# THE SOLUTION OF THE EXTENDED 16TH HILBERT PROBLEM FOR SOME CLASSES OF PIECEWISE DIFFERENTIAL SYSTEMS

Louiza Baymout[1], Rebiha Benterki[1] and Jaume Llibre[2]

[1] Department of Mathematics, University Mohamed El Bachir El Ibrahimi
of Bordj Bou Arréridj 34000, El Anasser, Algeria
E-mail: louiza.baymout@univ-bba.dz and r.benterki@univ-bba.dz

[2] Departament de Matematiques, Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail: jllibre@mat.uab.cat

**Abstract**. In this paper we provide the upper bounds for the maximum number of crossing limit cycles of certain classes of discontinuous piecewise differential systems separated by a straight line and consequently formed by two differential systems. There are six families of Hamiltonian nilpotent centers formed by a linear plus a cubic polynomials. First we study the crossing limit cycles of the discontinuous piecewise differential systems formed by an arbitrary linear center and one arbitrary of the six Hamiltonian nilpotent centers. These 6 classes of piecewise differential systems have at most one crossing limit cycle, and there are systems in each class with one limit cycle. Second we study the crossing limit cycles of the discontinuous piecewise differential systems formed by two arbitrary of the six Hamiltonian nilpotent centers. These 21 classes of piecewise differential systems have at most four crossing limit cycles and there are systems in each class with exactly four limit cycles. [1]

## 1. Introduction

In 1900, at the International Congress of Mathematicians in Paris, David Hilbert [17] posed a list of 23 problems, one of them is the 16th problem which remains open until now together with the Rieman conjecture. The 16th Hilbert problem ask for an upper bound for the maximum number of limit cycles of all planar polynomial differential systems of a given degree. We recall that a *limit cycle* of a planar differential system is an isolated periodic orbit in the set of all periodic orbits of this system. One of the main problems in the qualitative study of planar differential systems is to determine the existence and the number of their limit cycles, see [20, 21]. This importance comes from the main role of limit cycles in understanding and explaining the dynamics of a given differential system, as for example the limit cycle of the Van der Pol equation [26, 27], or the one of the Belousov Zhavotinskii model [3] etc.

In this paper, we are interesting in planar discontinuous piecewise vector fields with two pieces separated by the straight line $x = 0$. Then by following the Filippov [12] conventions for defining the discontinuous piecewise differential systems on the discontinuity line $x = 0$. These discontinuous piecewise differential systems can be written as follows

$$
\begin{cases} \dot{x} = f^+(x, y), \quad \dot{y} = g^+(x, y), \quad if x \in R_1; \\ \dot{x} = f^-(x, y), \quad \dot{y} = g^-(x, y), \quad if x \in R_2; \end{cases} \tag{1}
$$

where $R_1 = \{(x, y) : x \geq 0\}$, and $R_2 = \{(x, y) : x \leq 0\}$. These systems can exhibit either crossing limit cycles or sliding limit cycles. Here we are only interested on the crossing ones, which are isolated periodic orbits having exactly two crossing points, i.e. there exist two points $(0, y_i)$ with $i = 1, 2$ such that $f^+(0, y_i)g^+(0, y_i) \geq 0$ for $i = 0, 1$. For simplicity we shall say limit cycle instead of crossing limit cycle.

Andronov and coworkers [1] started around 1920's in a serious way the study of the discontinuous piecewise differential systems. Nowadays these systems still continue to be a more important subject of research

[1] *Acknowledgements.* The third author is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

*2010 Mathematics Subject Classification.* Primary 34C29, 34C25, 47H11.

*Key words and phrases.* Hamiltonian nilpotent centers, cubic polynomial, limit cycles, discontinuous piecewise differential systems, vector fields.

for many scientists. This is due to the relevant applications of the discontinuous piecewise differential systems to model many problems in control theory, mechanics, economics and biology, etc., as we can see in [5, 10, 15].

In recent years several authors paid more attention to the simplest class of discontinuous piecewise differential systems, the ones separated by a straight line and consequently formed by two pieces, in each piece there is a linear differential system, see [2, 6, 7, 13, 16, 19, 22, 23, 24, 25] and the references quoted in these papers. The results of all these papers only provide examples that the planar discontinuous piecewise linear differential systems separated by a straight line can have at most three limit cycles, but until now remains open to prove that three is the maximum number of limit cycles of this class of systems. Recently, Esteban et al in [11] studied the extension of the 16th Hilbert problem to discontinuous piecewise isochronous centers of degree one or two separated by a straight line. In the same year Benterki and Llibre [4] studied the same problem but now for some classes of discontinuous piecewise isochronous centers of degree one or three.

We denote by $\mathcal{F}_1$ the familly of discontinuous piecewise differential systems separated by the straight line $x=0$, and formed by two pieces, in one piece there is an arbitrary linear differential center, and in the other piece there is a Hamiltonian nilpotent center formed by a linear plus cubic homogeneous polynomials after an arbitrary affine change of variables.

We denote by $\mathcal{F}_2$ the family of discontinuous piecewise differential systems separated by the straight line $x=0$, and consequently formed by two pieces, in each piece there is an arbitrary Hamiltonian nilpotent center formed by a linear plus cubic homogeneous polynomials after an arbitrary affine change of variables.

For the first family of planar differential systems we will use the next lemma that gives a normal form for an arbitrary linear differential center.

**Lemma 1.** *By doing a linear change of variables and a rescaling of the independent variable every linear center in $\mathbb{R}^2$ can be written*

$$\dot{x} = -\frac{y\left(a^2+w^2\right)}{d} - ax + b, \quad \dot{y} = ay + c + dx, \text{ with } w > 0, d > 0. \tag{2}$$

*and its first integrale is*

$$H(x,y) = (ay+dx)^2 + 2d(cx-by) + w^2y^2. \tag{3}$$

For a proof of Lemma 1 see [18]. Our results use previous results of [9, 8] as the following one.

**Theorem 2.** *A Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms has a nilpotent center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

($\mathcal{C}_1$) $\dot{x} = ax+by, \dot{y} = \frac{-a^2}{b}x - ay + x^3$, *with* $b<0$.

($\mathcal{C}_2$) $\dot{x} = ax+by-x^3, \dot{y} = \frac{-a^2}{b}x - ay + 3x^2y$, *with* $a>0$ *and* $b\neq 0$.

($\mathcal{C}_3$) $\dot{x} = ax+by-3x^2y+y^3, \dot{y} = (c-\frac{a^2}{b+c})x - ay + 3xy^2$, *with either* $a=b=0$ *and* $c<0$, *or* $c=0$, $ab\neq 0$, *and* $a^2/b-6b>0$. *In this last case one can take* $a=1$.

($\mathcal{C}_4$) $\dot{x} = ax+by-3x^2y-y^3, \dot{y} = (c-\frac{a^2}{b+c})x - ay + 3xy^2$, *with either* $a=b=0$ *and* $c>0$, *or* $c=0$, $a\neq 0$, *and* $b<0$. *In this last case one can take* $a=1$.

($\mathcal{C}_5$) $\dot{x} = ax+by-3\mu x^2y+y^3, \dot{y} = (c-\frac{a^2}{b+c})x - ay + x^3 + 3\mu xy^2$, *with either* $a=b=0$ *and* $c<0$, *or* $c=0$, $b\neq 0$, *and* $(a^4-b^4-6\mu a^2b^2)/b>0$. *In this last case and when* $a\neq 0$ *one can take* $a=1$.

($\mathcal{C}_6$) $\dot{x} = ax + by - 3\mu x^2 y - y^3, \dot{y} = (c - \dfrac{a^2}{b+c})x - ay + x^3 + 3\mu x y^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $b \neq 0$, *and* $(a^4 + b^4 + 6\mu a^2 b^2)/b < 0$. *In this last case and when* $a \neq 0$ *one can take* $a = 1$. *Where* $a, b, c, \mu \in \mathbb{R}$.

For a proof of Theorem 2 see [9].

Our main results are the following seven theorems.

