SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 7, No. 1, pp. 1–17

J_2 Effect and Elliptic Inclined Periodic Orbits in the Collision Restricted Three-Body Problem*

E. Barrabés†, J. M. Cors‡, C. Pinyol§, and J. Soler†

Abstract. The existence of a new class of inclined periodic orbits of the collision restricted three-body problem is shown. The symmetric periodic solutions found are perturbations of elliptic Kepler orbits, and they exist only for special values of the inclination and are related to the motion of a satellite around an oblate planet.

Key words. collision restricted three-body problem, periodic orbits, symmetric orbits, critical inclination, continuation method

AMS subject classifications. 70F07, 70F15, 70H09, 70H12, 70M20

DOI. 10.1137/070683854

1. Introduction.

The launch of Sputnik in October 1957 opened the space age. The use of circular, elliptic, and synchronous orbits, combined with dynamical effects due to the Earth's equatorial bulge, gives rise to an array of orbits with specific properties to support various mission constraints. One example is the Molniya orbit, a highly elliptic 12-hour-period orbit the former USSR originally designed to observe the northern hemisphere. The orbital plane makes an angle of about 63 degrees with the equatorial plane of the Earth, and this is the only value that prevents the orbit itself from rotating slowly within its plane and around the focus.

In what follows we will introduce briefly a few common notions of orbital dynamics, together with the current terminology (sometimes a few centuries old), and state the aim of the paper.

The position of a body on a Keplerian elliptic orbit can be completely characterized by six parameters. One such set of parameters are the classical orbital elements. As the orbital plane is fixed in any inertial frame and passes through the origin, one should first give the position of this plane. In a Cartesian frame with axes xyz, this is given by the inclination i with respect to the xy-plane and the angle Ω from the positive x-axis to the intersection of the orbital plane with the xy-plane. In the classical terminology of astronomy this line is

*Received by the editors February 27, 2007; accepted for publication (in revised form) by J. Meiss May 26, 2007; published electronically January 16, 2008.

http://www.siam.org/journals/siads/7-1/68385.html

†Departament d'Informàtica i Matemàtica Aplicada, Universitat de Girona, Girona 17003, Spain (barrabes@ima.udg.edu, jaume.soler@ima.udg.edu). The first author was supported by grant MTM2006-05849/Consolider (including a FEDER contribution). The last author was supported by grant MTM2005-07660-C02-02.

‡Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya, Barcelona 08242, Spain (cors@epsem.upc.edu). This author was supported by MCYT grant MTM 2005-06098-C02-01 and by CIRIT grant 2001SGR00173.

§Departament d'Economia i Història Econòmica, Universitat Autònoma de Barcelona, Barcelona 08193, Spain (conxita.pinyol@uab.cat). This author was supported by grant SEJ2006/00712ECO.

known as the *line of nodes* (the nodes of the orbit being the two points of intersection with the xy-plane, and the ascending node that in which the body crosses from $z < 0$ to $z > 0$), and Ω is called the *longitude of the ascending node*.

Then we need the position of the ellipse on its plane. One focus is at the origin, and the line joining the pericenter and the apocenter (classically the *line of apsides*) forms an angle ω with the line of nodes which gives the position of the ellipse. Usually the half-line from the origin to the pericenter and the half-line from the origin to the ascending node are taken, and then we say that ω is the *longitude of the pericenter*.

The size and shape of the ellipse are given by the semimajor axis a (directly related to the energy) and the eccentricity e (related to the energy and the angular momentum).

Finally, the position of the body along the orbit is given either by the angle f (true anomaly) pericenter–origin–body, or by some other related angles such as E (eccentric anomaly) or M (mean anomaly). The three anomalies (a name already used in Greek astronomy) are related among themselves by the geometry and the dynamics of Keplerian motion. The position is actually a function of time, and the origin of time is called the *epoch* (see, for example, [2]).

Of course, Ω, ω, and f are not well defined for circular or zero-inclination orbits, a problem that can be solved in a variety of ways which go back to Laplace in the case of the classical elements and to Poincaré for the Hamiltonian formulation.

Thus the position and velocity of a point in space are completely characterized by the six orbital elements, which are constant (except f, or whichever anomaly is used) for a Keplerian orbit. This rather strange system of coordinates in phase space is useful because in most cases the non-Keplerian motion of a body subject to perturbations can be seen as a fast motion along a Keplerian orbit with slowly varying elements.

A set of variables closely related to the orbital elements are the *Delaunay elements*, which could be considered as the canonical (in the sense of Hamiltonian) version of the classical elements and will be defined in section 2.

Any small perturbation of the Keplerian motion has two kinds of effects on the motion: periodic and secular. An element subject to periodic perturbations simply oscillates around its central unperturbed value, while a secular perturbation is a steady, linear increase or decrease of its value. Of course, this is true only in a first order approximation, and it is a qualitative description, because a first order approximation is valid only on a finite interval of time, and the very concept of periodicity does not make sense. As for the secular effects we must remember that one of the major problems of the classical dynamical astronomy of the nineteenth century was the distinction between true secular effects and linearization of periodic effects of very long period, and that the whole matter has been settled only by the KAM theory.

In this sense, it is a result of classical astronomy that a, e, and i are subject to only periodic effects, while Ω, ω, and M display periodic and secular effects. In short, a perturbed Keplerian conic can be thought of, roughly speaking, as a conic which rotates slowly on its plane while the plane itself rotates around the z-axis.

The most common perturbations of the potential in celestial mechanics are due either to the presence of a third body or to lack of sphericity of the bodies. The latter can be dealt with by expanding the potential in spherical harmonics, so that if the body has axial symmetry, the potential can be seen as that of an inverse square distance central force plus other terms:

$$V(r,\phi) = -Gm\frac{r_{eq}}{r}\left(1 - \sum_{k=2}^{\infty} J_k \left(\frac{r_{eq}}{r}\right)^k P_k(\cos\phi)\right),$$

where G is the gravitational constant, m is the mass of the body, r_{eq} is its equatorial radius, (r,θ,ϕ) are spherical coordinates (θ does not appear because of the axial symmetry), P_k is the kth Legendre polynomial, and J_k are the coefficients defining the expansion (see, for example, [10]).

The third body perturbation is quite a different matter because the equations of motion must be supplemented with those of the new body. In the restricted three-body problem, it is assumed that one of the bodies is so small that it does not affect the motion of the other two (the *primaries*), and then we usually have a Keplerian motion plus a nonautonomous perturbation. If, however, we consider the potential in a region far away from the primaries and normalize this distance to unity, the velocities of the primaries are very high and heuristically we can somehow average their mass along their whole orbits, so that dynamically we are again in the case of a nonspherical potential.

For Earth-orbit design, the main perturbation is that of the J_2 term in the expansion of the potential of an oblate ellipsoid. This term perturbs the orbit in the sense explained above, resulting in a precession both of the line of nodes and of the pericenter. It is apparent (see, for example, pages 503–504 of [2]) that there exists a critical inclination angle, $i \simeq 63°$, such that the perigee is fixed in the first approximation because its secular terms are of opposite sign for inclinations above or under the critical value, irrespective of the eccentricity. The case of a prolate ellipsoid, though apparently not frequent in astronomy, could be treated in the same way, the only difference being that the J_2 term has the opposite sign, so that all the precessions are in the opposite direction.

