## Absorbing sets and Baker domains for holomorphic maps

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## Abstract

We consider holomorphic maps  $f: U \to U$  for a hyperbolic domain U in the complex plane, such that the iterates of f converge to a boundary point  $\zeta$  of U. By a previous result of the authors, for such maps there exist nice absorbing domains  $W \subset U$ . In this paper, we show that W can be chosen to be simply connected, if f has doubly parabolic type in the sense of the Baker–Pommerenke–Cowen classification of its lift by a universal covering (and  $\zeta$  is not an isolated boundary point of U). We also provide counterexamples for other types of the map f, and give an exact characterization of doubly parabolic type in terms of the dynamical behaviour of f.

1. Introduction

In this paper, we study iterates  $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$  of a holomorphic map

$$f: U \longrightarrow U$$

where U is a hyperbolic domain in the complex plane  $\mathbb{C}$  (that is, a domain whose complement in  $\mathbb{C}$  contains at least two points) and f has no fixed points, that is,  $f(z) \neq z$  for  $z \in U$ . In the special case when U is the unit disc  $\mathbb{D}$  (or, equivalently, the right half-plane  $\mathbb{H}$ ), the dynamical behaviour of f has been extensively studied, starting from the works of Denjoy, Valiron and Wolff in the 1920s and 1930s (see [14, 24–26] and a more detailed explanation in Section 2). In particular, the celebrated Denjoy–Wolff Theorem asserts that under this assumption, the iterates of f converge almost uniformly (that is, uniformly on compact subsets of U) as  $n \to \infty$  to a point  $\zeta$  in the boundary of U. Changing the coordinates by a Möbius map, we can conveniently assume in this case  $U = \mathbb{H}$ ,  $\zeta = \infty$ . Baker and Pommerenke [2, 19] and Cowen [13] proved that f on  $\mathbb{H}$  is semi-conjugate to a Möbius map  $T : \Omega \to \Omega$  by a holomorphic map  $\varphi : \mathbb{H} \to \Omega$ , where the following three cases can occur:

- (i)  $\Omega = \mathbb{H}, T(\omega) = a\omega$  for some a > 1 (hyperbolic type);
- (ii)  $\Omega = \mathbb{H}, T(\omega) = \omega \pm i$  (simply parabolic type);
- (iii)  $\Omega = \mathbb{C}, T(\omega) = \omega + 1$  (doubly parabolic type)

(see Section 2 for a precise formulation). The terms 'simply' and 'doubly' are used due to the following fact: if f has, respectively, simply or doubly parabolic type and extends holomorphically to a neighbourhood of infinity in the Riemann sphere, then  $\infty$  becomes a parabolic fixed point with one or two petals, respectively (see, for example, [9, 15]). An alternative terminology for simply and doubly parabolic types, used in [11], is 'parabolic type II' and 'parabolic type I', respectively.

For an arbitrary hyperbolic domain  $U \subset \mathbb{C}$ , the problem of describing the dynamics of a holomorphic map  $f: U \to U$  without fixed points is more complicated. To this aim, one can

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