**Theorem 3.** *The maximum number of crossing limit cycles for the six classes* ($\mathcal{C}_i$), $i = 1, ..., 6$, *of discontinuous piecewise differential systems of the family* $\mathcal{F}_1$, *affter an affine change of variables, which are separated by the straight line* $x = 0$ *is one. This maximum is reached in all the classes, see Figures 1, 2 and 3.*

**Theorem 4.** *The maximum number of crossing limit cycles of systems in family* $\mathcal{F}_2$ *formed by a Hamiltonian nilpotent center* ($\mathcal{C}_1$) *in one region, and by one of the Hamiltonian nilpotent centers* ($\mathcal{C}_i$) *with* $i = 1, \ldots, 6$ *in the other region is four. This maximum is reached for all the classes, see Figures 4, 5 and 6.*

**Theorem 5.** *The maximum number of crossing limit cycles of systems of the family* $\mathcal{F}_2$ *formed by a Hamiltonian nilpotent center* ($\mathcal{C}_2$) *in one region, and by one of the Hamiltonian nilpotent centers* ($\mathcal{C}_i$) *with* $i = 2, \ldots, 6$ *in the other region is four. This maximum is reached for all the classes, see Figures 7, 8 and 9.*

**Theorem 6.** *The maximum number of crossing limit cycles of discontinuous piecewise differential systems of the family* $\mathcal{F}_2$ *formed by a Hamiltonian nilpotent center* ($\mathcal{C}_3$) *in one region, and by one of the Hamiltonian nilpotent centers* ($\mathcal{C}_i$) *with* $i = 3, \ldots, 6$ *in the other region is four. This maximum is reached for all the classes, see Figures 9, 10 and 11.*

**Theorem 7.** *The maximum number of crossing limit cycles of discontinuous piecewise differential systems of the family* $\mathcal{F}_2$ *formed by a Hamiltonian nilpotent center* ($\mathcal{C}_4$) *in one region, and by one of the Hamiltonian nilpotent centers* ($\mathcal{C}_i$) *with* $i = 4, \ldots, 6$ *in the other region is four. This maximum is reached for all the classes.*

**Theorem 8.** *The maximum number of crossing limit cycles of discontinuous piecewise differential systems of the family* $\mathcal{F}_2$ *formed by a Hamiltonian nilpotent center* ($\mathcal{C}_5$) *in one region, and by one of the Hamiltonian nilpotent centers* ($\mathcal{C}_5$) *or* ($\mathcal{C}_6$) *in the other region is four. This maximum is reached for these two classes, see Figure 13.*

**Theorem 9.** *The maximum number of crossing limit cycles of discontinuous piecewise differential systems of the family* $\mathcal{F}_2$ *formed by a Hamiltonian nilpotent center* ($\mathcal{C}_6$) *in each region is four. This maximum is reached, see Figure 14.*

Theorems 3, 4, 5, 6, 7, 8 and 9 are proved in sections 3, 4, 5, 6, 7, 8 and 9, respectively.

## 2. The Hamiltonian nilpotent centers ($\mathcal{C}_1$), ($\mathcal{C}_2$), ($\mathcal{C}_3$), ($\mathcal{C}_4$), ($\mathcal{C}_5$) and ($\mathcal{C}_6$) after an affine change of variables

In this section we make a general affine change of variables in the expression of the Hamiltonian planar polynomial differential systems with a linear plus cubic homogeneous terms having a nilpotent center ($\mathcal{C}_1$), ($\mathcal{C}_2$), ($\mathcal{C}_3$), ($\mathcal{C}_4$), ($\mathcal{C}_5$) or ($\mathcal{C}_6$), and of their first integrals. We consider the change of variables

$$(x, y) = (a_1 x + b_1 y + c_1, \gamma_1 + \alpha_1 x + \beta_1 y).$$

Then after this change of variables the nilpotent center ($\mathcal{C}_1$) becomes

$$\begin{aligned}
\dot{x} = & \frac{1}{b(\alpha_1 b_1 - a_1\beta_1)}(-a^2 b_1(a_1 x + b_1 y + c_1) + b(3b_1^3 y^2(a_1 x + c_1) + 3b_1^2 y(a_1 x + c_1)^2 + b_1(a_1 \\
& x + c_1)^3 + b_1^4 y^3 - b\beta_1(\gamma_1 + \alpha_1 x + \beta_1 y)) - ab(\beta_1(a_1 x + c_1) + b_1(\gamma_1 + \\
& \alpha_1 x + 2\beta_1 y))), \\
\dot{y} = & \frac{1}{b(\alpha_1 b_1 - a_1\beta_1)}(a^2 a_1(a_1 x + b_1 y + c_1) + b(-a_1^4 x^3 - 3a_1^3 x^2(b_1 y + c_1) - 3a_1^2 x(b_1 y + c_1)^2 \\
& -a_1(b_1 y + c_1)^3 + \alpha_1 b(\gamma_1 + \alpha_1 x + \beta_1 y)) + ab(a_1(\gamma_1 + 2\alpha_1 x + \beta_1 y) + \\
& \alpha_1 b_1 y + \alpha_1 c_1)),
\end{aligned} \tag{4}$$

with its first integral

$$H_1(x,y) = \frac{a^2}{2b}(a_1x+b_1y+c_1)^2 + a(a_1x+b_1y+c_1)(\gamma_1+\alpha_1x+\beta_1y) - \frac{1}{4}(a_1x+b_1y+c_1)^4 + \frac{1}{2}b(\gamma_1+\alpha_1x+\beta_1y)^2.$$

The differential nilpotent center $(\mathcal{C}_2)$ becomes

$$\begin{aligned}\dot{x} = {} & \frac{1}{b(\alpha_1b_1-a_1\beta_1)}(bb_1^3y^2(3\gamma_1+3\alpha_1x+4\beta_1y) - b\beta_1(aa_1x - a_1^3x^3 - 3a_1c_1^2x - c_1^3 + c_1(a-3a_1^2 \\ & x^2) + b(\gamma_1+\alpha_1x+\beta_1y)) + b_1^2y\Big(3b(a_1x+c_1)(2\gamma_1+2\alpha_1x+3\beta_1y) - a^2\Big) \\ & +b_1(3bc_1^2(\gamma_1+\alpha_1x+2\beta_1y) - a^2a_1x + 3a_1^2bx^2(\gamma_1+\alpha_1x+2\beta_1y) - ab(\gamma_1 \\ & +\alpha_1x+2\beta_1y) + c_16a_1bx(\gamma_1+\alpha_1x+2\beta_1y) - a^2)), \\ \dot{y} = {} & \frac{1}{b(\alpha_1b_1-a_1\beta_1)}(\alpha_1b - a_1^3bx^2(4\alpha_1x+3(\gamma_1+\beta_1y))(ab_1y - b_1^3y^3 - 3b_1c_1^2y - c_1^3 + c_1(a-3b_1^2 \\ & y^2) + b(\gamma_1+\alpha_1x+\beta_1y)) - a_1^2x(3b(b_1y+c_1)(3\alpha_1x+2(\gamma_1+\beta_1y)) - a^2) + \\ & a_1(-3bc_1^2(\gamma_1+2\alpha_1x+\beta_1y) + a^2b_1y + c_1\left(a^2 - 6bb_1y(\gamma_1+2\alpha_1x+\beta_1y)\right) + \\ & ab(\gamma_1+2\alpha_1x+\beta_1y) - 3bb_1^2y^2(\gamma_1+2\alpha_1x+\beta_1y))),\end{aligned} \tag{5}$$

its first integral is

$$H_2(x,y) = \frac{a^2}{2b}(a_1x+b_1y+c_1)^2 + a(a_1x+b_1y+c_1)(\gamma_1+\alpha_1x+\beta_1y) + \frac{1}{2}b(\gamma_1+\alpha_1x+\beta_1y)^2 - (a_1x+b_1y+c_1)^3(\gamma_1+\alpha_1x+\beta_1y).$$

The differential center $(\mathcal{C}_3)$ becomes

$$\begin{aligned}\dot{x} = {} & \frac{-1}{(b+c)(\alpha_1b_1-a_1\beta_1)}(a^2b_1(a_1x+b_1y+c_1) + a(b+c)(\beta_1(a_1x+c_1) + b_1(\gamma_1+\alpha_1x+2\beta_1 \\ & y)) + (b+c)(\beta_1(\gamma_1+\alpha_1x+\beta_1y)(-3a_1^2x^2 - 6a_1c_1x + b - 3c_1^2 + (\gamma_1 \\ & +\alpha_1x+\beta_1y)^2) - b_1(a_1x+c_1)(c+3(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+\alpha_1x+3 \\ & \beta_1y)) - b_1^2y(c+3(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+\alpha_1x+2\beta_1y)))), \\ \dot{y} = {} & \frac{1}{(b+c)(\alpha_1b_1-a_1\beta_1)}(a^2a_1(a_1x+b_1y+c_1) + a(b+c)(a_1(\gamma_1+2\alpha_1x+\beta_1y) + \alpha_1b_1y + \alpha_1 \\ & c_1) - (b+c)(-\alpha_1(\gamma_1+\alpha_1x+\beta_1y)(b - 3b_1^2y^2 - 6b_1c_1y - 3c_1^2 + (\gamma_1+ \\ & \alpha_1x+\beta_1+y)^2)a_1^2x(c+3(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+2\alpha_1x+\beta_1y))(b_1y+ \\ & a_1+c_1)(c+3(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+3\alpha_1x+\beta_1y)))),\end{aligned} \tag{6}$$

its first integral is

$$H_3(x,y) = \frac{1}{2}\left(\frac{a^2}{b+c} - c\right)(a_1x+b_1y+c_1)^2 + a(a_1x+b_1y+c_1)(\gamma_1+\alpha_1x+\beta_1y) + \frac{1}{2}b(\gamma_1+\alpha_1x +\beta_1y)^2 - \frac{3}{2}(a_1x+b_1y+c_1)^2(\gamma_1+\alpha_1x+\beta_1y)^2 + \frac{1}{4}(\gamma_1+\alpha_1x+\beta_1y)^4.$$

