ \overset{\curvearrowleft}{P} H_{\curlywedge} - H,\ N^*; \\
\kappa_1 = 0,\ K_2 < 0 \neq \theta_6 & \Rightarrow & \binom{2}{2}\ H - H,\ N^d; \\
\kappa_1 = 0,\ K_2 < 0 = \theta_6 & \Rightarrow & \binom{2}{2}\ H - H,\ N^*; \\
\kappa_1 = 0,\ K_2 > 0 \neq \theta_6 & \Rightarrow & \binom{2}{2}\ \overset{\curvearrowright}{P}\ \overset{\curvearrowleft}{P} H - \overset{\curvearrowright}{P}\ \overset{\curvearrowleft}{P} H,\ N^d; \\
\kappa_1 = 0,\ K_2 > 0 = \theta_6 & \Rightarrow & \binom{2}{2}\ \overset{\curvearrowright}{P}\ \overset{\curvearrowleft}{P} H - \overset{\curvearrowright}{P}\ \overset{\curvearrowleft}{P} H,\ N^*; \\
\kappa_1 = 0,\ K_2 = 0 \neq \theta_6 & \Rightarrow & \binom{2}{2}\ \overset{\curvearrowright}{P} H - \overset{\curvearrowright}{P} H,\ N^d; \\
\kappa_1 = 0,\ K_2 = 0 = \theta_6 & \Rightarrow & \binom{2}{2}\ \overset{\curvearrowright}{P} H - \overset{\curvearrowright}{P} H,\ N^*.
\end{array}$$

b) Assume $\mu_2 = 0$ and $\mu_3 \neq 0$. In this case for systems (29) we obtain $f = 0$ and then

$$\mu_3 = de^2x^3, \quad \mu_4 = (b^2 + ae^2)x^4 - bdex^3y, \quad \kappa_1 = -32d, \quad K_1 = -ex^3. \tag{31}$$

The condition $\mu_3 \neq 0$ implies $de \neq 0$ and we may assume $d = 1$ and $a = 0$, due to a rescaling and a translation. Therefore the singular point $R_1(1,0,0)$ is a node N^d and $R_2(0,1,0)$ is a nilpotent singularity of multiplicity five. In order to examine the neighborhood of the second point we consider the corresponding systems (see (19))

$$\dot{v} = -z - v^2 + ev^2z + bvz^2, \quad \dot{z} = evz^2 + bz^3,$$

having R_2 at the origin of coordinates. Applying a blow–up we show that the behavior of the trajectories in the neighborhood of this point depends on the sign of the parameter e. More precisely we obtain the following types of the singularity R_2:

$$e<0 \;\Rightarrow\; \widehat{\binom{3}{2}}\, H_{\curlywedge} H H_{\curlywedge} - H;\quad e>0 \;\Rightarrow\; \widehat{\binom{3}{2}}\, \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H.$$

Since by (31), in the case $d=1$ we have $\operatorname{sign}(\mu_3 K_1) = -\operatorname{sign}(e)$, then we get the following configurations of the infinite singularities:

$$\mu_3 K_1<0 \;\Rightarrow\; \widehat{\binom{3}{2}}\, \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^d;\quad \mu_3 K_1>0 \;\Rightarrow\; \widehat{\binom{3}{2}}\, H_{\curlywedge} H H_{\curlywedge} - H,\ N^d.$$

***c*)** Admit now that $\mu_3=0$. Then $de=0$ and we shall consider two subcases: $d=0$ and $d\neq 0$. Clearly these cases are distinguished by the invariant polynomial κ_1 (see (31)).

***α*)** Assume first $\kappa_1\neq 0$, i.e. $d\neq 0$ and $e=0$. As it was mentioned above we can take $d=1$ and $a=0$ and therefore we get the systems

$$\dot{x} = y + x^2,\quad \dot{y} = b \tag{32}$$

for which $\mu_4 = b^2x^4$. If $\mu_4\neq 0$ then according to Remark 5, the elemental singular point $R_1(1,0,0)$ is a star node, whereas $R_2(0,1,0)$ is a nilpotent singularity of multiplicity six: two infinite and four finite singularities have coalesced all together. The corresponding systems (19) in this case are the systems

$$\dot{v} = -z - v^2 + bvz^2,\quad \dot{z} = bz^3,\quad b\neq 0,$$

having R_2 at the origin of coordinates. Applying a blow–up we find that the behavior of the trajectories in the neighborhood of this point is uniquely determined: $\widehat{\binom{4}{2}}\, \widehat{P}_{\curlywedge}\, \widehat{P} H_{\curlywedge} - H$.

Therefore considering the elemental star node in the case $\kappa = \widetilde{K} = \mu_2 = \mu_3 = 0$ and $\tilde{L}\kappa_1\mu_4\neq 0$ we obtain the following configuration of infinite singularities: $\widehat{\binom{4}{2}}\, \widehat{P}_{\curlywedge}\, \widehat{P} H_{\curlywedge} - H,\ N^*$.

Supposing $\mu_4=0$ we have $b=0$ and from (32) we obtain the degenerate system possessing the invariant parabola $y=-x^2$ filled with singularities. Considering the notations from Section 5 we get the configuration $N^*,\ (\ominus[\cup];\emptyset)$. On the other hand we observe that the phase portrait around infinity is topologically equivalent to the portrait QD^{∞}_{13} (see FIGURE 7).

***β*)** Suppose now $\kappa_1=0$. Then $d=0$ and we obtain the systems

$$\dot{x} = a + x^2,\quad \dot{y} = b + ex. \tag{33}$$

for which we have

$$\kappa_1=\mu_3=0,\quad \mu_4=(b^2+ae^2)x^4,\quad K_2=-384ax^2,\quad \theta_6=8ex^4. \tag{34}$$

By Remark 5 the elemental singular point $R_1(1,0,0)$ is a node N^d if $e\neq 0$ and it is a star node N^* if $e=0$.