The existence of a class of inclined periodic solutions of the circular three-body problem was shown by Jefferys in [5]. He showed the existence of families of elliptic orbits with inclination close to critical for any value of the eccentricity. His proof rests on a mirror theorem: in the rotating coordinate system of the restricted three-body problem any trajectory that hits twice perpendicularly a certain plane is a periodic solution. For an elliptic orbit, perpendicular crossing means that the body is at either the pericenter or the apocenter and the line of apsides lies on the mirror plane; for this situation to happen twice in time it is sufficient that the line of apsides does not have a secular motion, so the inclination must be near critical. Of course a precession of the line of nodes does exist, but it is hidden, as it were, in the rotating frame. It must be borne in mind that in celestial mechanics periodic usually means periodic in some rotating frame, and thus periodic or quasi-periodic in the inertial frame depending on whether the angle advanced by the rotating frame in a period is a rational multiple of π or not. The method used is the continuation method developed by Poincaré (see [8]), which is one of the most frequently used methods for proving the existence of periodic orbits.

The case dealt with in this paper is different from Jefferys's because the primaries move on an elliptic collision orbit along the z-axis. Heuristically speaking, however, it can be expected that far away from the primaries the potential will be similar to that of a very eccentric prolate ellipsoid, so that a J_2 effect, with its critical inclination, will exist. We show the existence of periodic solutions of Jefferys type: large semiaxis compared to the that of the primaries, arbitrary eccentricity, and inclination close to critical.

The problem can be seen as a perturbed Kepler problem, where the small parameter is the semimajor axis of the primaries' orbit after rescaling. The perturbed problem is degenerate due to the fast motion of the primaries, and the equations are no longer analytic when the parameter equals zero, which precludes the use of a standard implicit function theorem. We overcome the difficulty by using Arenstorf's theorem, where weaker assumptions of differentiability are needed (see [1]). A planar configuration of this problem is studied in [6].

In our case, the problem has a rotational symmetry around the z-axis (which contains the colliding primaries). This symmetry would be lost if we considered elliptic noncollision orbits for the primaries. See [3] and [4], where the elliptic restricted three-body problem is considered. In those papers, the periodic orbits are perturbations of the circular solutions of the Kepler problem having large radii on a plane perpendicular to that of the primaries. Periodic orbits in the spatial elliptic restricted three-body problem are also studied using double averaging in [7].

The paper is organized as follows. Section 2 describes the general setting of the collision restricted three-body problem. Section 3 shows how its solutions can be approximated through successive corrections to Keplerian motion. Section 4 deals with the continuation problem. The main result is the existence of quasi-elliptic orbits for discrete values of the semimajor axis of the primaries, with arbitrary eccentricity and inclination close to critical. A number of technical computations are presented in section 5.

Figure 1 shows one of the orbits predicted by the main theorem, numerically computed with initial values $r_0 = 0.621114405$, $\phi_0 = 1.116457610$, $\theta_0 = 0$, $p_r = 0$, $p_\phi = 0$, $p_\theta = 0.4098780306$, and $\mu = 30^{-2/3}$. The equations of motion and the first variational equations were numerically integrated with a Runge–Kutta 7-8 routine, and the equations defining the initial conditions were solved with a Newton method starting with the Keplerian orbit with $a = 1$, $e = 0.4$, and $\cos i = 1/\sqrt{5}$ (critical inclination).

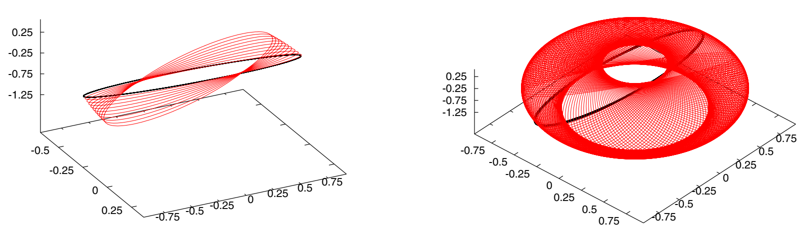

Figure 1. *Example of a quasi-periodic orbit in a Cartesian frame. The orbit is followed during* 6 *(plot on the left) and* 150 *(plot on the right) times the period of the primaries. The Keplerian orbit for* $\mu = 0$ *is plotted (black line). The primaries move along the vertical line passing through the origin.*

2. The collision restricted three-body problem.

The collision restricted three-body problem describes the motion of a massless particle under the attraction of two primaries with equal masses, $m_1 = m_2 = 1/2$, moving on a collision elliptic orbit. In order to avoid a triple collision, we consider that the third body is far from the primaries compared to the distance between them. This fact can be introduced in the equations of motion by making the primaries very close to each other and looking for solutions of the massless particle at distance of order unity to the primaries.

Let μ be a small parameter. The distance between both primaries is given by

$$\rho = \mu(1 - \cos E_p(t)), \tag{2.1}$$

where $E_p = E_p(t)$ is the eccentric anomaly of m_1 and it is related to its mean anomaly ℓ_p through Kepler's equation

$$E_p - \sin E_p = \ell_p, \tag{2.2}$$

where $\ell_p = \mu^{-3/2}t$. The period of the motion of the primaries is $T_p = 2\pi\mu^{3/2}$, so that $E_p = k\pi$ when $t = \pi k \mu^{3/2}$.

Equation (2.2) is a particular case (for $e = 1$) of Kepler's equation $\ell = E - e \sin E$, where e is the eccentricity, ℓ is the mean anomaly (real time measured in such units that the period is 2π), and E is the eccentric anomaly, which is related to the angular position f of the body on the orbit (from the pericenter) through $\tan f/2 = \sqrt{(1+e)/(1-e)}\tan E/2$. The latter equation results from the geometry of elliptic orbits, and Kepler's equation is just the mathematical expression of the law of areas, i.e., the conservation of the angular momentum (see [10]).

We consider a fixed coordinate system (q_1, q_2, q_3) (see Figure 2) with origin at the center of mass of m_1 and m_2 in such a way that the primaries move along the q_3-axis. Their positions are given by $\mathbf{r}_1 = (0, 0, \frac{\mu}{2}(1 - \cos E_p))$ and $\mathbf{r}_2 = -\mathbf{r}_1$. Let $\mathbf{q} = (q_1, q_2, q_3)$ and $\mathbf{p} = (p_1, p_2, p_3) = (\dot q_1, \dot q_2, \dot q_3)$ be the position and momentum of the infinitesimal body m_3. The problem of describing its motion is known as the *three-dimensional collision restricted three-body problem*.