The differential nilpotent center $(\mathcal{C}_4)$ becomes

$$\begin{aligned}\dot{x} = {} & -\frac{1}{(b+c)(\alpha_1b_1-a_1\beta_1)}(a^2b_1(a_1x+b_1y+c_1) + a(b+c)(\beta_1(a_1x+c_1) + b_1(\gamma_1+\alpha_1x+2\beta_1 \\ & y)) + (-\beta_1(\gamma_1+\alpha_1x+\beta_1y)(3a_1^2x^2 + 6a_1c_1x - b + 3c_1^2 + (\gamma_1+\alpha_1x \\ & +\beta_1y)^2) + b_1^2y(c+3(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+\alpha_1x+2\beta_1y)) - b_1(a_1x \\ & +c_1)(c+3(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+\alpha_1x+3\beta_1y))), \\ \dot{y} = {} & \frac{1}{(b+c)(\alpha_1b_1-a_1\beta_1)}(a^2a_1(a_1x+b_1y+c_1) + a(b+c)(a_1(\gamma_1+2\alpha_1x+\beta_1y) + \alpha_1b_1y + \alpha_1c_1) \\ & -(b+c)(\alpha_1(\gamma_1+\alpha_1x+\beta_1y)(-b + 3b_1^2y^2 + 6b_1c_1y + 3c_1^2 + (\gamma_1+\alpha_1x \\ & +\beta_1y)^2) + a_1^2x(c+3(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+2\alpha_1x+\beta_1y)) + a_1(b_1y+ \\ & c_1)(c+3)(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+3\alpha_1x+\beta_1y))),\end{aligned} \tag{7}$$

its first integral is

$$H_4(x,y) = -\frac{1}{2}\left(c-\frac{a^2}{b+c}\right)(a_1x+b_1y+c_1)^2+a(a_1x+b_1y+c_1)(\gamma_1+\alpha_1x+\beta_1y)+\frac{1}{2}b(\gamma_1 +\alpha_1x+\beta_1y)^2-\frac{3}{2}(a_1x+b_1y+c_1)^2(\gamma_1+\alpha_1x+\beta_1y)^2-\frac{1}{4}(\gamma_1+\alpha_1x+\beta_1y)^4.$$

The differential nilpotent center $(\mathcal{C}_5)$ becomes

$$(8)\quad \begin{aligned}\dot{x} = & -\frac{1}{(b+c)(\alpha_1b_1-a_1\beta_1)}(a^2b_1(a_1x+b_1y+c_1)+a(b+c)(\beta_1(a_1x+c_1)+b_1(\gamma_1+\alpha_1x+2\beta_1 y))+(b+c)(-3b_1^3y^2(a_1x+c_1)+\beta_1(\gamma_1+\alpha_1x+\beta_1y)(-3\mu(a_1x+c_1)^2 \\ & +b+(\gamma_1+\alpha_1x+\beta_1y)^2)-b_1^4y^3-b_1(a_1x+c_1)((a_1x+c_1)^2+c+3\mu(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+\alpha_1x+3\beta_1y))-b_1^2y(3((a_1x+c_1)^2+\mu(\gamma_1+\alpha_1x+\beta_1y)(\gamma_1+\alpha_1x+2\beta_1y))+c))),\\ \dot{y} = & \frac{1}{(b+c)(\alpha_1b_1-a_1\beta_1)}(a^2a_1(a_1x+b_1y+c_1)+a(b+c)(a_1(\gamma_1+2\alpha_1x+\beta_1y)+\alpha_1b_1y+\alpha_1c_1) \\ & -(b+c)(+3a_1^3x^2(b_1y+c_1)+\alpha_1(\gamma_1+\alpha_1x+\beta_1y)(-b+3\mu(b_1y+c_1)^2 -(\gamma_1+\alpha_1x+\beta_1y)^2)+a_1^4x^3+a_1(b_1y+c_1)((b_1y+c_1)^2+c+3\mu(\gamma_1 +\alpha_1x+\beta_1y)(\gamma_1+3\alpha_1x+\beta_1y))+a_1^2x(3((b_1y+c_1)^2+\mu(\gamma_1+\alpha_1x +\beta_1y)(\gamma_1+2\alpha_1x+\beta_1y))+c))),\end{aligned}$$

its first integral is

$$H_5(x,y) = \frac{1}{2}\left(\frac{a^2}{b+c}-c\right)(a_1x+b_1y+c_1)^2+a(a_1x+b_1y+c_1)(\gamma_1+\alpha_1x+\beta_1y)-\frac{1}{4}(a_1x+b_1y+c_1)^4 -\frac{3}{2}\mu(a_1x+b_1y+c_1)^2(\gamma_1+\alpha_1x+\beta_1y)^2+\frac{1}{2}b(\gamma_1+\alpha_1x+\beta_1y)^2+\frac{1}{4}(\gamma_1+\alpha_1x+\beta_1y)^4.$$

Finally the differential nilpotent center $(\mathcal{C}_6)$ becomes

$$(9)\quad \begin{aligned}\dot{x} = & \frac{-1}{(b+c)(\alpha_1b_1-a_1\beta_1)}(a^2b_1(a_1x+b_1y+c_1)+a(b+c)(\beta_1(a_1x+c_1)+b_1(\gamma_1+\alpha_1x+2\beta_1y)) \\ & +(b+c)(-\beta_1\gamma_1+\alpha_1x+\beta_1y(3\mu(a_1x+c_1)^2-b+(\gamma_1+\alpha_1x+\beta_1y)^2 -3b_1^3y^2(a_1x+c_1)-b_1^4y^3-b_1(a_1x+c_1)((a_1x+c_1)^2+c+3\mu(\gamma_1+\alpha_1 x+\beta_1y)(\gamma_1+\alpha_1x+\beta_1y))-b_1^2y(3((a_1x+c_1)^2+\mu(\gamma_1+\alpha_1x+\beta_1y) (\gamma_1+\alpha_1x+2\beta_1y))+c))),\\ \dot{y} = & \frac{1}{(b+c)(\alpha_1b_1-a_1\beta_1)}(a^2a_1(a_1x+b_1y+c_1)+a(b+c)(a_1(\gamma_1+2\alpha_1x+\beta_1y)+\alpha_1b_1y+\alpha_1c_1) \\ & -(b+c)(+3a_1^3x^2(b_1y+c_1)+\alpha_1(\gamma_1+\alpha_1x+\beta_1y)(-b+3\mu(b_1y+c_1)^2 +(\gamma_1+\alpha_1x+\beta_1y)^2)+a_1^4x^3+a_1(b_1y+c_1)((b_1y+c_1)^2+c+3\mu(\gamma_1+ \alpha_1x+\beta_1y)(\gamma_1+3\alpha_1x+\beta_1y))+a_1^2x(3((b_1y+c_1)^2+\mu(\gamma_1+\alpha_1x+\beta_1 y)(\gamma_1+2\alpha_1x+\beta_1y))+c))),\end{aligned}$$

its first integral is

$$H_6(x,y) = \frac{1}{2}\left(\frac{a^2}{b+c}-c\right)(a_1x+b_1y+c_1)^2+a(a_1x+b_1y+c_1)(\gamma_1+\alpha_1x+\beta_1y)-\frac{1}{4}(a_1x+b_1y+c_1)^4 -\frac{3}{2}\mu(a_1x+b_1y+c_1)^2(\gamma_1+\alpha_1x+\beta_1y)^2+\frac{1}{2}b(\gamma_1+\alpha_1x+\beta_1y)^2-\frac{1}{4}(\gamma_1+\alpha_1x+\beta_1y)^4.$$

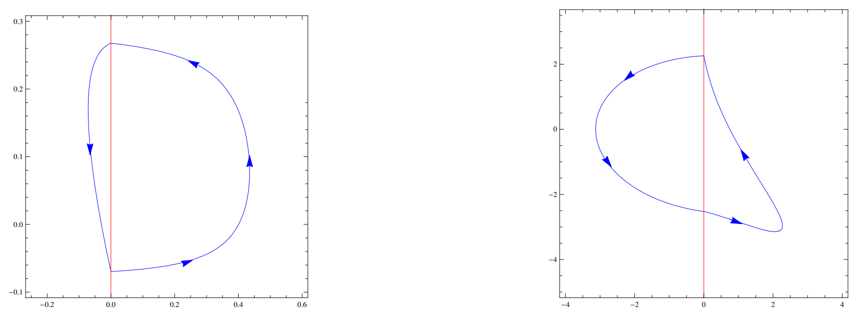
## 3. Proof of Theorem 3

Now we have to prove Theorem 3 for the class of discontinuous piecewise differential systems separated by the straight line $x=0$ and formed by the pair of differential systems (2)–($i$) for $i \in \{4,..,9\}$.