If $\mu_4\neq 0$ then $R_2(0,1,0)$ is an intricate singularity of multiplicity six: two infinite and four finite singularities have coalesced all together. To describe the neighborhood of R_2 we need to examine the neighborhood of the origin in the corresponding systems

$$\dot{v} = -v^2 - az^2 + ev^2z + bvz^2,\quad \dot{z} = evz^2 + bz^3,$$

where $b^2+ae^2\neq 0$. Applying a blow–up we find that the behavior of the trajectories in the neighborhood of this point depends on the parameters a, b and e. More exactly, using our notation from Section 5 we obtain the following types of the singularity R_2:

$$\begin{array}{lcl}
a<0,\ b^2+ae^2<0 & \Rightarrow & \binom{4}{2}\ \widehat{P}E\widehat{P}-HHH;\\
a=0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\widehat{P}_{\curlywedge}H-H\widehat{P}_{\curlywedge}\widehat{P};\\
a<0,\ b^2+ae^2>0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\widehat{P}H-H\widehat{P}\widehat{P};\\
a>0 & \Rightarrow & \binom{4}{2}\ H-H.
\end{array}$$

By (34) we have $\operatorname{sign}(a)=-\operatorname{sign}(K_2)$ and $\operatorname{sign}(b^2+ae^2)=\operatorname{sign}(\mu_4)$, and since the condition $\theta_6=0$ (i.e. $e=0$) implies $\mu_4>0$, we get the following configurations of the infinite singularities:

$$\begin{array}{lcl}
K_2<0,\ \theta_6\neq 0 & \Rightarrow & \binom{4}{2}\ H-H,\ N^d;\\
K_2<0,\ \theta_6=0 & \Rightarrow & \binom{4}{2}\ H-H,\ N^*;\\
K_2>0,\ \mu_4<0 & \Rightarrow & \binom{4}{2}\ \widehat{P}E\widehat{P}-HHH,\ N^d;\\
K_2>0,\ \mu_4>0\neq\theta_6 & \Rightarrow & \binom{4}{2}\ \widehat{P}\widehat{P}H-H\widehat{P}\widehat{P},\ N^d;\\
K_2>0,\ \mu_4>0=\theta_6 & \Rightarrow & \binom{4}{2}\ \widehat{P}\widehat{P}H-H\widehat{P}\widehat{P},\ N^*;\\
K_2=0,\ \theta_6\neq 0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\widehat{P}_{\curlywedge}H-H\widehat{P}_{\curlywedge}\widehat{P},\ N^d;\\
K_2=0,\ \theta_6=0 & \Rightarrow & \binom{4}{2}\ \widehat{P}\widehat{P}_{\curlywedge}H-H\widehat{P}_{\curlywedge}\widehat{P},\ N^*.
\end{array}$$

Assume now $\mu_4=0$, i.e. $b^2+ae^2=0$. We shall consider two subcases: $e\neq 0$ and $e=0$ (these conditions are governed by the invariant polynomial θ_6).

$\boldsymbol{\beta_1}$**)** If $\theta_6\neq 0$ then $e\neq 0$ and we may assume $e=1$ due to a rescaling. Then $a=-b^2$ and we obtain the degenerate systems

$$\dot{x}=(b+x)(x-b),\quad \dot{y}=b+x, \tag{35}$$

possessing the invariant line $x=-b$ filled with singularities. We note that the linear systems $\dot{x}=-b+x$, $\dot{y}=1$ have the invariant line $x=b$ and two infinite singularities $R_1(1,0,0)$ and $R_2(0,1,0)$ with the following matrices:

$$R_1\ \Rightarrow\ \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix};\qquad R_2\ \Rightarrow\ \begin{pmatrix}-1 & b\\ 0 & 0\end{pmatrix}.$$

So the elemental singular point R_1 is a node N^d and the double singular point R_2 is a semi–elemental saddle–node. Moreover we observe that the line $x=-b$ is different from the invariant line $x=b$ of the linear systems if $b\neq 0$ and they coincide if $b=0$.

Therefore considering our notations (see Section 5) we clearly get either the configuration N^d, $\big(\ominus[|];\overline{\binom{1}{1}}SN_3\big)$ if $b\neq 0$, or N^d, $\big(\ominus[|];\overline{\binom{1}{1}}SN_2\big)$ if $b=0$.

$\boldsymbol{\beta_2}$**)** If $\theta_6=0$ we have $e=b=0$ and the systems (33) have the form

$$\dot{x}=a+x^2,\quad \dot{y}=0. \tag{36}$$

So in this case we have an invariant conic filled with singularities which splits into two parallel complex (respectively real) lines if $a>0$ (respectively $a<0$) and it is a double real line if $a=0$. Clearly at infinity the corresponding constant system possesses one singular point which is a star node. Therefore considering the notations from Section 5 we arrive at the next configurations of the singular points for the above systems:

$$\begin{array}{lcl}
a<0 & \Rightarrow & N^*,\ \big(\ominus[||];\emptyset\big);\\
a=0 & \Rightarrow & N^*,\ \big(\ominus[|2];\emptyset\big);\\
a>0 & \Rightarrow & N^*,\ \big(\ominus[||^c];\emptyset\big).
\end{array}$$

On the other hand, for systems (35) (respectively (36)) we have $K_2=384b^2x^2\geq 0$ (respectively $K_2=-384ax^2$). Therefore the invariant polynomials K_2 and θ_6 distinguish the configurations of the infinite singularities as well as the phase portraits around infinity for the degenerate systems $(\mathbf{S}_{III})$ in the case $\kappa=\widetilde{K}=0$ and $\tilde{L}\neq 0$ as follows:

$$\begin{array}{llll}
K_2<0 & \Rightarrow & N^*,\ \big(\ominus\,[|\,|^c];\emptyset\big) & QD^{\infty}_{30};\\
K_2>0\neq\theta_6 & \Rightarrow & N^d,\ \big(\ominus\,[|\,|];\overline{\binom{1}{1}}SN_3\big) & QD^{\infty}_{14};\\
K_2>0=\theta_6 & \Rightarrow & N^*,\ \big(\ominus\,[|\,|\,|];\emptyset\big) & QD^{\infty}_{15};\\
K_2=0\neq\theta_6 & \Rightarrow & N^d,\ \big(\ominus\,[|\,|];\overline{\binom{1}{1}}SN_2\big) & QD^{\infty}_{16};\\
K_2=0=\theta_6 & \Rightarrow & N^*,\ \big(\ominus\,[|2];\emptyset\big) & QD^{\infty}_{17}.
\end{array}$$