The equations of motion for the infinitesimal body can be written as a nonautonomous Hamiltonian system depending on the parameter μ as

$$\dot q_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot p_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad i = 1, 2, 3,$$

where

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^3)^2 - \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{2.3}$$

and R_1 and R_2 are given by

$$\begin{aligned} R_1^2 &= q_1^2 + q_2^2 + \left(q_3 - \tfrac{\mu}{2}(1 - \cos E_p)\right)^2, \\ R_2^2 &= q_1^2 + q_2^2 + \left(q_3 + \tfrac{\mu}{2}(1 - \cos E_p)\right)^2. \end{aligned}$$

Let us introduce spherical coordinates (r, ϕ, θ), and (p_r, p_ϕ, p_θ) by means of the canonical change

$$\begin{aligned} q_1 &= r\cos\phi\cos\theta, & p_1 &= p_r\cos\phi\cos\theta - \tfrac{p_\phi}{r}\sin\phi\cos\theta - \tfrac{p_\theta}{r\cos\phi}\sin\theta, \\ q_2 &= r\cos\phi\sin\theta, & p_2 &= p_r\cos\phi\sin\theta - \tfrac{p_\phi}{r}\sin\phi\sin\theta + \tfrac{p_\theta}{r\cos\phi}\cos\theta, \\ q_3 &= r\sin\phi, & p_3 &= p_r\sin\phi + \tfrac{p_\phi}{r}\cos\phi. \end{aligned}$$

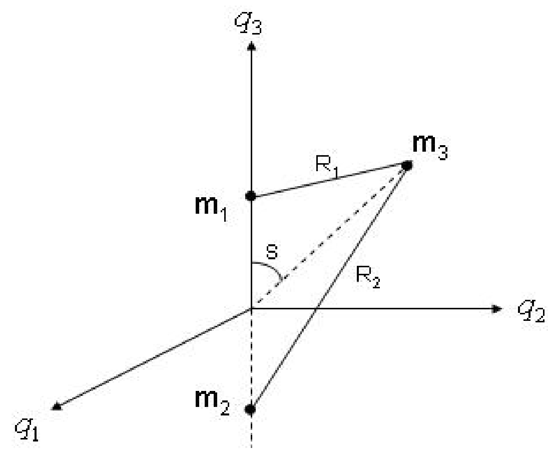

Figure 2. *Collision restricted three-body problem.*

In the new variables, the Hamiltonian (2.3) becomes

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2\cos^2\phi}\right) - \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{2.4}$$

with R_1 and R_2 given by

$$\begin{array}{rcl} R_1^2 & = & r^2 + \left(\frac{\mu}{2}\right)^2(1-\cos E_p)^2 - r\mu(1-\cos E_p)\sin\phi, \\ R_2^2 & = & r^2 + \left(\frac{\mu}{2}\right)^2(1-\cos E_p)^2 + r\mu(1-\cos E_p)\sin\phi. \end{array} \tag{2.5}$$

Notice that E_p as given by (2.2) is a function of time t and μ, which is not defined for $\mu = 0$. So, neither the Hamiltonian (2.3) nor (2.4) is defined.

The equations of motion for the infinitesimal mass in spherical coordinates are

$$\begin{array}{rclrcl} \dot{r} & = & p_r, & \dot{p_r} & = & -\frac{\partial\mathcal{H}}{\partial r}, \\ \dot{\theta} & = & \frac{p_\theta}{r^2\cos^2\phi}, & \dot{p_\theta} & = & 0, \\ \dot{\phi} & = & \frac{p_\phi}{r^2}, & \dot{p_\phi} & = & -\frac{\partial\mathcal{H}}{\partial\phi}. \end{array} \tag{2.6}$$

Since R_1 and R_2 do not depend on θ, $\dot{p_\theta} = 0$ and the angular momentum $p_\theta = \Theta$ is constant. Thus, it can be calculated from the initial conditions, and the equation for θ can be decoupled from the other equations. In this way, we can consider the system of equations

$$\begin{array}{rclrcl} \dot{r} & = & p_r, & \dot{p_r} & = & -\frac{\partial\mathcal{H}}{\partial r}, \\ \dot{\phi} & = & \frac{p_\phi}{r^2}, & \dot{p_\phi} & = & -\frac{\partial\mathcal{H}}{\partial\phi}. \end{array} \tag{2.7}$$

Once r and ϕ are obtained, we will get θ from its equation in (2.6).

From now on, *reduced problem* means the problem given by (2.7), and *complete problem* means the whole set of equations (2.6). Our aim is to find periodic solutions of the reduced problem that will be periodic or quasi-periodic solutions of the complete problem.

It is easy to see that the equations of the reduced problem are invariant by the symmetry

$$S : (t, r, \phi, p_r, p_\phi, E_p) \quad \longrightarrow \quad (-t, r, \phi, -p_r, -p_\phi, -E_p),$$

so that, if

$$\gamma(t) = (r(t), \phi(t), p_r(t), p_\phi(t), E_p(t))$$

is a particular solution of (2.7), then so is

$$(r(-t), \phi(-t), -p_r(-t), -p_\phi(-t), -E_p(-t)),$$

and we have the following well-known result.

Proposition 2.1. *Let $\gamma(t) = (r(t), \phi(t), p_r(t), p_\phi(t), E_p(t))$ be a solution of the reduced problem given by (2.7). If $\gamma(t)$ satisfies $(p_r(0), p_\phi(0)) = (0, 0)$, $(p_r(T/2), p_\phi(T/2)) = (0, 0)$, and $E_p(T/2) = k\pi$, then $\gamma(t)$ is a periodic solution of period T.*

In order to find elliptic orbits we will introduce Delaunay variables (l, g, h) and (L, G, H), where

$$L = \sqrt{a}, \qquad H = G\cos i, \qquad G = \sqrt{a(1-e^2)},$$

a is the semimajor axis of the infinitesimal mass, G its angular momentum, i the inclination of its orbital plane with respect to the q_1q_2 reference plane, l the mean anomaly, g the argument of the pericenter measured from the ascending node, and h the longitude of the ascending node (see, for example, [9]).

We will use the symmetry conditions stated in Proposition 2.1 to obtain periodic solutions of the reduced problem. These conditions can be expressed in Delaunay variables as

$$l(t) = 0 \mod \pi, \qquad g(t) = \pi/2 \mod \pi \tag{2.8}$$

for epochs $t = 0$ and $t = T/2$, where $T = 2k\pi\mu^{3/2}$ in order to satisfy $E_p(T/2) = k\pi$.

3. Approximate solutions. In this section we will show how those solutions of the three-dimensional collision elliptic restricted three-body problem in which the infinitesimal body keeps moving far away from the primaries can be approximated through successive corrections to the Keplerian motion. In section 4, we will use these approximations to continue some elliptic solutions of the Kepler problem to the case $\mu \neq 0$.

As the Hamiltonian (2.3) is not defined when $\mu = 0$, instead of expansions in power series (which are no longer available) we use asymptotic series. Using expressions (2.5) and (2.1), we can write

$$R_1 = r\sqrt{1 + \frac{\rho^2}{4r^2} - \frac{\rho}{r}\cos\mathcal{S}},$$

where $\mathcal{S} = \frac{\pi}{2} - \phi$ is the angle between the position vectors of m_1 and m_3 (see Figure 2). We assume that the distance from the origin to the primaries $(\mu/2)$ is small compared to the distance from the origin to the infinitesimal body, so that $\rho \ll r$ and we can expand R_1^{-1} as a power series in $\frac{\rho}{2r}$ by using Legendre polynomials. Then

$$\frac{1}{R_1} = \frac{1}{r}\sum_{j=0}^{\infty} P_j(\cos\mathcal{S})\left(\frac{\rho}{2r}\right)^j = \frac{1}{r}\left[1+\sum_{j=1}^{\infty}\mu^j P_j(\cos\mathcal{S})\left(\frac{1-\cos E_p}{2r}\right)^j\right],$$

where $P_j(\cos\mathcal{S})$ is the jth Legendre polynomial. Expanding R_2^{-1} in a similar way, the Hamiltonian (2.4) becomes