In one region we consider the linear differential center (2) with its first integral $H(x,y)$. In the other region we consider the Hamiltonian differential nilpotent center ($i$) with its first integral $H_{i-3}(x,y)$. If there exists a limit cycle of the discontinuous piecewise differential system (2)–($i$), it must intersects the

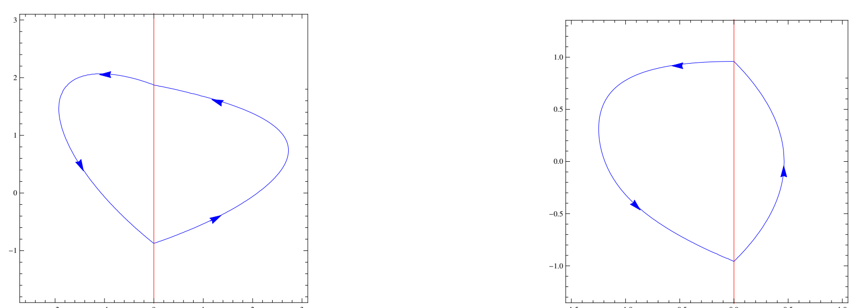
(a) (b)

FIGURE 1. $(a)$ The unique limit cycle of the discontinuous piecewise differential system (L)–(1), and $(b)$ is the unique limit cycle of the discontinuous piecewise differential system (L)–(2).



(a) (b)

FIGURE 2. $(a)$ The unique limit cycle of the discontinuous piecewise differential system (L)–(3), and $(b)$ is the unique limit cycle of the discontinuous piecewise differential system (L)–(4).



(a) (b)

FIGURE 3. $(a)$ The unique limit cycle of the discontinuous piecewise differential system (L)–(5) and, $(b)$ is the unique limit cycle of the discontinuous piecewise differential system (L)–(6).

separation line $x = 0$ in two distinct points $(0, y_1)$ and $(0, y_2)$ where $y_1 < y_2$. These two points must satisfy the system of equations

$$\begin{aligned} H(0,y_1) - H(0,y_2) =& \quad (y_2 - y_1)(-a^2 y_1 - a^2 y_2 + 2bd - w^2 y_1 - w^2 y_2), \\ =& \quad (y_1 - y_2) h(y_1, y_2), \\ H_{i-3}(0,y_1) - H_{i-3}(0,y_2) =& \quad h_{i-3}(y_1, y_2), \end{aligned} \tag{10}$$

where the two polynomials $h(y_1, y_2)$ and $h_{i-3}(y_1, y_2)$ for all $i \in \{4, .., 9\}$, are of degrees one and three, respectively. Since $y_1 < y_2$, by solving $h(y_1, y_2) = 0$, we get $y_2 = g(y_1)$ which is a function of the variable $y_1$. Substituting $y_2$ in $h_{i-3}(y_1, y_2) = 0$ we obtain a quadratic equation in the variable $y_1$. This equation has at most two real solutions $(y, f(y))$ and $(Y, f(Y))$. In fact, these two solutions represent the same solution of (10), due to the symmetry $(y, f(y)) = (f(Y), Y)$. Then both solutions provide the same limit cycle for the discontinuous piecewise differential system (2)–($i$). So, we have proved that there is at most one limit cycle for the discontinuous piecewise differential system (2)–($i$).

**One limit cycle for the class formed by system** (2)–(4)**.** In what follows we give a discontinuous piecewise differential system of the class (2)–(4) with one limit cycle. In the region $R_2$ we consider the Hamiltonian nilpotent center

$$(11)\quad \begin{aligned} \dot{x} = & \frac{1}{1500100}(100\left(-x^3 - 30x^2(y-1) + x(13714 - 300(y-2)y) - 1000(y-3)y^2\right) - 1302001y \\ & +1050950), \\ \dot{y} = & \frac{1}{150010}(x^3 + 30x^2(y-1) + 100x(3(y-2)y + 1963) + 20(y(50(y-3)y - 6857) + 6600)), \end{aligned}$$

this system has the first integral

$$\begin{aligned} H_1(x,y) = & -\frac{1}{4}\left(\frac{x}{10} + y - 1\right)^4 - 5\left(\frac{x}{10} + y - 1\right)^2 - \left(\frac{y+50}{100} - 15x\right)\left(\frac{x}{10} + y - 1\right) - \frac{1}{200000}(-1500x \\ & +y + 50)^2. \end{aligned}$$

In the region $R_1$ we consider the linear differential center

$$(12)\qquad \dot{x} = -\frac{x}{10} - \frac{101y}{1000} + \frac{1}{10}, \quad \dot{y} = \frac{x}{10} + \frac{y}{10} + \frac{1}{2},$$

with the first integral

$$H(x,y) = \frac{1}{10000}(200(x-1)y + 100x(x+10) + 101y^2).$$

Now we focus in the solutions of system (10) with $i = 4$ satisfying $y_1 < y_2$, and it has the unique solution $(y_1, y_2) = (-5.45516.., 7.43536..)$, which provides the unique limit cycle of the piecewise discontinuous differential system (11)–(12), see Figure 1(a).

**One limit cycle for the class formed by system** (2)–(5)**.**We consider the differential center (5) in the region $R_2$

$$(13)\quad \begin{aligned} \dot{x} = & \frac{1}{290000}(91x^3 + 60x^2(455 - 31y) + 300x(y(33y - 950) + 7722) - 1000(y(4(y-30)y \\ & +981) + 585)), \\ \dot{y} = & \frac{1}{290000}(12x^3 + 4035x^2 + 60(31x + 2375)y^2 - 39(7x(x+200) + 59400)y + 1294510x \\ & -3300y^3 + 7844000, \end{aligned}$$

this system has the first integral

$$\begin{aligned} H_2(x,y) = & \frac{1}{10}\left(-\frac{x}{100} + \frac{y}{10} - \frac{3}{2}\right)(-3x + y + 5) - \frac{1}{20}\left(-\frac{x}{100} + \frac{y}{10} - \frac{3}{2}\right)^2 - \frac{1}{20}(-3x + y + 5)^2 \\ & -\left(-\frac{x}{100} + \frac{y}{10} - \frac{3}{2}\right)^3 (-3x + y + 5). \end{aligned}$$

In the other region $R_1$ we consider the linear differential center

$$(14)\qquad \dot{x} = -\frac{x}{10} - \frac{101y}{100} - \frac{1}{10}, \quad \dot{y} = x + \frac{y}{10} - \frac{3}{10},$$

with its first integral

$$H(x,y) = \left(x + \frac{y}{10}\right)^2 + 2\left(\frac{y}{10} - \frac{3x}{10}\right) + y^2.$$

For the discontinuous piecewise differential centers (13)–(14), the unique solution of system (10) with $i = 5$ such that $y_1 < y_2$ is $(y_1, y_2) = (-3.57003.., 3.37201..)$. This proves the uniqueness of the limit cycle of the discontinuous differential system (13)–(14), see Figure 1(b).

**One limit cycle for the class formed by system** (2)–(6). We consider the differential cubic nilpotent center (6) in the region $R_1$

$$(15)\quad \begin{aligned} \dot{x} =& \frac{1}{6990000}(-1472099x^3 - 210x^2(704000y - 80207) - 100x(600y(1051y - 3650) + 212897) \\ & -1000\left(100y\left(50y^2 - 456y + 809\right) - 7149\right)), \\ \dot{y} =& \frac{1}{699000000}(2939999x^3 + 30x^2(14720990y - 1574999) + 700x(60y(352000y - 80207) \\ & -428929) + 1000(10y(200y(1051y - 5475) + 212897) + 237501)), \end{aligned}$$

this system has the first integral

$$H_3(x,y) = \frac{1}{20}\left(5\left(\frac{x-10}{100}+y\right)^4 + (7x+y-5)^2 - \frac{3(7x+y-5)^2(x+100y-10)^2}{1000}\right).$$

In the region $R_2$ we consider the linear differential center

$$(16)\quad \dot{x} = -\frac{x}{10} - \frac{101y}{1000} + \frac{1}{100}, \quad \dot{y} = \frac{x}{10} + \frac{y}{10} - \frac{3}{100},$$

with the first integral

$$H(x,y) = \left(\frac{x}{10}+\frac{y}{10}\right)^2 + \frac{1}{5}\left(-\frac{3x}{100} - \frac{y}{100}\right) + \frac{y^2}{10000}.$$

This discontinuous piecewise differential system formed by the differential centers (16)–(15) has exactly one limit cycle, because system (10) when $i = 6$. has exactly one real solution $(y_1, y_2)$ satisfying $y_1 < y_2$, namely $(y_1, y_2) = (-0.0697386.., 0.267758..)$, that provides the limit cycle shown in Figure 2(a).

**One limit cycle for the class formed by system** (2)–(7). We consider the differential cubic nilpotent center (7) in the region $R_1$

$$(17)\quad \begin{aligned} \dot{x} =& \frac{1}{980}(6(737x+3094)y^2 - (33x(31x+364)+33292)y - 10(x(x(191x+1785)+4732)+2892) \\ & -2344y^3), \\ \dot{y} =& \frac{1}{980}(10\left(573x^2+3570x+4732\right)y + 33(31x+182)y^2 + 5(x(5x(97x+798)+14042)+20924) \\ & -1474y^3), \end{aligned}$$

this system has the first integral

$$\begin{aligned} H_4(x,y) =& -5\left(2x - \frac{9y}{10} + 7\right)^2 - \frac{1}{80}(x+2y+2)^2 - \frac{1}{20}(20x-9y+70)(x+2y+2)\frac{1}{800}(-3)(20x-9y \\ & +70)^2(x+2y+2)^2 - \frac{1}{64}(x+2y+2)^4. \end{aligned}$$

In the region $R_2$ we consider the linear differential center

$$(18)\quad \dot{x} = -\frac{x}{10} - \frac{113y}{50} - \frac{3}{10}, \quad \dot{y} = x + \frac{y}{10} - \frac{1}{2},$$

with the first integral

$$H(x,y) = \left(x + \frac{y}{10}\right)^2 + 2\left(\frac{3y}{10} - \frac{x}{2}\right) + \frac{9y^2}{4}.$$

This discontinuous piecewise differential system formed by the differential centers (17)–(18) has exactly one limit cycle, because system (10), when $i = 7$ has exactly one real solution $(y_1, y_2)$ satisfying $y_1 < y_2$, namely $(y_1, y_2) = (-2.06113.., 1.79564..)$, that provides the limit cycle shown in Figure 2(b).