2) Consider now the case $\tilde{L}=0$. Then $g=0$ and we may assume $e=f=0$ due to a translation. So we get the systems

$$\dot{x}=a+cx+dy,\quad \dot{y}=b-xy \tag{37}$$

for which we have

$$\kappa=\mu_0=\mu_1=\widetilde{K}=\tilde{L}=0,\quad \mu_2=-cdxy,\quad \kappa_1=-32d.$$

***a*)** Suppose $\mu_2\neq 0$, i.e. $cd\neq 0$. Then via a rescaling we may assume $d=1$ and considering (20) and Theorem 1 we deduce that the singular point $R_1(1,0,0)$ is a double semi–elemental saddle–node, whereas $R_2(0,1,0)$ is a nilpotent singularity of multiplicity three: two infinite and one finite singularities have coalesced all together. To determine the geometric structure of the neighborhood of $R_2(0,1,0)$ we shall examine the singular point $(0,0)$ of the systems

$$\dot{v}=-z-v^2-cvz-az^2+bvz^2,\quad \dot{z}=-vz+bz^3,$$

where $c\neq 0$.

Remark 6. *Doing a blow–up we detect a unique type for* $(0,0)$*:* $\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H$*, independently of the values of the parameters* a*,* b *and* c*.*

So considering the saddle–node $R_1(1,0,0)$ we obtain that the systems (37) possess at infinity the configuration of singularities $\widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ \overline{\binom{1}{1}}SN$.

***b*)** Assume $\mu_2=0$. Then $cd=0$ and we examine two subcases: $d\neq 0$ and $d=0$.

$\boldsymbol{\alpha}$) Admit first $\kappa_1\neq 0$, i.e. $d\neq 0$ and $c=0$. Due to a rescaling we may take $d=1$ and for systems (37) we calculate:

$$\mu_2=0,\quad \mu_3=axy^2,\quad \kappa_1=-32\neq 0,\quad K_1=-xy^2.$$

$\boldsymbol{\alpha_1}$) If $\mu_3\neq 0$ then according to Theorem 1 the singular point $R_1(1,0,0)$ becomes a triple semi–elemental singularity: two finite singularities have coalesced with an infinite one. Moreover R_1 is a saddle if $a<0$ and it is a node if $a>0$. We note that in this case the triple nilpotent point R_2 is of the type indicated in Remark 6.

Thus taking into account that $\operatorname{sign}(a)=-\operatorname{sign}(\mu_3K_1)$, for systems (37), at infinity we obtain
$\widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ \overline{\binom{2}{1}}N$ if $\mu_3K_1<0$ and $\widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ \overline{\binom{2}{1}}S$ if $\mu_3K_1>0$.

$\boldsymbol{\alpha_2}$) Suppose now $\mu_3=0$. Then $a=0$ and we obtain the systems

$$\dot{x}=y,\quad \dot{y}=b-xy \tag{38}$$

for which we have $\mu_4=-bxy^3$. If $\mu_4\neq 0$ then by Theorem 1 the singularity R_1 increases its multiplicity and it becomes a semi–elemental saddle–node of multiplicity four. By Remark 6, we get the following configuration of infinite singularities of systems (38): $\widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ \overline{\binom{3}{1}}SN$.

Supposing $\mu_4=0$ we have $b=0$ and we get the degenerate system $\dot{x}=y$, $\dot{y}=-xy$, possessing the invariant line $y=0$ filled with singularities. The reduced system is linear, having the unique infinite singularity $[0:1:0]$ which is a nilpotent elliptic–saddle. More exactly as this system is linear we have $\widehat{\binom{1}{2}}\ E-H$. Considering the line $y=0$, for the above degenerate system, we get at infinity $\widehat{\binom{1}{2}}\, E-H$, $(\ominus\, [|];\emptyset)$. On the other hand we observe that the phase portrait around infinity is topologically equivalent to the portrait QD^{∞}_{18} in FIGURE 7.

***β*)** In the case $\kappa_1=0$ we have $d=0$ and for systems (37) we calculate:

$$\mu_2=0,\quad \mu_3=-acx^2y,\quad K_1=-cx^2y. \tag{39}$$

***β*₁)** If $\mu_3\neq 0$ then $ac\neq 0$ and we may assume $c=1$ due to a rescaling. By Theorem 1 the singular point $R_1(1,0,0)$ is a double semi–elemental saddle–node. At the same time the singular point $R_2(0,1,0)$ is an intricate singularity of multiplicity four: two finite points have coalesced with two infinite ones. To determine the geometric structure of the neighborhood of R_2 we examine the singular point $(0,0)$ of the systems

$$\dot v=-v^2-vz-az^2+bvz^2,\quad \dot z=-vz+bz^3,$$

where $a\neq 0$. Applying a blow–up we show that the behavior of the trajectories in the neighborhood of this point depends on the parameter a. More exactly, using our notation from Section 5 we obtain the following types for the singularity R_2:

$$a<0\ \Rightarrow\ \binom{2}{2}\ \widehat{P}\,E-\widehat{P}\,E;\quad a>0\ \Rightarrow\ \binom{2}{2}\ \widehat{P}\,H-\widehat{P}\,H.$$

Since by (39) we have $\operatorname{sign}(a)=\operatorname{sign}(\mu_3K_1)$, considering the singularity R_1 we get the following configurations of infinite singularities for systems (37) in the case $d=0$ (i.e. $\kappa_1=0$):

$$\mu_3K_1<0\ \Rightarrow\ \binom{2}{2}\ \widehat{P}\,E-\widehat{P}\,E,\overline{\binom{1}{1}}SN;\quad \mu_3K_1>0\ \Rightarrow\ \binom{2}{2}\ \widehat{P}\,H-\widehat{P}\,H,\overline{\binom{1}{1}}SN.$$