$$\mathcal{H}(\mathbf{q},\mathbf{p},t,\mu) = \mathcal{H}_0(\mathbf{q},\mathbf{p}) + \mu^2\mathcal{H}_1(\mathbf{q},\mathbf{p},t,\mu) + \mu^4\mathcal{H}_R(\mathbf{q},\mathbf{p},t,\mu), \tag{3.1}$$

where

$$\mathcal{H}_0(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2\cos^2\phi}\right) - \frac{1}{r},$$

$$\mathcal{H}_1(\mathbf{q},\mathbf{p},t,\mu) = \frac{-1}{r}P_2(\cos S)\left(\frac{1-\cos E_p}{2r}\right)^2 = \frac{(1-\cos E_p)^2}{8r^3}\left(1-3\cos^2\mathcal{S}\right),$$

and

$$\mathcal{H}_R(\mathbf{q},\mathbf{p},t,\mu) = \frac{-1}{r}\sum_{k=2}^{\infty}\mu^{2(k-2)}P_{2k}(\cos\mathcal{S})\left(\frac{1-\cos E_p}{2r}\right)^{2k}.$$

The dependence on (t,μ) comes from the eccentric anomaly $E_p = E_p(t,\mu)$ given by (2.2). Notice that if $r \geq \delta$ for some fixed $\delta > 0$, then $\mathcal{H}_1(\mathbf{q},\mathbf{p},t,\mu)$ and $\mathcal{H}_R(\mathbf{q},\mathbf{p},t,\mu)$ are bounded. Thus, $\mu^2\mathcal{H}_1$ and $\mu^4\mathcal{H}_R$ are continuous at $\mu = 0$, although $\mathcal{H}_1$ and $\mathcal{H}_R$ are not so. This is the reason why expansions as power series in μ cannot be used.

Let us denote $\mathbf{z} = (l,g,h,L,G,H)$. Applying the corresponding symplectic change of variables, Hamiltonian (3.1) becomes

$$\mathcal{H}(\mathbf{z},t,\mu) = \mathcal{H}_0(\mathbf{z}) + \mu^2\mathcal{H}_1(\mathbf{z},t,\mu) + \mu^4\mathcal{H}_R(\mathbf{z},t,\mu), \tag{3.2}$$

where

$$\mathcal{H}_0(\mathbf{z}) = -\frac{1}{2L^2}, \tag{3.3}$$

$$\mathcal{H}_1(\mathbf{z},t,\mu) = \frac{(1-\cos E_p)^2}{8r^3}\left[1-3\left(1-\frac{H^2}{G^2}\right)\sin^2(g+f)\right]. \tag{3.4}$$

In (3.4) we have used the true anomaly f of the motion of the infinitesimal mass in order to write $q_3 = r\sin i\sin(f+g)$ and

$$\cos^2(\mathcal{S}) = \frac{q_3^2}{r^2} = \sin^2 i\sin^2(f+g) = \left(1-\frac{H^2}{G^2}\right)\sin^2(g+f).$$

Observe that $\mathcal{H}_0$ is the Hamiltonian of the Kepler problem and that, despite that Hamiltonian (3.2) is not defined for $\mu = 0$, the limit when $\mu \to 0$ exists and

$$\lim_{\mu\to 0}\mathcal{H}(\mathbf{z},t,\mu) = \mathcal{H}_0(\mathbf{z}).$$

Therefore, the equations of motion can be written as

$$\dot{\mathbf{z}} = \mathcal{F}(\mathbf{z}, t, \mu), \tag{3.5}$$

where $\mathcal{F} = J \cdot \nabla\mathcal{H}$, $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, and I is the identity matrix of dimension 3×3. Using (3.2), the vector field $\mathcal{F}$ is given by

$$\mathcal{F}(\mathbf{z}, t, \mu) = \mathcal{F}_0(\mathbf{z}) + \mu^2\mathcal{F}_1(\mathbf{z}, t, \mu) + \mu^4\mathcal{F}_R(\mathbf{z}, t, \mu),$$

where

$$\begin{aligned} \mathcal{F}_0(\mathbf{z}) &= (L^{-3}, 0, 0, 0, 0, 0)^t, \\ \mathcal{F}_1(\mathbf{z}, t, \mu) &= J \cdot \nabla\mathcal{H}_1, \\ \mathcal{F}_R(\mathbf{z}, t, \mu) &= J \cdot \nabla\mathcal{H}_R, \end{aligned}$$

and $\mathcal{H}_1$ and $\mathcal{H}_R$ are the terms in (3.2).

The next lemma shows that the solutions of (3.5) can be written as the solutions of the Kepler problem plus terms of order μ^2, and the same is true of its partial derivatives with respect to the initial conditions.

Lemma 3.1. *Let $\mathbf{z}_0$ be an initial condition and $\mathbf{z}^{(0)}(t, \mathbf{z}_0)$ a solution of*

$$\dot{\mathbf{z}} = \mathcal{F}_0(\mathbf{z})$$

with $\mathbf{z}^{(0)}(0, \mathbf{z}_0) = \mathbf{z}_0$ such that it remains bounded and bounded away from the singularities of $\mathcal{F}(\mathbf{z}, t, \mu)$. Let $\mathbf{z}(t, \mathbf{z}_0, \mu)$ be a solution of (3.5) with the same initial condition $\mathbf{z}_0$. Then we can write

$$\mathbf{z}(t, \mathbf{z}_0, \mu) = \mathbf{z}^{(0)}(t, \mathbf{z}_0) + \mu^2\mathbf{z}^{(1)}(t, \mathbf{z}_0, \mu) + \mathbf{z}_R(t, \mathbf{z}_0, \mu),$$

where $\mathbf{z}^{(1)}(t, \mathbf{z}_0, \mu)$ is the solution of

$$\dot{\mathbf{z}} = \mathcal{F}_1(\mathbf{z}^{(0)}(t, \mathbf{z}_0), t, \mu) + D\mathcal{F}_0(\mathbf{z}^{(0)}(t, \mathbf{z}_0))\,\mathbf{z}$$

with initial condition $\mathbf{z}^{(1)}(0, \mathbf{z}_0, \mu) = 0$, and $D\mathcal{F}$ is the matrix whose entries are the partial derivatives of $\mathcal{F}$ with respect to the $\mathbf{z}$ variables. Furthermore, $\mathbf{z}_R(t, \mathbf{z}_0, \mu)$ and $D_{\mathbf{z}_0}\mathbf{z}_R(t, \mathbf{z}_0, \mu)$ are $\mathcal{O}(\mu^4)$ in a finite interval of time.

These results can be obtained by using Taylor's expansions and Gronwall's inequality (see [4]). They are also valid for any initial conditions in a compact neighborhood of $\mathbf{z}_0$ satisfying the hypothesis of the lemma.

4. Continuation of symmetric periodic solutions.

In this section we use the results of section 3 to show the existence of symmetric periodic solutions of the reduced problem.

Let us start by considering the Kepler problem given by Hamiltonian (3.3), whose solution with initial condition $\mathbf{z}_0$ is

$$\mathbf{z}^{(0)}(t, \mathbf{z}_0) = (l_0 + L_0^{-3}t, g_0, h_0, L_0, G_0, H_0).$$

Clearly, the orbit with initial conditions $\mathbf{z}_0^* = (0, \pi/2, h_0^*, 1, G_0^*, H_0^*)$ is symmetric and periodic of period $T = 2\pi$. We want to find periodic symmetric solutions close to $\mathbf{z}^{(0)}(t, \mathbf{z}_0^*)$.