**One limit cycle for the class formed by system** (2)–(8). We consider the differential center (8) in the region $R_2$

$$\begin{aligned} \dot{x} = & \frac{1}{10640000}(2307285x^3 - 3x^2(1602696995y + 417255538) + 3x(5y(224663690145y+ \\ & 81962755532) + 793469623268) - 5(y(25y(6388790068751y + 1159455046638) \\ & +499170228203556) - 446074006985264)), \\ \dot{y} = & \frac{1}{53200000}(-15425x^3 + x^2(572720190 - 34609275y) + x(15y(1602696995y+ \\ & 834511076) - 3519832393532) + 7135887059545552 - 15y(5y(74887896715y \\ & +40981377766) + 793469623268)), \end{aligned} \tag{19}$$

this system has the first integral

$$\begin{aligned} H_5(x, y) = & \frac{1}{2500}((-3x + 5y + 18118)^4) - \frac{1}{500000}(3(-5x + 3555y + 212)^2(-3x + 5y + 18118)^2) \\ & + \left(-\frac{x}{2} + \frac{711y}{2} + \frac{106}{5}\right)^2 - \frac{1}{40000}((-5x + 3555y + 212)^4). \end{aligned}$$

In the other region $R_1$ we consider the linear differential center

$$\dot{x} = -\frac{x}{10} - \frac{y}{5} + \frac{1}{10}, \quad \dot{y} = \frac{x}{10} + \frac{y}{10} - \frac{1}{10}, \tag{20}$$

with its first integral

$$H(x, y) = \left(\frac{x}{10} + \frac{y}{10}\right)^2 + \frac{1}{5}\left(-\frac{x}{10} - \frac{y}{10}\right) + \frac{y^2}{100}.$$

For the discontinuous piecewise differential centers (19)–(20), the unique solution of system (10) with $i = 8$ such that $y_1 < y_2$ is $(y_1, y_2) = (-0.875101.., 1.8751..)$. This proves the uniqueness of the limit cycle of the discontinuous differential system (20)–(20), see Figure 3(a).

**One limit cycle for the class formed by system** (2)–(9). We consider the differential cubic nilpotent center (9) in the region $R_1$

$$\begin{aligned} \dot{x} = & \frac{1}{490000}(10(-765(4932x + 343)y^2 + 54x(11711 - 15395x)y + x(1020(147 - 65x)x - 436303) \\ & -6251501y^3 + 879650 - 18870103y), \\ \dot{y} = & \frac{1}{49000}(3900x^2(51y - 49) + 10x(9y(9237y - 3332) + 82033) + y(27y(46580y - 11711) \\ & +436303) + 26000x^3 - 1247640), \end{aligned} \tag{21}$$

this system has the first integral

$$H_6(x, y) = \frac{1}{20}(-5\left(x + \frac{y}{10} - 5\right)^4 - 3\left(x + 5y + \frac{1}{10}\right)^2\left(x + \frac{y}{10} - 5\right)^2 - \left(x + \frac{y}{10} - 5\right)^2 - 5(x + 5y + \frac{1}{10})^4).$$

In the region $R_2$ we consider the linear differential center

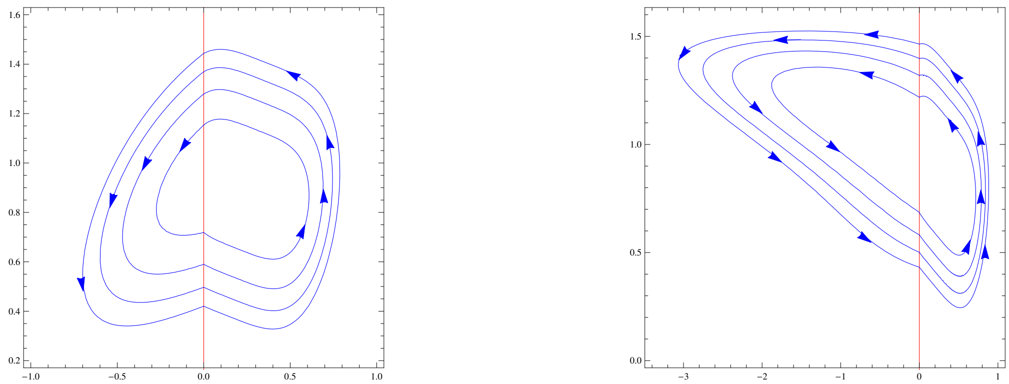
$$\dot{x} = -\frac{x}{10} - \frac{101y}{10} + \frac{1}{100}, \quad \dot{y} = \frac{x}{10} + \frac{y}{10} + 10, \tag{22}$$

with the first integral

$$H(x, y) = \left(\frac{x}{10} + \frac{y}{10}\right)^2 + \frac{1}{5}\left(-\frac{x}{10} - \frac{y}{10}\right) + \frac{y^2}{100}.$$

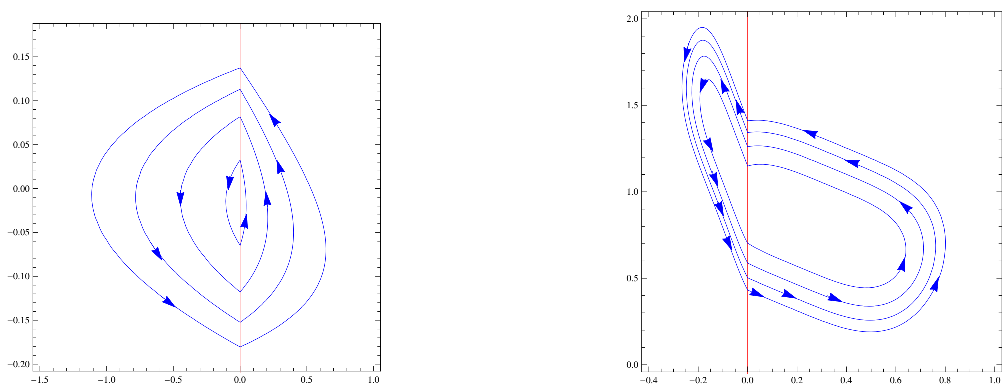
This discontinuous piecewise differential system formed by the differential centers (21)–(22) has exactly one limit cycle, because system (10), when $i = 9$ has exactly one real solution $(y_1, y_2)$ satisfying $y_1 < y_2$, namely $(y_1, y_2) = (-0.95781.., 0.95979..)$, that provides the limit cycle shown in Figure 3(b).

The proof of Theorem 3 is done.




$(a)$ $(b)$

FIGURE 4. $(a)$ The four limit cycles of the discontinuous piecewise differential system (4)–$(\widetilde{4})$, $(b)$ is the four limit cycles of the discontinuous piecewise differential system (4)–(5).




$(a)$ $(b)$

FIGURE 5. $(a)$ The four limit cycles of the discontinuous piecewise differential system (4)–(6), $(b)$ is the four limit cycles of the discontinuous piecewise differential system (4)–(7).

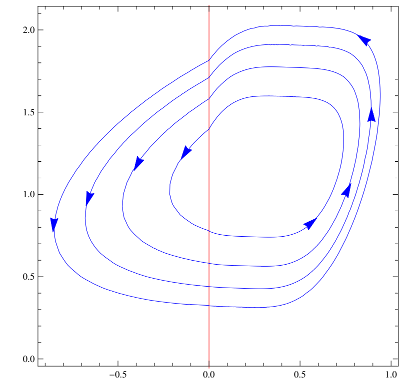
## 4. Proof of Theorem 4

In this section we prove Theorems 4, 5, 6, 7, 8 and 9 for the class formed by system $(i)$–$(\widetilde{i})$ and by system $(i)$–$(j)$ where $i \in \{4, .., 9\}$ and $j \in \{i+1, .., 9\}$.