***β*₂)** Supposing $\mu_3=0$, by (39) we obtain $ac=0$ and we consider two subcases: $c\neq 0$ and $c=0$.

***i*)** If $K_1\neq 0$ then $c\neq 0$ and as above, we assume $c=1$. Then $a=0$ and we arrive at the systems $\dot x=x,\quad \dot y=b-xy$ for which we calculate:

$$\mu_2=\mu_3=0,\quad \mu_4=-bx^3y,\quad K_1=-x^2y.$$

If $\mu_4\neq 0$ then by Theorem 1 besides the double semi–elemental saddle–node $R_1(1,0,0)$ the above systems possess the intricate singular point $R_2(0,1,0)$ of multiplicity five: three finite singularities have coalesced with two infinite ones. In order to determine the geometric structure of the neighborhood of R_2 we examine the singular point $(0,0)$ of the systems

$$\dot v=-v^2-vz+bvz^2,\quad \dot z=-vz+bz^3,\quad b\neq 0.$$

Applying a blow–up we determine that the behavior of the trajectories in the neighborhood of this point could be described as $\binom{3}{2}\,E\,\widehat{P}-\widehat{P}\,H$. Considering the saddle–node R_1 we obtain $\binom{3}{2}\,E\,\widehat{P}-\widehat{P}\,H,\ \overline{\binom{1}{1}}SN$.

Assuming $\mu_4=0$ we obtain $b=0$ and this leads to the degenerate system $\dot x=x$, $\dot y=-xy$ possessing the invariant line $x=0$ filled with singularities. It can easily be determined that the reduced linear system has two singular points at infinity: (i) the semi–elemental saddle–node $R_1(1,0,0)$ (which corresponds to the singular point R_1 of the degenerate quadratic systems); (ii) the singular point $R_2(0,1,0)$ which is a node N^d.

Therefore considering our notations (see Section 5) for degenerate systems $(\mathbf{S}_{III})$ in the case the configuration of infinite singularities is $\overline{\binom{1}{1}}SN$, $(\ominus\,[|];N^d)$, and the phase portrait around infinity is topologically equivalent to the portrait QD^{∞}_{19} (see FIGURE 7).

***ii*)** Suppose now $K_1=0$, i.e. $c=0$. Then we obtain the systems

$$\dot x=a,\quad \dot y=b-xy \tag{40}$$

for which we calculate:

$$\mu_2=\mu_3=0,\quad \mu_4=a^2x^2y^2,\quad \kappa_2=-a,\quad L_1=8ax^2,\quad L_2=-3b.$$

If $\mu_4\neq 0$ then according to Theorem 1 the singular point $R_1(1,0,0)$ is semi–elemental of multiplicity three: two finite singularities have coalesced with one infinite singularity. Moreover it is a node if $\kappa_2<0$ (i.e. $a>0$)

and it is a saddle if $\kappa_2 > 0$ (i.e. $a < 0$). At the same time $R_2(0,1,0)$ is an intricate singular point of multiplicity four: two finite singularities have coalesced with two infinite ones. In order to examine the neighborhood of the second point we consider the corresponding systems

$$\dot{v} = -v^2 - az^2 + bvz^2, \quad \dot{z} = -vz + bz^3, \quad a \neq 0,$$

having R_2 at the origin of coordinates. Applying a blow–up we show that the behavior of the trajectories in the neighborhood of this point depends on the sign of the parameter a. More precisely we obtain the following types of the singularity R_2:

$$a < 0 \ \Rightarrow \ \widehat{\tbinom{2}{2}}\, E{-}E; \qquad a > 0 \ \Rightarrow \ \widehat{\tbinom{2}{2}}\, H{-}H.$$

Since $\operatorname{sign}(a) = \operatorname{sign}(L_1)$, considering the semi–elemental singular point R_1 we get the following configurations of infinite singularities for systems (40) in the case $a \neq 0$ (i.e. $\mu_4 \neq 0$):

$$L_1 < 0 \ \Rightarrow \ \widehat{\binom{2}{2}}\, E{-}E, \overline{\binom{2}{1}} S; \ \ L_1 > 0 \ \Rightarrow \ \widehat{\binom{2}{2}}\, H{-}H, \overline{\binom{2}{1}} N.$$

Supposing $\mu_4 = 0$ we obtain $a = 0$ and this leads to the degenerate systems $\dot{x} = 0$, $\dot{y} = b - xy$ possessing the invariant conic $xy = b$ filled with singularities. Clearly this conic splits in two lines if $b = 0$ and this situation is governed by the invariant polynomial L_2. We observe that both infinite singularities of the degenerate systems are non-isolated ones.

Thus applying the respective notations (see Section 5) for degenerate systems $(\mathbf{S}_{III})$ in the case $\kappa = \tilde{L} = \kappa_1 = K_1 = 0$ we obtain the following configurations of infinite singularities (see FIGURE 7):

$$L_2 \neq 0 \ \Rightarrow \ \big(\ominus\, [\,)(\,]; N^*, \emptyset\big)\ QD^{\infty}_{20}; \ \ L_2 = 0 \ \Rightarrow \ \big(\ominus\, [\times]; N^*, \emptyset\big)\ QD^{\infty}_{21}.$$

Since all the cases are examined, the Main Theorem is proved for the family of systems $(\mathbf{S}_{III})$.