From Lemma 3.1, any solution of the reduced problem can be written as the solution of the Kepler problem plus a perturbation, provided that $\mathbf{z}^{(0)}(t, \mathbf{z}_0^*)$ remains bounded and bounded away from the singularities. To ensure that $\mathbf{z}_0^*$ satisfies these conditions, it is sufficient that $G_0^* > 0$; that is, the eccentricity is less than 1. Also, notice that we are dealing with elliptic orbits, so $G_0^* < 1$.

Thus, we will look for initial conditions $\mathbf{z}_0 = (0, \pi/2, h_0, L_0, G_0, H_0)$ in a neighborhood of a fixed $\mathbf{z}_0^*$ with $0 < G_0^* < 1$, in such a way that the solution $\mathbf{z}(t, \mathbf{z}_0, \mu)$ of (3.5), with $\mu \neq 0$ small enough, is a symmetric periodic orbit of the reduced problem.

For a fixed $\mathbf{z}_0^*$, let $\mathcal{D}$ be a compact neighborhood of $\mathbf{z}_0^*$ where the conditions of Lemma 3.1 hold and $0 < G_0 < 1$.

Then, given $\mathbf{z}_0 \in \mathcal{D}$, we know that

$$\mathbf{z}(t, \mathbf{z}_0, \mu) = \mathbf{z}^{(0)}(t, \mathbf{z}_0) + \mu^2 \mathbf{z}^{(1)}(t, \mathbf{z}_0, \mu) + \mathcal{O}(\mu^4), \tag{4.1}$$

where

$$\mathbf{z}^{(0)}(t, \mathbf{z}_0) = (L_0^{-3} t, \pi/2, h_0, L_0, G_0, H_0), \tag{4.2}$$

$$\mathbf{z}^{(1)}(t, \mathbf{z}_0, \mu) = \mathcal{Z}(t, \mathbf{z}_0) \int_0^t \mathcal{Z}^{-1}(s, \mathbf{z}_0) \mathcal{F}_1(\mathbf{z}^{(0)}(s, \mathbf{z}_0), s, \mu)\, ds, \tag{4.3}$$

and

$$\mathcal{Z}(t, \mathbf{z}_0) = \left. \frac{\partial \mathbf{z}^{(0)}(t, \xi)}{\partial \xi} \right|_{\xi = \mathbf{z}_0}.$$

From (4.1),

$$\begin{aligned} l(t, \mathbf{z}_0, \mu) &= L_0^{-3} t + \mu^2 l^{(1)}(t, \mathbf{z}_0, \mu) + \mathcal{O}(\mu^4), \\ g(t, \mathbf{z}_0, \mu) &= \pi/2 + \mu^2 g^{(1)}(t, \mathbf{z}_0, \mu) + \mathcal{O}(\mu^4), \end{aligned}$$

where $l^{(1)}$ and $g^{(1)}$ are the first and second coordinates of $\mathbf{z}^{(1)}$, given by (4.3), respectively.

Obviously, the symmetry conditions given by (2.8) are fulfilled at $t = 0$. They also must be satisfied at $t = T/2 = k\pi\mu^{3/2}$ in order to have $E_p(T/2) = k\pi$. We consider k a natural number and $\mu > 0$ such that $\mu = k^{-2/3}$. Then, $T/2 = \pi$ and we have to find initial conditions $\mathbf{z}_0 \in \mathcal{D}$ satisfying

$$\begin{aligned} L_0^{-3}\pi - \pi + \mu^2 l^{(1)}(\pi, \mathbf{z}_0, \mu) + \mathcal{O}(\mu^4) &= 0, \\ g^{(1)}(\pi, \mathbf{z}_0, \mu) + \mathcal{O}(\mu^2) &= 0. \end{aligned} \tag{4.4}$$

A natural way to solve (4.4) is to find a solution for the case $\mu = 0$ and then to apply an implicit function theorem. The first handicap is that neither $l^{(1)}(\pi, \mathbf{z}_0, \mu)$ nor $g^{(1)}(\pi, \mathbf{z}_0, \mu)$ is defined for $\mu = 0$, and neither is the Hamiltonian $\mathcal{H}_1$. Moreover, they do not satisfy the differentiability conditions of the standard implicit function theorem. In order to overcome these difficulties, we will see first that both equations can be extended to the case $\mu = 0$. Second, we will use Arenstorf's theorem, which requires weaker conditions (see [1]).

Let us start by extending (4.4) to the case $\mu = 0$. Observe that, from (4.3), $l^{(1)}$ and $g^{(1)}$ are bounded. This fact will be sufficient to define the first equation in (4.4) for $\mu = 0$. As for the second one, we show that $g^{(1)}(\pi, \mathbf{z}_0, \mu)$ can be written in terms of L_0, H_0, and G_0 plus a term of order $\mu^{3/2}$. In order to prove this we need the following technical lemma.

Lemma 4.1. *Given $\mathbf{z}_0^*$, let $\mathbf{z}_0 = (0, \pi/2, h_0, L_0, G_0, H_0) \in \mathcal{D}$. Let be $\varphi(t, \mathbf{z}_0)$ be a function bounded in $\mathcal{D}$, such that $\frac{d\varphi}{dt}(t, \mathbf{z}_0)$ is also bounded in $\mathcal{D}$. Then,*

$$\int_0^\pi (1 - \cos E_p)^2 \varphi(t, \mathbf{z}_0)\, dt = \frac{5}{2} \int_0^\pi \varphi(t, \mathbf{z}_0)\, dt + \mathcal{R}(\mathbf{z}_0, \mu),$$

where $|\mathcal{R}(\mathbf{z}_0, \mu)| \leq K\mu^{3/2}$ for a certain constant K.

Proof. From (2.2), the function $(1 - \cos E_p)^2$ is even and 2π-periodic with respect to the variable ℓ_p. Its Fourier series is given by

$$(1 - \cos E_p)^2 = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\ell_p),$$

where

$$a_k = \frac{2}{\pi} \int_0^\pi (1 - \cos E_p)^2 \cos(k\ell_p)\, d\ell_p.$$

From the fact that $(1 - \cos E_p)\, dE_p = d\ell_p$, the zero coefficient is

$$a_0 = \frac{2}{\pi} \int_0^\pi (1 - \cos E_p)^3\, dE_p = 5.$$

Then, using that $\ell_p = \mu^{-3/2} t$,

$$\int_0^\pi (1 - \cos E_p)^2 \varphi(t, \mathbf{z}_0)\, dt = \frac{5}{2} \int_0^\pi \varphi(t, \mathbf{z}_0)\, dt + \underbrace{\sum_{k=1}^{\infty} a_k \int_0^\pi \varphi(t, \mathbf{z}_0) \cos(k\mu^{-3/2} t)\, dt}_{\mathcal{R}(\mathbf{z}_0, \mu)}.$$