In one region we consider the Hamiltonian cubic differential system with a nilpotent center after an affine change of variables $(i)$ such that $i \in \{4, .., 9\}$ with its first integral $H_{i-3}(x, y)$. By changing the parameters $(a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1)$ with the parameters $(a_2, b_2, c_2, \alpha_2, \beta_2, \gamma_2)$ in system $(i)$ and in its first integral, we get the second Hamiltonian cubic differential system with a nilpotent center $(\widetilde{i})$ with its corresponding first integral $\widetilde{H}_{i-3}\ (x, y)$ in the other region. If there exists a limit cycle of this discontinuous piecewise differential system $(i)$–$(\widetilde{i})$, it must intersect the separation line $x = 0$ in two distinct points $(0, y_1)$ and $(0, y_2)$, with $y_1 < y_2$. These two points must satisfy the system of equations

$$
\begin{aligned}
H_{i-3}(0, y_1) - H_{i-3}(0, y_2) &= \quad h_{i-3}(y, Y), \\
\widetilde{H}_{i-3}\ (0, y_1) - \widetilde{H}_{i-3}\ (0, y_2) &= \quad \widetilde{h}_{i-3}\ (y_1, y_2),
\end{aligned} \tag{23}
$$

where the two polynomials $h_{i-3}(y_1, y_2)$ and $\widetilde{h}_{i-3}\ (y_1, y_2)$ for all $i \in \{4, .., 9\}$ are of degree three. By Bézout Theorem (see for instance [14]), the maximum number of solutions of system (23) is nine. According to the symmetry of the solutions of this system we know that the maximum number of solutions satisfying $y_1 < y_2$ is four. Hence the discontinuous piecewise differential system $(i)$–$(\widetilde{i})$ with $i \in \{4, .., 9\}$ has at most four limit cycles.




$(a)$ $(b)$

FIGURE 6. $(a)$ The four limit cycles of the discontinuous piecewise differential system (4)–(8), $(b)$ is the four limit cycles of the discontinuous piecewise differential system (4)–(9).

For the class $(i)$–$(j)$, with $i \in \{4, .., 9\}$ and $j \in \{i+1, .., 9\}$, we consider in the first region the Hamiltonian cubic nilpotent center $(i)$ with its first integral $H_{i-3}(x, y)$. In the second region we consider the Hamiltonian cubic nilpotent center $(j)$ with its first integral $H_{j-3}(x, y)$. If there exists a limit cycle of the discontinuous piecewise differential systems $(i)$–$(j)$, it must intersects the separation line $x = 0$ in two distinct points $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. These two points must satisfy the system of equations

$$
\begin{array}{rcl} H_{i-3}(0, y_1) - H_{i-3}(0, y_2) & = & h_{i-3}(y_1, y_2), \\ H_{j-3}(0, y_1) - H_{j-3}(0, y_2) & = & h_{j-3}(y_1, y_2), \end{array} \tag{24}
$$

where the two polynomials $h_{i-3}(y_1, y_2)$ and $h_{j-3}(y_1, y_2)$ are of degree three. By using Bézout Theorem and taking into account the symmetry of the solutions of this system, the maximum number of solutions satisfying $y_1 < y_2$ is four. Hence the discontinuous piecewise differential system $(i)$–$(j)$ has at most four limit cycles.

For Theorems 5, 6, 7, 8 and 9 the proof that four is an upper bound for the maximum number of limit cycles that the corresponding discontinuous piecewise differential systems can exhibit is similar and we will not say in the proof of those respective theorems anything about this upper bound.

**Four limit cycles for the class formed by** (4)–$(\widetilde{4})$**.** In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$
\begin{array}{rl} \dot{x} = & 0.148627..x^3 + x^2(-0.200453..y - 0.037585..) + y((-0.0135046..y - 0.00759635..)y \\ & -0.755137..) + 0.725322.. + x((0.0901174..y + 0.033794..)y + 0.214281..), \\ \dot{y} = & 0.330599..x^3 + x^2(-0.44588..y - 0.0836025..) + y((-0.0300391..y - 0.016897..)y \\ & -0.214281..) + 0.202988.. + x((0.200453..y + 0.07517..)y + 0.0661793..), \end{array} \tag{25}
$$

with its first integral

$$
\begin{array}{rcl} H_1(x, y) & = & 0.1024..x^4 + x^3(-0.184143..y - 0.0345268..) + x^2((0.124177..y + 0.0465664..)y \\ & & +0.0409968..) + 0.43251.. + x(y((-0.0372174..y - 0.0209348..)y - 0.265486..) \\ & & +0.251494..) + y(y((0.00418293..y + 0.0031372..)y + 0.467793..) - 0.898647..). \end{array}
$$

In the region $R_2$ we consider the Hamiltonian differential cubic nilpotent center

$$(26)\quad \begin{aligned} \dot{x} = &\ \frac{1}{1883}(5(\frac{7}{10}(442x-224y+91)+\frac{4}{25}(70x-16y-3)-7(\frac{8}{5}\left(-7x+\frac{8y}{5}+\frac{3}{10}\right)^3+49(3x \\ &+7y-7)))), \\ \dot{y} = &\ \frac{5}{269}\left(-7\left(-7x+\frac{8y}{5}+\frac{3}{10}\right)^3+21(3x+7y-7)-35x-49(y-1)+\frac{16y}{5}+\frac{3}{5}\right), \end{aligned}$$

wich has the first integral

$$\begin{aligned} \widetilde{H}_1(x,y) = &\ \frac{1}{4}(\left(7x-\frac{8y}{5}\right)^4+\frac{27}{50}\left(7x-\frac{8y}{5}\right)^2+\frac{6}{5}\left(\frac{8y}{5}-7x\right)^3+\frac{27}{250}\left(\frac{8y}{5}-7x\right)+\frac{81}{10000}+14(3x+ \\ &7y-7)^2-\frac{2}{5}(70x-16y-3)(3x+7y-7)+\frac{1}{350}(-70x+16y+3)^2). \end{aligned}$$

The four real solutions of system (23) with $i=4$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (25)–(26) shown in Figure 4(a) are the set $S_{1,1}$ given by

$$S_{1,1}=\{(0.420187..,1.44392..),(0.496882..,1.3699..),(0.589761..,1.27971..),(0.718353..,1.15381..)\}.$$

**Four limit cycles for the class formed by system** (4)—(5)**.** In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(27)\quad \begin{aligned} \dot{x} = &\ \frac{1}{8676}(25(\frac{1}{2000}(31(600x-155y-34))-\frac{36}{5}(\frac{31}{20}\left(-6x+\frac{31y}{20}+\frac{17}{50}\right)^3+\frac{126}{25}(40x+70y \\ &-71))+\frac{9}{50}(1432x-868y+345))), \\ \dot{y} = &\ \frac{1}{1446}(5(\frac{1}{25}(-6)(1200x+895y-1099)-6(6\left(-6x+\frac{31y}{20}+\frac{17}{50}\right)^3-\frac{72}{25}(40x+70y \\ &-71))+30x-\frac{31y}{4}-\frac{17}{10}, \end{aligned}$$

of type (4) with the first integral

$$\begin{aligned} H_1(x,y) = &\ \frac{1}{4}(\left(6x-\frac{31y}{20}\right)^4+\frac{867}{1250}\left(6x-\frac{31y}{20}\right)^2+\frac{34}{25}\left(\frac{31y}{20}-6x\right)^3+\frac{4913}{31250}\left(\frac{31y}{20}-6x\right) \\ &+\frac{83521}{6250000}+\frac{5}{18}\left(-6x+\frac{31y}{20}+\frac{17}{50}\right)^2+\frac{18}{125}(40x+70y-71)^2 \\ &-\frac{1}{250}(600x-155y-34)(40x+70y-71)). \end{aligned}$$

In the half-plane $R_2$ we consider the Hamiltonian planar polynomial systems with linear plus cubic homogeneous terms has a nilpotent center

$$(28)\quad \begin{aligned} \dot{x} = &\ x^2(1.19984..-1.26886..y)+0.0300114..x^3+y((-0.107242..y-0.0705721..)y-6.97976..) \\ &+6.80631..+x(y(13.4167..y-25.3561..)+10.6326..), \\ \dot{y} = &\ x^2(0.0851568..-0.0900341..y)+y(y(12.678..-4.47222..y)-10.6326..)+0.00189308..x^3 \\ &+2.48977..+x(y(1.26886..y-2.39968..)+1.3799..), \end{aligned}$$

of type (5) with the first integral

$$\begin{aligned} H_2(x,y) = &\ x^2(y(126.761..-67.0262..y)-72.8918..)-0.05..x^4+x^3(3.17064..y-2.99889..)-345.933.. \\ &+x(y(y(472.482..y-1339.41..)+1123.31..)-263.039..)+y(y((-2.83248..y-2.48527..)y \\ &-368.699..)+719.073..). \end{aligned}$$

The four real solutions of system (24) with $i=4$ and $j=5$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (27)–(28) shown in Figure 4(b) are the set $S_{1,2}$ given by

$$S_{1,2}=\{(0.432123..,1.46424..),(0.500649..,1.39791..),(0.581365..,1.31939..),(0.684976..,1.21799..)\}.$$

**Four limit cycles for the class formed by system** (4)–(6)**.** In the half-plane $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(29)\quad \begin{aligned}\dot{x} = {} & 0.0000231195..x^3 + x^2(-0.00693586..y - 0.00667458..) + y((-23.1195..y - 66.7458..)y \\ & -95.7292..) - 1.47427.. + x((0.693586..y + 1.33492..)y - 0.693397..), \\ \dot{y} = {} & 2.3119546737227093.. - 7x^3 + x^2(-0.0000693586..y - 0.0000667458..) + y((-0.231195..y \\ & -0.667458..)y + 0.693397..) - 1.01726.. + x((0.00693586..y + 0.0133492..)y + 0.063066..),\end{aligned}$$

of type (4) with the first integral

$$\begin{aligned} H_1(x,y) = {} & -2.5 - 9(-1.x + 100.y + 96.2329)^4 - 0.000625(-1.x + 100.y + 96.2329)^2) - 0.0005(x \\ & -100.y - 96.2329)(1.x + 332.534y + 190.456 - 0.0001(1.x + 332.534y + 190.456)^2. \end{aligned}$$