8.4. **The family of systems** $(\mathbf{S}_{IV})$**.** For these systems we have $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$ and according to Lemma 1, at infinity we have one real singularity of multiplicity greater than or equal to three. As $C_2 = yp_2(x,y) - xq_2(x,y) = x^3$ this singularity is $R_2(0,1,0)$ and by Theorem 1 the divisor encoding the multiplicities of infinite singular points have the form $\binom{i}{3}u$ with $i \in \{0,1,\ldots,4\}$. Constructing the corresponding systems at infinity (possessing the singular point R_2 at the origin of coordinates) we obtain

$$\begin{cases} \dot{v} = -dz - (c-f)vz - az^2 - v^3 + ev^2z + bvz^2, \\ \dot{z} = hz + gvz + fz^2 - v^2z + evz^2 + bz^3, \end{cases} \tag{41}$$

with the matrix $\Rightarrow \begin{pmatrix} 0 & -d \\ 0 & h \end{pmatrix}$ of their linear parts. So R_2 is a triple semi–elemental singular point if $h \neq 0$. For these systems we have $\mu_0 = -h^3$ and by Theorem 1, R_2 is a saddle if $\mu_0 < 0$ and it is a node if $\mu_0 > 0$.

Thus in the case $\mu_0 \neq 0$ the configuration of infinite singularities for systems $(\mathbf{S}_{IV})$ is $\overline{\tbinom{0}{3}}S$ if $\mu_0 < 0$ and it is $\overline{\tbinom{0}{3}}N$ if $\mu_0 > 0$.

In what follows we assume $\mu_0 = 0$. Then $h = 0$ and for systems $(\mathbf{S}_{IV})$ we calculate

$$\mu_0 = 0, \quad \mu_1 = dg^3x, \quad \widetilde{K} = 2g^2x^2. \tag{42}$$

8.4.1. *The case* $\boldsymbol{\mu_1 \neq 0}$. Then $dg \neq 0$ and considering (41) and Theorem 1, the singular point $R_2(0,1,0)$ is a nilpotent singular point of multiplicity four. To determine the behavior of the trajectories in its neighborhood we consider systems (41) with $h = 0$, having R_2 at the origin of coordinates. Doing a blow–up and using our notation from Section 5 we obtain the configuration $\widehat{\tbinom{1}{3}}H_{\curlywedge}H\widehat{P}_{\curlywedge}{-}\widehat{P}$.

8.4.2. *The case* $\boldsymbol{\mu_1 = 0}$. In this case we get $dg = 0$ and we examine two subcases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

8.4.2.1. *The subcase* $\widetilde{K} \neq 0$. Then by (42) we have $g \neq 0$, $d = 0$ and we may assume $e = f = 0$ (doing a translation) and $g = 1$ (doing a rescaling). So we get the systems

$$\dot{x} = a + cx + x^2, \quad \dot{y} = b - x^2 + xy \tag{43}$$

for which we calculate

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = ax^2, \quad K_2 = 48(c^2 - 4a)x^2. \tag{44}$$

8.4.2.1.1. If $\mu_2 \neq 0$ then by Theorem 1, the singular point $R_2(0,1,0)$ is an intricate singular point of multiplicity five: two finite singularities have coalesced with three infinite ones. In this case in order to determine the geometrical type of this point considering (41) we examine the systems

$$\dot{v} = -cvz - az^2 - v^3 + bvz^2, \quad \dot{z} = vz - v^2z + bz^3 \tag{45}$$

having R_2 at the origin of coordinates. Applying a blow–up we determine that the behavior of the trajectories in the neighborhood of this point depends on the sign of the parameter a. More precisely we obtain the following types of the singularity R_2:

$$\begin{array}{lcl}
a < 0 & \Rightarrow & \binom{2}{3}\, HH\overset{\curvearrowright}{P} - \overset{\curvearrowleft}{P}HH; \\
a > 0,\ c^2 - 4a < 0 & \Rightarrow & \binom{2}{3}\, \overset{\curvearrowright}{P} - \overset{\curvearrowleft}{P}; \\
a > 0,\ c^2 - 4a > 0 & \Rightarrow & \binom{2}{3}\, H\overset{\curvearrowright}{P}E - \overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}\,\overset{\curvearrowleft}{P}; \\
a > 0,\ c^2 - 4a = 0 & \Rightarrow & \binom{2}{3}\, HE - \overset{\curvearrowleft}{P}\,\overset{\curvearrowleft}{P}.
\end{array}$$

It remains to note that by (44) we obtain $\operatorname{sign}(a) = \operatorname{sign}(\mu_2)$ and $\operatorname{sign}(c^2 - 4a) = \operatorname{sign}(K_2)$. Therefore we arrive at the corresponding conditions indicated in the DIAGRAM 4 (see the Main Theorem).

8.4.2.1.2. Assuming $\mu_2 = 0$ by (44) we get $a = 0$ and then we calculate

$$\mu_0 = \mu_1 = \mu_2 = 0, \quad \mu_3 = -bcx^3, \quad \mu_4 = bx^3(bx - c^2x + c^2y), \quad K_3 = -6bx^6. \tag{46}$$

1) If $\mu_3 \neq 0$ then $bc \neq 0$ and we may assume $c = 1$ doing a rescaling. In this case the intricate singular point $R_2(0,1,0)$ has multiplicity six and to determine its geometric type we consider again the systems (45) with $a = 0$ and $c = 1$. In this case doing a blow–up we find that the behavior of the trajectories in the neighborhood of R_2 depends on the sign of the parameter b. And as $\operatorname{sign}(b) = -\operatorname{sign}(K_3)$ we obtain the following types of the singularity R_2 with conditions:

$$K_3 < 0 \ \Rightarrow\ \binom{3}{3}\, H\overset{\curvearrowright}{P}E - \overset{\curvearrowleft}{P}HH; \quad K_3 > 0 \ \Rightarrow\ \binom{3}{3}\, HH\overset{\curvearrowright}{P} - \overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}\,\overset{\curvearrowleft}{P}.$$

2) Suppose now $\mu_3 = 0$, i.e. $bc = 0$. If $\mu_4 \neq 0$ considering (46) we obtain $b \neq 0$ and this implies $c = 0$. So we get the systems $\dot{x} = x^2, \quad \dot{y} = b - x^2 + xy$ for which we have

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \quad \mu_4 = b^2x^4, \quad K_3 = -6bx^6.$$