Integrating by parts,

$$\int_0^\pi \varphi(t, \mathbf{z}_0) \cos(k\mu^{-3/2} t)\, dt = \frac{\mu^{3/2}}{k} \left(\varphi(\pi, \mathbf{z}_0) \sin(k\mu^{-3/2}\pi) - \int_0^\pi \frac{d\varphi}{dt}(t, \mathbf{z}_0) \sin(k\mu^{-3/2} t)\, dt \right).$$

Since $\varphi(t, \mathbf{z}_0)$, $\frac{d\varphi}{dt}(t, \mathbf{z}_0)$ are bounded for all $(t, \mathbf{z}_0)$ with $\mathbf{z}_0 \in \mathcal{D}$ (say, $|\varphi(t, \mathbf{z}_0)| \leq k_1$ and $|\frac{d\varphi}{dt}(t, \mathbf{z}_0)| \leq k_2$), we have that for a certain constant C

$$\left| \int_0^\pi \varphi(t, \mathbf{z}_0) \cos(k\mu^{-3/2} t)\, dt \right| \leq \frac{\mu^{3/2}}{k} \left(k_1 + 2k_2 \frac{\mu^{3/2}}{k} \right) = \frac{\mu^{3/2}}{k} C,$$

and

$$|\mathcal{R}(\mathbf{z}_0, \mu)| \leq \mu^{3/2} C \sum_{k=1}^{\infty} \frac{|a_k|}{k}.$$

The series on the right-hand side converges because a_k are Fourier coefficients of a regular function, and so the lemma is proved. ∎

Lemma 4.2. *Given $\mathbf{z}_0^*$, let $\mathbf{z}_0 = (0, \pi/2, h_0, L_0, G_0, H_0) \in \mathcal{D}$. Then, for $\mu > 0$ and small enough, $g^{(1)}(\pi, \mathbf{z}_0, \mu)$ is of type $\mathcal{C}^1$ in $\mathcal{D}$ and there exist functions $I_1(L_0, G_0)$ and $I_2(L_0, G_0)$ such that*

$$g^{(1)}(\pi, \mathbf{z}_0, \mu) = -\frac{15}{16}(I_1(L_0, G_0) - H_0^2 I_2(L_0, G_0)) + \mathcal{O}(\mu^{3/2}).$$

Furthermore, if $L_0 = 1$, then

$$g^{(1)}(\pi, \mathbf{z}_0, \mu) = -\frac{15\pi}{32 G_0^4}\left(5\frac{H_0^2}{G_0^2} - 1\right) + \mathcal{O}(\mu^{3/2}).$$

Proof. From (4.3), we have that

$$g^{(1)}(t, \mathbf{z}_0, \mu) = \int_0^t \left.\frac{\partial \mathcal{H}_1}{\partial G}\right|_{\mathbf{z}^{(0)}(s, \mathbf{z}_0)} ds,$$

where $\mathbf{z}^{(0)}(s, \mathbf{z}_0)$ is the function defined in (4.2). Then, using (3.4) we obtain that

$$g^{(1)}(\pi, \mathbf{z}_0, \mu) = \frac{-3}{8}\int_0^\pi (1 - \cos E_p)^2 \varphi(t, \mathbf{z}_0)\, dt,$$

where

$$\varphi(t, \mathbf{z}_0) = \frac{\partial}{\partial G}\left(\frac{1}{r^3}\left(\cos^2 f - \frac{1}{3}\right) - \frac{H^2 \cos^2 f}{r^3 G^2}\right)\bigg|_{\mathbf{z}^{(0)}(t, \mathbf{z}_0)}. \tag{4.5}$$

Since $\mathbf{z}_0 \in \mathcal{D}$, it is clear that $g^{(1)}(\pi, \mathbf{z}_0, \mu)$ is of type $\mathcal{C}^1$ in $\mathcal{D}$.

From Lemma 4.1 and (4.5), we have that

$$\begin{aligned} g^{(1)}(\pi, \mathbf{z}_0, \mu) &= \frac{-15}{16}\int_0^\pi \varphi(t, \mathbf{z}_0)\, dt + \mathcal{O}(\mu^{3/2}) \\ &= \frac{-15}{16}\left(\int_0^\pi \frac{\partial}{\partial G}\left(\frac{1}{r^3}\left(\cos^2 f - \frac{1}{3}\right)\right)\bigg|_{\mathbf{z}^{(0)}(t, \mathbf{z}_0)} dt \right. \\ &\qquad \left. - H_0^2 \int_0^\pi \frac{\partial}{\partial G}\left(\frac{\cos^2 f}{G^2 r^3}\right)\bigg|_{\mathbf{z}^{(0)}(t, \mathbf{z}_0)} dt\right) + \mathcal{O}(\mu^{3/2}) \\ &= -\frac{15}{16}(I_1(L_0, G_0) - H_0^2 I_2(L_0, G_0)) + \mathcal{O}(\mu^{3/2}), \end{aligned}$$

where

$$I_1(L_0, G_0) = \frac{G_0}{e_0 L_0^5}\int_0^{E(e_0, L_0)} (e_0 - \cos E)\frac{5e_0^2 - 5 - 2e_0\cos E + (7 - 3e_0^2)\cos^2 E - 2e_0 \cos^3 E}{(1 - e_0\cos E)^6}\, dE,$$

$$\begin{aligned} I_2(L_0, G_0) = \frac{1}{G_0 e_0 L_0^5}&\left(\int_0^{E(e_0, L_0)} \frac{-2e_0}{1 - e_0^2}\frac{(\cos E - e_0)^2}{(1 - e_0\cos E)^4}\, dE\right. \\ &\left. + \int_0^{E(e_0, L_0)} (\cos E - e_0)\, \frac{4 - 5e_0^2 + 4e_0 \cos E + (2e_0^2 - 7)\cos^2 E + 2e_0\cos^3 E}{(1 - e_0\cos E)^6}\, dE\right), \end{aligned}$$

and $E(e_0, L_0)$ is the solution of the equation $\pi = L_0^3(E - e_0 \sin E)$ (see section 5 for the details). In particular, when $L_0 = 1$, both integrals can be calculated explicitly and

$$I_1(1, G_0) = \frac{-\pi}{2G_0^4}, \qquad I_2(1, G_0) = \frac{-5\pi}{2G_0^6},$$

which ensures the last statement of the lemma. ■

The next lemma shows that derivatives of $g^{(1)}$ satisfy conditions similar to those in Lemma 4.2; i.e., they can be written in terms of L_0, H_0, and G_0 plus terms of order $\mu^{3/2}$.

