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Brushing the hairs of transcendental entire functions ☆

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ABSTRACT

Let f be a transcendental entire function of finite order in the Eremenko–Lyubich class $\mathcal{B}$ (or a finite composition of such maps), and suppose that f is hyperbolic and has a unique Fatou component. We show that the Julia set of f is a *Cantor bouquet*; i.e. is ambiently homeomorphic to a straight brush in the sense of Aarts and Oversteegen. In particular, we show that any two such Julia sets are ambiently homeomorphic.

We also show that if $f \in \mathcal{B}$ has finite order (or is a finite composition of such maps), but is not necessarily hyperbolic with connected Fatou set, then the Julia set of f *contains* a Cantor bouquet.

As part of our proof, we describe, for an arbitrary function $f \in \mathcal{B}$, a natural compactification of the dynamical plane by adding a "circle of addresses" at infinity.

1. Introduction

In recent decades there has been an increasing interest in studying the dynamics generated by the iterates of transcendental entire functions. For these dynamical systems, the presence of an essential singularity at infinity creates some major differences to the dynamics of polynomials or rational maps.

More precisely, let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. Then the *Julia set* $J(f)$ is the set of points $z \in \mathbb{C}$ at which the family $\{f^n\}_{n>0}$ is not equicontinuous with respect to the spherical metric (this is where the dynamics of f is "chaotic"). The complement $F(f) = \mathbb{C} \setminus J(f)$—i.e. the set of stable behaviour—is called the *Fatou set*. Due to the essential singularity at infinity, the Julia sets of transcendental entire functions tend to be far more complicated than for rational maps, and one of the main problems in the field is to understand these sets (and the dynamics thereon) from a topological and geometric point of view.

A cornerstone example of transcendental entire maps is given by the complex exponential family $E_\lambda(z) = \lambda \exp(z)$, $\lambda \in \mathbb{C}$. It is often considered the simplest possible family of transcendental entire functions, and has received considerable attention since at least the 1980s. For real parameter values $\lambda \in (0, 1/e)$, the Fatou set has a unique connected component, consisting of all points that converge to an attracting fixed point $z_0^\lambda \in \mathbb{R}$ under iteration. In [8] it was proved that for these parameters, the Julia set is given by a union of pairwise disjoint arcs to ∞, which are called *hairs* or *dynamic rays*. Each point z on each hair, except possibly the finite endpoint of the hair, satisfies $\mathrm{Re}(E_\lambda^j)(z) \to +\infty$ when $j \to \infty$. Due to its appearance,

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