In this case all four finite singularities of systems $(\mathbf{S}_{IV})$ have coalesced with the triple infinite one and hence the intricate singular point $R_2(0,1,0)$ of the above systems has the multiplicity seven. Doing a blow–up for the singular point $(0,0)$ of the corresponding systems

$$\dot{v} = -v^3 + bvz^2, \quad \dot{z} = vz - v^2z + bz^3$$

we show that the geometric structure of the neighborhood of $(0,0)$ depends again on the parameter b. As $\operatorname{sign}(b) = -\operatorname{sign}(K_3)$ we arrive at the configurations

$$K_3 < 0 \ \Rightarrow\ \binom{4}{3}\, E\overset{\curvearrowleft}{P}_{\curlywedge}H - H\overset{\curvearrowright}{P}_{\curlywedge}E; \quad K_3 > 0 \ \Rightarrow\ \binom{4}{3}\, \overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}_{\curlywedge}\overset{\curvearrowleft}{P} - \overset{\curvearrowright}{P}\,\overset{\curvearrowleft}{P}_{\curlywedge}\overset{\curvearrowright}{P}.$$

In order to finish the case $\widetilde{K}\neq 0$ it remains to examine the degenerate systems $(\mathbf{S}_{IV})$. Considering (46) we observe that for systems (43) with $a=0$ the condition $\mu_3=\mu_4=0$ implies $b=0$. Therefore we obtain the degenerate systems

$$\dot{x}=x(c+x),\quad \dot{y}=x(y-x)$$

possessing the invariant line $x=0$ filled with singularities. We observe that the reduced systems are linear having the infinite singular point $R_2(0,1,0)$ which is a double semi–elemental saddle–node. Moreover the reduced systems have the invariant line $x=-c$. Clearly this line coincides with the line $x=0$ of systems (43) if and only if $c=0$. This condition is governed by the invariant polynomial K_2 as for these systems we have $K_2=48c^2x^2$. So using the notations given in Section 5 we get the following configurations of infinite singularities (see FIGURE 7) for degenerate systems $(\mathbf{S}_{IV})$ in the case $\widetilde{K}\neq 0$:

$$\begin{array}{lll} K_2\neq 0 & \Rightarrow & \big(\ominus\,[|];\overline{\tbinom{0}{2}}SN_3\big) \quad QD^{\infty}_{22};\\ K_2=0 & \Rightarrow & \big(\ominus\,[|];\overline{\tbinom{0}{2}}SN_2\big) \quad QD^{\infty}_{23}.\end{array}$$

8.4.2.2. *The subcase* $\widetilde{K}=0$. Then for systems $(\mathbf{S}_{IV})$ with $h=0$ we have $g=0$ and assuming $e=0$ (doing a translation if necessary) we get the systems

$$\dot{x}=a+cx+dy,\quad \dot{y}=b+fy-x^2. \tag{47}$$

We calculate

$$\mu_0=\mu_1=0,\quad \mu_2=d^2x^2.$$

8.4.2.2.1. If $\mu_2\neq 0$ (i.e. $d\neq 0$) by Theorem 1, the singular point $R_2(0,1,0)$ is a nilpotent singular point of multiplicity $\binom{2}{3}$. Its geometrical type can be determined by doing a blow–up for $(0,0)$ for the systems

$$\dot{v}=-dz+(f-c)vz-az^2-v^3+bvz^2,\quad \dot{z}=fz^2-v^2z+bz^3$$

having R_2 at the origin of coordinates. We obtain univocally that the singularity R_2 is geometrically equivalent to $\widehat{\binom{2}{3}}\ \overset{\curvearrowright}{P}_{\curlywedge}\,\overset{\curvearrowleft}{P}-\overset{\curvearrowright}{P}_{\curlywedge}\,\overset{\curvearrowleft}{P}$.

8.4.2.2.2. Assume $\mu_2=0$. Then $d=0$ and for systems (47) calculations yield:

$$\mu_3=-c^2fx^3,\quad K_1=-cx^3,\quad K_3=6(2c-f)fx^6.$$

1) If $\mu_3\neq 0$ then $cf\neq 0$ and we may assume $c=1$ (doing a rescaling) and $b=0$. In this case the intricate singular point $R_2(0,1,0)$ has multiplicity six and to determine its geometric type we consider the systems

$$\dot{v}=(f-1)vz-az^2-v^3,\quad \dot{z}=fz^2-v^2z$$

having R_2 at the origin of coordinates. Using a blow–up, we find that the behavior of the trajectories in the neighborhood of R_2 depends on the sign of the parameter f. More exactly we obtain the following types of the singularity R_2:

$$\begin{array}{lll} f<0 & \Rightarrow & \binom{3}{3}\ \overset{\curvearrowright}{P}_{\curlywedge}EE\,\overset{\curvearrowleft}{P}_{\curlywedge}-\overset{\curvearrowright}{P}\,\overset{\curvearrowleft}{P};\\ 0<f<2 & \Rightarrow & \binom{3}{3}\,\overset{\curvearrowright}{P}_{\curlywedge}\,\overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}\,\overset{\curvearrowleft}{P}_{\curlywedge}-HH;\\ f=2 & \Rightarrow & \binom{3}{3}\,HH-\overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P};\\ f>2 & \Rightarrow & \binom{3}{3}\,H_{\curlywedge}\,\overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}\,H_{\curlywedge}-\overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}.\end{array}$$

On the other hand we observe that in the case $c=1$ we have $\mathrm{sign}\,(f)=\mathrm{sign}\,(\mu_3K_1)$ and $\mathrm{sign}\,(f(f-2))=-\mathrm{sign}\,(K_3)$. Moreover as $\mu_3\neq 0$, the condition $f=2$ is equivalent to $K_3=0$. So we get the following configurations of the infinite singularities for systems $(\mathbf{S}_{IV})$:

$$\begin{array}{lll} \mu_3K_1<0 & \Rightarrow & \binom{3}{3}\ \overset{\curvearrowright}{P}_{\curlywedge}EE\,\overset{\curvearrowleft}{P}_{\curlywedge}-\overset{\curvearrowright}{P}\,\overset{\curvearrowleft}{P};\\ \mu_3K_1>0,\ K_3>0 & \Rightarrow & \binom{3}{3}\,\overset{\curvearrowright}{P}_{\curlywedge}\,\overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}\,\overset{\curvearrowleft}{P}_{\curlywedge}-HH;\\ \mu_3K_1>0,\ K_3=0 & \Rightarrow & \binom{3}{3}\,HH-\overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P};\\ \mu_3K_1>0,\ K_3<0 & \Rightarrow & \binom{3}{3}\,H_{\curlywedge}\,\overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}\,H_{\curlywedge}-\overset{\curvearrowleft}{P}\,\overset{\curvearrowright}{P}.\end{array}$$

2) Suppose now $\mu_3 = 0$. Then for systems (47) with $d = 0$ we obtain cf=0 and we consider two subcases: $c \neq 0$ and $c = 0$. Clearly these possibilities are distinguished by the invariant polynomial K_1.