Lemma 4.3. *Under the same hypothesis as in Lemma 4.2,*

$$\frac{\partial g^{(1)}}{\partial L_0}(\pi, \mathbf{z}_0, \mu) = -\frac{15}{16}\left(\frac{\partial I_1}{\partial L_0}(L_0, G_0) - H_0^2 \frac{\partial I_2}{\partial L_0}(L_0, G_0)\right) + \mathcal{O}(\mu^{3/2}),$$
$$\frac{\partial g^{(1)}}{\partial H_0}(\pi, \mathbf{z}_0, \mu) = \frac{15}{8} H_0 I_2(L_0, G_0) + \mathcal{O}(\mu^{3/2}).$$

Proof. The result is straightforward using arguments similar to those of Lemma 4.2. ■

In order to extend the symmetry equation to $\mu = 0$, let

$$\Omega = \{(L_0, H_0);\, \mathbf{z}_0 = (0, \pi/2, h_0, L_0, G_0, H_0) \in \mathcal{D}\}.$$

We define the function

$$\Phi(\xi, \mu) = (\Phi_1(\xi, \mu), \Phi_2(\xi, \mu))$$

for $\xi = (L_0, H_0) \in \Omega$ and $\mu \geq 0$ as

$$\Phi_1(\xi, \mu) = \begin{cases} L_0^{-3}\pi - \pi + \mu^2 l^{(1)}(\pi, \mathbf{z}_0, \mu) + \mathcal{O}(\mu^4) & \text{if } \mu \neq 0, \\ L_0^{-3}\pi - \pi & \text{if } \mu = 0, \end{cases} \tag{4.6}$$

and

$$\Phi_2(\xi, \mu) = \begin{cases} g^{(1)}(\pi, \mathbf{z}_0, \mu) + \mathcal{O}(\mu^2) & \text{if } \mu \neq 0, \\ -\frac{15}{16}(I_1(L_0, G_0) - H_0^2 I_2(L_0, G_0)) & \text{if } \mu = 0, \end{cases} \tag{4.7}$$

where $I_1(L_0, G_0)$ and $I_2(L_0, G_0)$ are the functions stated in the proof of Lemma 4.2. Then, (4.4) can be written as

$$\Phi(\xi, \mu) = (0, 0)$$

for $\mu \geq 0$, and for $\mu = 0$, $L_0 = 1$, satisfies

$$\Phi_1(\xi, 0) = 0, \qquad \Phi_2(\xi, 0) = -\frac{15\pi}{32G_0^4}\left(5\frac{H_0^2}{G_0^2} - 1\right).$$

Thus, for each fixed value of G_0, $\Phi(\xi_0, 0) = (0, 0)$ if $\xi_0 = (1, G_0/\sqrt{5})$. Observe that $H_0^2/G_0^2 = \cos^2 i$, and so the solution for $\mu = 0$ corresponds to an inclination i with $\cos i = 1/\sqrt{5}$, which is the critical inclination angle obtained in the case of the problem of an Earth-centered orbit when the effects of J_2 are considered.

In order to show that there exist symmetric periodic solutions of the reduced problem for $\mu \neq 0$, we need to show that there exist solutions of $\Phi(\xi,\mu) = (0,0)$. In this case, we will use the next proposition, proved in [4], which is a sufficient condition for Arenstorf's theorem.

Proposition 4.4. *Let U be an open domain in R^n, $I \subset R$ an open neighborhood of the origin, $f : U \times I \to R^n$ with $f(0,0) = 0$, differentiable with respect to $x \in U$, and $D_x f(0,0)$ nonsingular. Assume that there exist $c > 0$, $k > 0$ such that for $x \in U$, $\epsilon \in I$,*

1. $\|D_x f(x,\epsilon) - D_x f(0,0)\| \leq c(\|x\| + \epsilon)$,
2. $\|f(0,\epsilon)\| \leq k\epsilon$.

Then there exists a function $x(\epsilon) \in U$, defined for $\epsilon \in I' \subset I$, such that $f(x(\epsilon),\epsilon) = 0$ and $x(0) = 0$.

In order to apply Proposition 4.4 we need to prove that the function Φ satisfies some properties.

Proposition 4.5. *Let G_0 be fixed and $\xi_0 = (1, G_0/\sqrt{5})$. For μ small enough, there exists η such that the function $\Phi(\xi,\mu)$ is differentiable with respect to ξ in $\mathcal{B} = \{\xi \in \Omega;\ \|\xi - \xi_0\| \leq \eta\}$ and satisfies the three properties*

(i) $\|\Phi(\xi_0,\mu)\| \leq C_0\mu^{3/2}$,
(ii) $\|(D_\xi\Phi)^{-1}(\xi_0,0)\| \leq M$,
(iii) $\|D_\xi\Phi(\xi,\mu) - D_\xi\Phi(\xi_0,0)\| \leq C_1(\|\xi - \xi_0\| + \mu^{3/2})$,

where M, C_0, and C_1 are constants independent of μ and $D_\xi\Phi(\xi,\mu)$ denotes the Jacobi matrix of Φ with respect to the variables ξ.

Proof. Statement (i) is a direct consequence of the definition of Φ (see (4.6) and (4.7)), the fact that $l^{(1)}$ is a bounded function, and Lemma 4.2.

Using that the derivatives of $l^{(1)}$ are also bounded and Lemma 4.3, we have that

$$
\begin{aligned}
D_\xi\Phi(\xi,\mu) &= \begin{pmatrix} \frac{-3\pi}{L_0^4} + \mathcal{O}(\mu^2) & \mathcal{O}(\mu^2) \\ \mathcal{J}(L_0,G_0) + \mathcal{O}(\mu^{3/2}) & \frac{15}{8}H_0 I_2(L_0,G_0) + \mathcal{O}(\mu^{3/2}) \end{pmatrix}, \\
D_\xi\Phi(\xi,0) &= \begin{pmatrix} \frac{-3\pi}{L_0^4} & 0 \\ \mathcal{J}(L_0,G_0) & \frac{15}{8}H_0 I_2(L_0,G_0) \end{pmatrix},
\end{aligned}
\tag{4.8}
$$

where $\mathcal{J}(L_0,G_0) = \frac{-15}{16}\frac{\partial(I_1 - H_0^2 I_2)}{\partial L_0}$. Then, as $I_2(1,G_0) = -5\pi/(2G_0^6) \neq 0$, $D_\xi\Phi(\xi_0,0)$ can be inverted, and item (ii) is proved.

Let us prove (iii). First, we have that

$$
\|D_\xi\Phi(\xi,\mu) - D_\xi\Phi(\xi_0,0)\| \leq \|D_\xi\Phi(\xi,\mu) - D_\xi\Phi(\xi,0)\| + \|D_\xi\Phi(\xi,0) - D_\xi\Phi(\xi_0,0)\|. \tag{4.9}
$$

On one hand, as the components of $\Phi(\xi,0)$ are of type $\mathcal{C}^1$ with respect to ξ, we get that

$$
\|D_\xi\Phi(\xi,0) - D_\xi\Phi(\xi_0,0)\| \leq c_0\|\xi - \xi_0\|. \tag{4.10}
$$

On the other hand,

$$
\|D_\xi\Phi(\xi,\mu) - D_\xi\Phi(\xi,0)\| \leq \sum_{i=1}^{2} \|D_\xi\Phi_i(\xi,\mu) - D_\xi\Phi_i(\xi,0)\| \leq c_1\mu^2 + c_2\mu^{3/2} \tag{4.11}
$$

by expressions (4.8). Substituting (4.10) and (4.11) into (4.9), we prove item (iii). ■

Notice that, given h_0, G_0, and $\mathbf{z}_0^* = (0, \pi/2, h_0, 1, G_0, G_0/\sqrt{5})$, the solution $\mathbf{z}^{(0)}(t, \mathbf{z}_0^*)$ is a solution of the Kepler problem lying on a plane of the critical inclination i with $\cos i = 1/\sqrt{5}$. Finally, let us prove that there exist periodic symmetric solutions of the perturbed reduced problem close to $\mathbf{z}^{(0)}(t, \mathbf{z}_0^*)$.