In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(30)\quad \begin{aligned}\dot{x} = {} & \frac{1}{280}(3x^2(983 - 468y) + 3x\left(6200y^2 + 5812y - 3425\right) - 2(y(50y(292y + 843) + 60453) + 931) \\ & -298x^3), \\ \dot{y} = {} & \frac{1}{560}(-46x^3 + 3x^2(596y + 283) + x(12y(234y - 983) - 4171) - 4y^2(3100y + 4359) + 20550y \\ & +6594),\end{aligned}$$

of type (6) with the first integral

$$\begin{aligned} H_3(x,y) = {} & \frac{1}{4}(-(3x + 50y - 1)^2 + 4(x - 2(y + 2))(3x + 50y - 1) - 4(x - 2(y + 2))^2(x - 2(y + 2))^4 \\ & -\frac{3}{50}(3x + 50y - 1)^2(x - 2(y + 2))^2). \end{aligned}$$

The four real solutions of system (24) with $i = 4$ and $j = 6$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (29)–(30) shown in Figure 5(a) are the set $S_{1,3}$ given by

$$S_{1,3} = \{(-0.18047.., 0.13754..), (-0.15244.., 0.11308..), (-0.11784.., 0.08204..), (-0.064769.., 0.03252..)\}.$$

**Four limit cycles for the class formed by system** (4)–(7)**.** In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(31)\quad \begin{aligned}\dot{x} = {} & \frac{1}{4890600000}(32574287591 - 34036948070y + 800(49432500x^3 - 228150x^2(170y + 37) + 13x \\ & (9180y(85y + 37) - 1786537) - 5202y^2(170y + 111))), \\ \dot{y} = {} & \frac{1}{75240000}(303711290y - 298134377 + 50(49432500x^3 - 228150x^2(170y + 37) + 13x(9180y \\ & (85y + 37) + 320183) - 5202y^2(170y + 111))),\end{aligned}$$

of type (4) with the first integral

$$\begin{aligned} H_1(x,y) = {} & \frac{1}{3600000000}(625767092449 - 1302971503640y + 200(8032781250x^4 - 49432500x^3(170y + 37) \\ & +4225x^2(9180y(85y + 37) + 320183) + y^2(442170y(85y + 74) + 3403694807) \\ & -13x(170y(1530y(170y + 111) - 1786537) + 298134377))). \end{aligned}$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(32)\quad \begin{aligned}\dot{x} = {} & x^2(6.84222.. - 6.08187..y) + x(y(0.193773.. - 0.416004..y) - 0.804278..) - 0.787102..x^3 \\ & +0.27556.. + y((-0.00635887..y - 0.00415197..)y - 0.287934..), \\ \dot{y} = {} & 8.13871..x^3 + x^2(2.36131..y - 1.18342..) + y((0.138668..y - 0.0968865..)y + 0.804278..) \\ & -0.740904.. + x(y(6.08187..y - 13.6844..) + 12.0021..),\end{aligned}$$

of type (7) with the first integral

$$\begin{aligned} H_4(x,y) = {} & -4.06(x^4 + x^3(0.386844..y - 0.193876..) + x^2(y(1.49455..y - 3.3628..) + 2.9494..) + 0.06387.. \\ & +x(y((0.0681523..y - 0.0476176..)y + 0.395285..) - 0.364138..) + y(y((0.000781312..y \\ & +0.000680201..).y + 0.0707566..) - 0.135432..). \end{aligned}$$

The four real solutions of system (24) with $i=4$ and $j=7$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (31)–(32) shown in Figure 5(b) are the set $S_{1,4}$ given by

$$S_{1,4}=\{(0.43188..,1.41107..),(0.502868..,1.34301..),(0.588021..,1.26079..),(0.702739..,1.14902..)\}.$$

**Four limit cycles for the class formed by system** (4)–(8)**.** In the half-plane $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$\begin{aligned}\dot{x}=&\ \frac{1}{8151}(2(\frac{17}{100}(650x-170y-37)-75(\frac{17}{10000000}(-650x+170y+37)^3+\frac{21}{2}(26x+35y\\&-25))+\frac{3}{4}(3666x-2380y+591))),\\ \dot{y}=&\ \frac{1}{25080000}(50(16477500x^3-76050x^2(170y+37)+13x(3060y(85y+37)+112321)\\&-1734y^2(170y+111))+106305430y-74411959),\end{aligned}\tag{33}$$

of type (4) with the first integral

$$\begin{aligned}H_1(x,y)\ =&\ \frac{1}{1200000000}(110296574483-321888027880y+200(2677593750x^4-16477500x^3(170y+37)\\&+4225x^2(3060y(85y+37)+112321)+y^2(147390y(85y+74)+1178616769)-\\&13x(10y(8670y(170y+111)-10630543)+74411959))).\end{aligned}$$

In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$\begin{aligned}\dot{x}=&\ x^2(7.93667..-11.9089..y)+x(y(7.98447..-5.95413..y)-0.58277..)-6.24628..x^3\\&-0.696039..+y((0.021644..y+0.0142604..)y+1.01937..),\\ \dot{y}=&\ 8.60663..x^3+x^2(18.7389..y-12.4536..)+y(y(1.98471..y-3.99223..)+0.58277..)\\&+0.805031..+x(y(11.9089..y-15.8733..)+1.17967..),\end{aligned}\tag{34}$$

of type (8) with the first integral

$$\begin{aligned}H_5(x,y)\ =&\ 3.98198..x^4+x^3(11.5597..y-7.68244..)+x^2(y(11.0197..y-14.688..)+1.09158..)-0.4349..\\&+x(y(y(3.67301..y-7.38825..)+1.07851..)+1.48983..)+y(y((-0.0100139..y-0.008797..)\\&y-0.943252..)+1.28813..).\end{aligned}$$

The four real solutions of system (24) with $i=4$ and $j=8$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (33)–(34) shown in Figure 6(a) are the set $S_{1,5}$ given by

$$S_{1,5}=\{(0.239326..,1.09438..),(0.321936..,1.01403..),(0.321936..,1.01403..),(0.431104..,0.907126..)\}.$$

**Four limit cycles for the class formed by system** (4)–(9)**.** In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$\begin{aligned}\dot{x}=&\ \frac{1}{113200}(-80802x^2(15y+4)+67x(270y(15y+8)-37)-5y(810y(5y+4)+113089)\\&+1804578x^3+662916),\\ \dot{y}=&\ \frac{1}{566000}(-1804578x^2(15y+4)+x(404010y(15y+8)+435169)-335y(270y(5y+4)-37)\\&+40302242x^3-137276),\end{aligned}\tag{35}$$

of type (4) with the first integral

$$\begin{aligned}H_1(x,y)\ =&\ \frac{1}{40000}(20151121x^4-1203052x^3(15y+4)+x^2(404010y(15y+8)+435169)+3920656\\&-2x(335y(270y(5y+4)-37)+137276)+5y(5y(135y(15y+16)+113089)\\&-1325832)).\end{aligned}$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$
(36)\qquad
\begin{aligned}
\dot{x} = {} & 21.3589..x^3 + x^2(-56.2053..y - 45.3121..) + y((-21.1376..y - 16.9101..)y - 590.23..) \\
& +691.974.. + x(y(54.2197..y + 66.1925..) + 64.3626..), \\
\dot{y} = {} & 28.0462..x^3 + x^2(-64.0768..y - 72.6467..) + y((-18.0732..y - 33.0963..)y - 64.3626..) \\
& +0.0391091.. + x(y(56.2053..y + 90.6242..) + 77.2486..),
\end{aligned}
$$

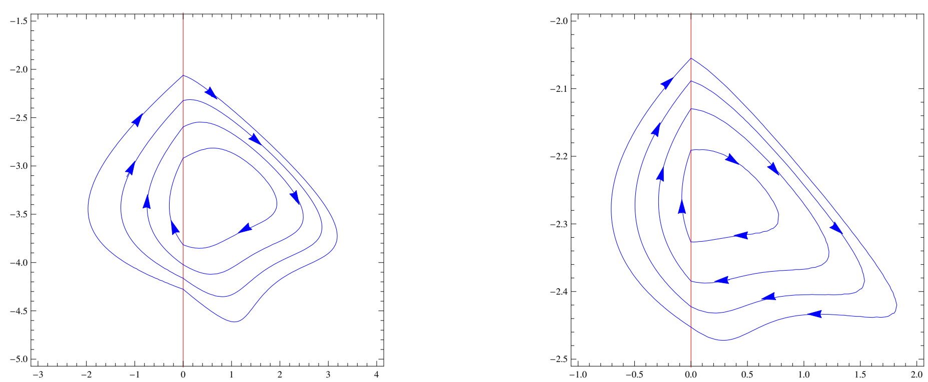
of type (9) with the first integral

$$
\begin{aligned}
H_6(x,y) = {} & -64.0648(x^4 + x^3(-3.04625..y - 3.45366..) + x^2(y(4.00804..y + 6.46248..) + 5.50866..) \\
& +65.0845.. + x(y((-2.57763..y - 4.72024..)y - 9.1795..) + 0.0055778..) + y(y((0.75367..y \\
& +0.803915..)y + 42.0898..) - 98.6904..).
\end{aligned}
$$