***a*)** If $K_1 \neq 0$ then $c \neq 0$, $f = 0$ and we may assume $c = 1$ due to a rescaling. So we get the systems

$$\dot{x} = a + x, \quad \dot{y} = b - x^2$$

for which $\mu_4 = (a^2 - b)x^4$. If $\mu_4 \neq 0$ all four finite singularities of the systems $(\mathbf{S}_{IV})$ have coalesced with the triple infinite one and hence the intricate singular point $R_2(0,1,0)$ of the above systems has multiplicity seven. Doing a blow–up at the singular point $(0,0)$ of the corresponding systems

$$\dot{v} = -vz - az^2 - v^3 + bvz^2, \quad \dot{z} = -v^2z + bz^3$$

we determine that the geometric structure of R_2 depends on the sign of the expression $a^2 - b$. Since sign $(a^2 - b) =$ sign (μ_4) we arrive at the configurations

$$\mu_4 < 0 \ \Rightarrow \ \binom{4}{3}\,\widehat{P}_{\curlywedge} E E \widehat{P}_{\curlywedge} - H H; \quad \mu_4 > 0 \ \Rightarrow \ \binom{4}{3}\,\widehat{P}_{\curlywedge}\widehat{P}\,\widehat{P}\,\widehat{P}_{\curlywedge} - \widehat{P}\,\widehat{P}.$$

In the case $\mu_4 = 0$ we obtain $b = a^2$ and this leads to the degenerate systems

$$\dot{x} = a + x, \quad \dot{y} = (a + x)(a - x)$$

possessing the invariant line $x = -a$ filled with singularities. It can be easily determined that the reduced linear systems possess one nilpotent singular point of multiplicity three at infinity: a finite singular point has coalesced with two infinite ones. In our notations for the above degenerate systems we obtain the configuration $\big(\ominus\, [||];\ \widehat{\tbinom{1}{2}\ E} - H\big)$. On the other hand we observe that the phase portrait around infinity is topologically equivalent to the portrait QD_{24}^{∞} (see FIGURE 7).

***b*)** Assume $K_1 = 0$. In this case we have $c = 0$ and we obtain the systems

$$\dot{x} = a, \quad \dot{y} = b + fy - x^2 \tag{48}$$

for which we calculate $\mu_4 = a^2x^4$. If $\mu_4 \neq 0$ then $a \neq 0$ and we may assume $a = 1$ due to a rescaling. Similarly to the previous case, we show that the intricate singular point $R_2(0,1,0)$ of the above systems has multiplicity seven. Doing a blow–up at $(0,0)$ of the respective systems

$$\dot{v} = fvz - z^2 - v^3 + bvz^2, \quad \dot{z} = fz^2 - v^2z + bz^3 \tag{49}$$

we find that the geometric structure of R_2 depends on the parameter f. More precisely we have $\binom{4}{3}\,\widehat{P}_{\curlywedge} E H_{\curlywedge} - \widehat{P}$ if $f \neq 0$ and $\binom{4}{3}\,\widehat{P}_{\curlywedge}\widehat{P} - \widehat{P}\,\widehat{P}_{\curlywedge}$ if $f = 0$. On the other hand for systems (48) we have $K_3 = -6f^2x^6$. So we obtain the following configurations:

$$K_3 \neq 0 \ \Rightarrow \ \binom{4}{3}\,\widehat{P}_{\curlywedge} E H_{\curlywedge} - \widehat{P}; \quad K_3 = 0 \ \Rightarrow \ \binom{4}{3}\,\widehat{P}_{\curlywedge}\widehat{P} - \widehat{P}\,\widehat{P}_{\curlywedge}.$$

In the case $\mu_4 = 0$ we have $a = 0$ and we get the degenerate systems

$$\dot{x} = 0, \quad \dot{y} = b + fy - x^2 \tag{50}$$

possessing the invariant conic $b + fy - x^2 = 0$ filled with singularities. Clearly the type of this conic depends on the parameters b and f. More exactly, if $f \neq 0$ we have a parabola. In the case $f = 0$ this conic splits in two parallel lines, which are real if $b > 0$, complex if $b < 0$ and a double real line if $b = 0$.

On the other hand for systems (50) we have $K_3 = -6f^2x^6$ and in the case $f = 0$ (i.e. $K_3 = 0$) we obtain $L_3 = 4bx^4$. So clearly the conditions above are governed by these two invariant polynomials. We observe also that for the reduced constant system the infinite singular point $(0,1,0)$ is a star node.

Thus for degenerate systems $(\mathbf{S}_{IV})$ in the case $\widetilde{K} = K_1 = 0$ we get the following configurations of infinite singularities and the corresponding phase portraits around the infinity (see FIGURE 7):

$$\begin{array}{lcll} K_3 \neq 0 & \Rightarrow & \big(\ominus\,[\cup];\, N^*\big) & QD^{\infty}_{25}; \\ K_3 = 0,\ L_3 < 0 & \Rightarrow & \big(\ominus\,[\|^c];\, N^*\big) & QD^{\infty}_{30}; \\ K_3 = 0,\ L_3 > 0 & \Rightarrow & \big(\ominus\,[\|\|];\, N^*\big) & QD^{\infty}_{26}; \\ K_3 = 0,\ L_3 = 0 & \Rightarrow & \big(\ominus\,[|_2];\, N^*\big) & QD^{\infty}_{27}. \end{array}$$

Since all the cases are examined, the Main Theorem is proved for the family of systems $(\mathbf{S}_{IV})$.