Theorem 4.6. *Consider the three-dimensional collision restricted three-body problem with masses $m_1 = m_2 = 1/2$, and primaries' semimajor axis $\mu/2$. If $\mu = k^{-2/3}$, where k is a positive integer large enough, there exist initial conditions such that the infinitesimal body moves in a symmetric periodic orbit of the* reduced problem*, of period 2π, near a Keplerian elliptic orbit. The inclination of the orbit is close to the "critical value"* $\cos i = 1/\sqrt{5}$.

Proof. Let us consider initial values h_0, G_0, and $\xi_0 = (1, G_0/\sqrt{5})$. It is clear that $\Phi(\xi_0, 0) = (0, 0)$. Given $\xi \in \Omega$, we define $f(x, \mu) = \Phi(x + \xi_0, \mu)$, where $x = \xi - \xi_0$. From Proposition 4.5 we can easily prove that $f(x, \mu)$ is under the hypothesis of Proposition 4.4. Then there exists a function $x(\mu)$ such that $f(x(\mu), \mu) = (0, 0)$ and $x(0) = 0$.

This yields a continuum of solutions of system $\Phi(\xi, \mu) = (0, 0)$. These conditions must be satisfied simultaneously with $E_p(T/2) = k\pi$, which is equivalent to $T = 2k\pi\mu^{3/2}$. Thus, for each $\mu = k^{-2/3}$, k a large positive integer, a periodic solution of the reduced problem exists. ■

Remark. All the orbits found are on an integral resonance with the motion of the primaries; i.e., the primaries undergo k complete orbits in one orbit of the infinitesimal body. If $k = p/q$ is an irreducible rational, then similar arguments show that in q complete orbits of the infinitesimal the primaries undergo p complete orbits.

5. Appendix. Here we develop the calculations needed in the proof of Lemma 4.2. We want to compute

$$\begin{aligned}\int_0^\pi \varphi(t, \mathbf{z}_0)dt &= \int_0^\pi \left.\frac{\partial \Delta_1}{\partial G}\right|_{\mathbf{z}^{(0)}(t,\mathbf{z}_0)} dt - H_0^2 \int_0^\pi \left.\frac{\partial \Delta_2}{\partial G}\right|_{\mathbf{z}^{(0)}(t,\mathbf{z}_0)} dt \\ &= I_1(L_0, G_0) - H_0^2 I_2(L_0, G_0),\end{aligned}$$

where $\Delta_1 = \frac{\cos^2 f - 1/3}{r^3}$ and $\Delta_2 = \frac{\cos^2 f}{G^2 r^3}$.

We will introduce the change of variables given by $t = L_0^3(E - e_0 \sin E)$, where L_0^2 and e_0 correspond to the semimajor axis and the eccentricity of the Keplerian orbit $\mathbf{z}^{(0)}(t, \mathbf{z}_0)$, respectively. As the new variable to integrate will be E, we use the rule

$$\frac{\partial \Delta_i}{\partial G} = \frac{\partial \Delta_i}{\partial E}\frac{dE}{de}\frac{de}{dE} \tag{5.1}$$

for $i = 1, 2$. On one hand, from Kepler's equation $t = a^{3/2}(E - e \sin E)$, we have that

$$0 = a^{3/2}\left(\frac{dE}{de} - \sin E - e \cos E\,\frac{dE}{de}\right) \quad \text{and} \quad \frac{dE}{de} = \frac{\sin E}{1 - e \cos e}.$$

On the other hand, as $G^2 = a(1 - e^2)$, we have that $\frac{de}{dG} = \frac{-G}{ae}$. Substituting into (5.1), we have that

$$\frac{\partial \Delta_i}{\partial G} = \frac{-G \sin E}{ae(1 - e \cos E)}\frac{\partial \Delta_i}{\partial E}. \tag{5.2}$$

Next, as $r = a(1 - e\cos E) = \frac{a(1-e^2)}{1+e\cos f}$, we have that $\cos f = \frac{\cos E - e}{1-e\cos E}$, and deriving both expressions we obtain

$$\frac{\partial r}{\partial E} = a\,\frac{e - \cos E}{\sin E}, \qquad -\sin f\,\frac{df}{dE} = \frac{\sin E(e^2 - 2 + e\cos E)}{(1 - e\cos E)^2}. \tag{5.3}$$

Thus, using the expressions (5.2) and (5.3), we have that

$$\begin{aligned}
\frac{\partial \Delta_1}{\partial G} &= \frac{G(e - \cos E)(5e^2 - 5 - 2\,e\cos E + (7 - 3\,e^2)\cos^2 E - 2\,e\cos^3 E)}{ea^4(1 - e\cos E)^7}, \\
\frac{\partial \Delta_2}{\partial G} &= \frac{-2(\cos E - e)^2}{G^3 a^3 (1 - e\cos E)^5} \\
&\quad + \frac{(\cos E - e)}{Gea^4}\,\frac{4 - 5e^2 + 4e\cos E + (2e^2 - 7)\cos^2 E + 2e\cos^3 E}{(1 - e\cos E)^7}.
\end{aligned}$$

Finally, evaluating both expressions on the solution $\mathbf{z}^{(0)}(t, \mathbf{z}_0)$ of the Kepler problem, we obtain the expressions for the functions I_1 and I_2 as

$$\begin{aligned}
\int_0^\pi \left.\frac{\partial \Delta_1}{\partial G}\right|_{\mathbf{z}^{(0)}(t,\mathbf{z}_0)} dt &= \frac{G_0}{e_0 L_0^5} \int_0^{E(e_0,L_0)} f_1(e_0, E)\,dE = I_1(L_0, G_0), \\
\int_0^\pi \left.\frac{\partial \Delta_2}{\partial G}\right|_{\mathbf{z}^{(0)}(t,\mathbf{z}_0)} dt &= \frac{1}{G_0 e_0 L_0^5} \int_0^{E(e_0,L_0)} f_2(e_0, E)\,dE = I_2(L_0, G_0),
\end{aligned}$$

where $E(e_0, L_0)$ is the solution of the equation $\pi = L_0^3(E - e_0 \sin E)$ and

$$\begin{aligned}
f_1(e_0, E) &= (e_0 - \cos E)\frac{5e_0^2 - 5 - 2e_0\cos E + (7 - 3e_0^2)\cos^2 E - 2e_0\cos^3 E}{(1 - e_0\cos E)^6}, \\
f_2(e_0, E) &= \frac{-2e_0}{1 - e_0^2}\,\frac{(\cos E - e_0)^2}{(1 - e_0\cos E)^4} \\
&\quad + (\cos E - e_0)\,\frac{4 - 5e_0^2 + 4e_0\cos E + (2e_0^2 - 7)\cos^2 E + 2e_0\cos^3 E}{(1 - e_0\cos E)^6}.
\end{aligned}$$

Furthermore, it is clear that for a fixed value of $e_0 < 1$ the functions f_1 and f_2 are continuous and differentiable with respect to E and e_0, and so $I_1(L_0, G_0)$ and $I_2(L_0, G_0)$ are functions of type $\mathcal{C}^1$ with respect to e_0. In particular, when $L_0 = 1$, $E(e_0, L_0) = \pi$, and both integrals can be calculated explicitly:

$$I_1(1, G_0) = \frac{-\pi}{2G_0^4}, \qquad I_2(1, G_0) = \frac{-5\pi}{2G_0^6}.$$

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