8.5. **The family of systems $(\mathbf{S}_V)$.** For these systems we have $C_2 = 0$ (this implies $\eta = \widetilde{M} = 0$) and we may consider $e = f = 0$ due to a translation. So in what follows we shall consider the systems

$$\dot{x} = a + cx + dy + x^2, \quad \dot{y} = b + xy, \tag{51}$$

for which we have $\mu_0 = 0$ and $\mu_1 = dx$. The line at infinity of systems (51) is filled up with singularities, and removing the degeneracy in the systems obtained on the local charts at infinity we get the following two systems

$$\begin{cases} \dot{u} = cu - bz + du^2 + auz, \\ \dot{z} = 1 + cz + duz + az^2; \end{cases}$$

$$\begin{cases} \dot{v} = -d - cv - az + bvz, \\ \dot{z} = v + bz^2, \end{cases}$$

which we call reduced systems. As we could observe, the first systems could not have singular points on the line $z = 0$, whereas the second ones could possess such a point if $d = 0$. So in what follows we concentrate our attention on the quadratic systems

$$\dot{v} = -d - cv - az + bvz, \quad \dot{z} = v + bz^2. \tag{52}$$

8.5.1. *The case* $\boldsymbol{\mu_1 \neq 0}$. Then $d \neq 0$ and systems (52) do not have any singular point on the line $z = 0$. This means that after removal of the degeneracy, similarly to what we did above, the systems (51) do not have infinite singularities. According to our notations (see Section 5) we have the configuration $[\infty; \emptyset]$.

8.5.2. *The case* $\boldsymbol{\mu_1 = 0,\ \mu_2 \neq 0}$. This implies $d = 0$ and then for systems (51) we have $\mu_2 = ax^2 \neq 0$. On the other hand, for the singular point $(0, 0)$ the systems (52) have the following matrix of their linearization at $(0, 0)$ and the corresponding eigenvalues:

$$\begin{pmatrix} -c & -a \\ 1 & 0 \end{pmatrix}, \qquad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4a}}{2},$$

and then $\lambda_1\lambda_2 = a \neq 0$. Therefore this singular point is a saddle if $a < 0$; if $a > 0$ and $c^2 - 4a > 0$ it is a generic node with both directions transversal to the line $z = 0$; if $a > 0$ and $c^2 - 4a = 0$ it is a one direction node; and if $c^2 - 4a < 0$ it is either a focus or a center.

On the other hand for systems (51) we have $K_2 = 48(c^2 - 4a)x^2$ and hence we obtain

$$\operatorname{sign}(a) = \operatorname{sign}(\mu_2), \quad \operatorname{sign}(c^2 - 4a) = \operatorname{sign}(K_2).$$

So using the notations given in Section 5 we obtain the following configurations of infinite singularities for the systems $(\mathbf{S}_V)$ in the case $\mu_1 = 0$ and $\mu_2 \neq 0$:

$$\begin{array}{lcl} \mu_2 < 0 & \Rightarrow & [\infty;\, S]; \\ \mu_2 > 0,\ K_2 < 0 & \Rightarrow & [\infty;\, C]; \\ \mu_2 > 0,\ K_2 > 0 & \Rightarrow & [\infty;\, N]; \\ \mu_2 > 0,\ K_2 = 0 & \Rightarrow & [\infty;\, N^d]. \end{array}$$

8.5.3. *The case* $\boldsymbol{\mu_2 = 0}$. Then $a = 0$ and systems (51) become

$$\dot{x} = cx + x^2, \quad \dot{y} = b + xy, \tag{53}$$

with the respective reduced systems at infinity

$$\dot{v} = -cv + bvz, \quad \dot{z} = v + bz^2.$$

Clearly the singular point $(0, 0)$ of the last systems is a semi–elemental double saddle–node if $c \neq 0$ and it is a triple nilpotent point (which is an elliptic–saddle) if $c = 0$ and $b \neq 0$. As for systems (53) we have

$$\mu_1 = \mu_2 = 0, \quad \mu_3 = -bcx^3, \quad \mu_4 = bx^3(bx + c^2y),$$

we arrive at the following configurations of infinite singularities for systems (53):

$$\mu_3 \neq 0 \;\Rightarrow\; \left[\infty; \overline{\binom{2}{0}} SN\right]; \quad \mu_3 = 0,\, \mu_4 \neq 0 \;\Rightarrow\; \left[\infty; \widehat{\binom{3}{0}}\, ES\right].$$

Assuming $\mu_4 = 0$ (i.e. $b = 0$) we get the degenerate systems

$$\dot{x} = x(c + x), \quad \dot{y} = xy, \tag{54}$$

possessing the invariant line $x = 0$ filled with singularities. We observe that the phase portraits on the whole Poincaré disk for the above systems are described in Section 5. More exactly in Figure 5 are indicated the phase portraits of systems (54), which correspond to *(c)* if $c \neq 0$ and to *(d)* if $c = 0$. In Section 5 the notations for both the finite part and at infinity are described in detail. Using the notations for infinite singularities and considering that for systems (54) we have $K_2 = 48c^2x^2$, we obtain the following configurations of singularities for degenerate systems $(\mathbf{S}_V)$ and their respective phase portraits around infinity (see Figure 7):

$$\begin{array}{lll} K_2 \neq 0 & \Rightarrow \big[\infty; \;\; (\ominus\,[|]; \emptyset_3)\big] & QD_{28}^{\infty}; \\ K_2 = 0 & \Rightarrow \big[\infty; \;\; (\ominus\,[|]; \emptyset_2)\big] & QD_{29}^{\infty}. \end{array}$$

Thus all the families of the quadratic systems given by Lemma 1 are examined and hence the Main Theorem is completely proved. ∎

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