The four real solutions of system (24) with $i = 4$ and $j = 9$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (35)–(36) shown in Figure 6(b) are the set $S_{1,6}$ given by

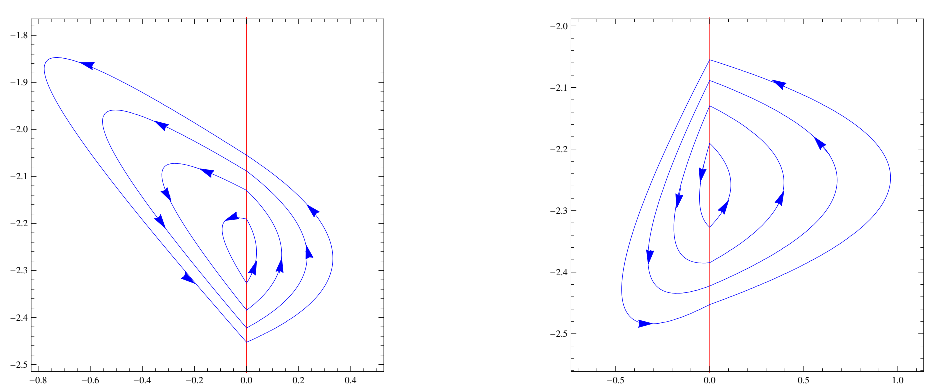
$$
S_{1,6} = \{(0.322324.., 1.81602..), (0.439432.., 1.71116..), (0.580949.., 1.582..), (0.777225.., 1.39821..)\}.
$$

The proof of Theorem 4 is done.



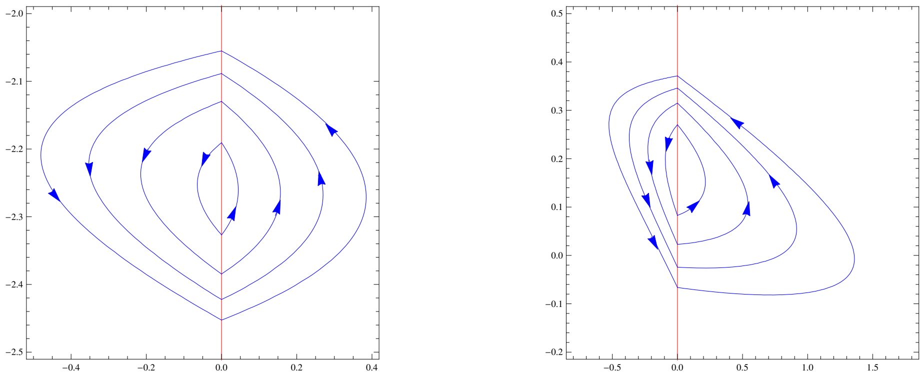
(a) (b)

FIGURE 7. $(a)$ The four limit cycles of the discontinuous piecewise differential system (5)–$(\widetilde{5})$, $(b)$ is the four limit cycles of the discontinuous piecewise differential system (5)–(6).



(a) (b)

FIGURE 8. $(a)$ The four limit cycles of the discontinuous piecewise differential system (5)–(7), $(b)$ is the four limit cycles of the discontinuous piecewise differential system (5)–(8).



$(a)$ $(b)$

FIGURE 9. $(a)$ The four limit cycles of the discontinuous piecewise differential system (5)–(9), $(b)$ is the four limit cycles of the discontinuous piecewise differential system (6)–$(\widetilde{6})$.

## 5. Proof of Theorem 5

**Four limit cycles for the class formed by system** $(5)$–$(\widetilde{5})$**.** In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$
(37)\quad \begin{aligned}
\dot{x} = & \frac{1}{2040000}(-44800x^3 + 960x^2(3190y + 9887) - 60x(220y(1199y + 3965) - 5617) + 35727973 \\
& +2y(2420y(4400y + 14907) + 4609389)), \\
\dot{y} = & \frac{1}{5100000}(256000x^3 + 4800x^2(70y + 1793) - 480x(10y(1595y + 9887) + 1331) - 9154111 \\
& +50y(110y(2398y + 11895) - 16851)),
\end{aligned}
$$

with its first integral

$$
\begin{aligned}
H_2(x,y) = & -\left(\frac{x}{10} + y + \frac{449}{100}\right)\left(-\frac{2x}{5} + \frac{11y}{10} + \frac{1}{100}\right)^3 - \left(\frac{x}{10} + y + \frac{449}{100}\right)^2 - \frac{1}{400000000}(-40x + 110y + 1)^2 \\
& -\frac{1}{1000000}(40x - 110y - 1)(10x + 100y + 449).
\end{aligned}
$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$
(38)\quad \begin{aligned}
\dot{x} = & \ 0.276953..x^3 + x^2(1.68257..y + 5.21482..) + y((0.187723..y + 0.635998..)y + 0.0812631..) \\
& +0.31494.. + x(y(2.61772..y + 15.8565..) + 23.5608..), \\
\dot{y} = & -0.122318..x^3 + x^2(-0.830858..y - 2.59563..) + x((-1.68257..y - 10.4296..)y - 16.2432..) \\
& -22.7062.. + y((-0.872574..y - 7.92823..)y - 23.5608..),
\end{aligned}
$$

which has the first integral

$$
\begin{aligned}
\widetilde{H}_2(x,y) = & -0.0135..x^4 + x^3(-0.122267..y - 0.381968..) + x^2((-0.371406..y - 2.30221..)y - 3.58548..) \\
& -1.17785.. + x(y((-0.385219..y - 3.50011..)y - 10.4015..) - 10.0242..) + y(y((-0.0207187..y \\
& -0.0935921..)y - 0.0179378..) - 0.139038..).
\end{aligned}
$$

The four real solutions of system (23) with $i = 5$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (37)–(38) shown in Figure 7(a) are the set $S_{2,2}$ given by

$$
S_{2,2} = \{(-4.27681.., -2.06171..), (-4.16402.., -2.32352..), (-4.02224.., -2.59599..), (-3.81634.., -2.91886..)\}.
$$

**Four limit cycles for the class formed by system** (5)—(6)**.** In the half-plane $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$
(39)\quad \begin{aligned}
\dot{x} =&\ \frac{1}{318000}(-8820(187x+2446)y^2+1260x(67x+856)y+x(45x(734-27x)+5560472)\\
&-41906248y+9878400y^3+129395280),\\
\dot{y} =&\ \frac{1}{318000}(-1260(67x+428)y^2+45x(81x-1468)y+2x(238679-15(x-249)x)\\
&+549780y^3-5560472y+2421420),
\end{aligned}
$$

of type (5) with the first integral

$$
\begin{aligned}
H_2(x,y) =&\ \frac{1}{12000000}(-72962(x-14y+30)^2+22920(x-120y-422)(x-14y+30)-1800(x-120y\\
&-422)^2-15(x-120y-422)(x-14y+30)^3).
\end{aligned}
$$

In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$
(40)\quad \begin{aligned}
\dot{x} =&\ 0.0920409..x^3+x^2(1.19586..y+2.68903..)+y((0.0174702..y-0.0381536..)y-0.0741121..)\\
&+0.228838..+x(y(3.68178..y+16.5284..)+18.5409..),\\
\dot{y} =&\ -0.0172082..x^3+x^2(-0.276123..y-0.621409..)+y((-1.22726..y-8.26419..)y-18.5409..)\\
&-13.8587..+x((-1.19586..y-5.37806..)y-6.04679..),
\end{aligned}
$$

of type (5) with the first integral

$$
\begin{aligned}
H_3(x,y) =&\ \frac{1}{4}(-6(-0.01..x+0.407922..y+0.921687..)^2(0.2..x+y+2.24993..)^2+(0.2..x+y+2.24993..)^4\\
&+0.00004..(-1.x+40.7922..y+92.1687..)^2.
\end{aligned}
$$

The four real solutions of system (24) with $i=5$ and $j=6$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (39)–(40) shown in Figure 7(b) are the set $S_{2,3}$ given by

$$
S_{2,3}=\{(-2.45273..,-2.05487..),(-2.42253..,-2.0884..),(-2.38468..,-2.12957..),(-2.32697..,-2.19059..)\}.
$$

**Four limit cycles for the class formed by system** (5)–(7)**.** In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$
(41)\quad \begin{aligned}
\dot{x} =&\ \frac{1}{169500}(2205(2446-367x)y^2+630x(751-127x)y+x(2865001-45x(47x+944))\\
&+10476562y2469600y^3-32348820),\\
\dot{y} =&\ \frac{1}{169500}(45x^2(141y+377)+x(90y(889y+944)-487498)+y(105y(2569y-2253)\\
&-2865001)+60x^3+985485),
\end{aligned}
$$

of type (5) with the first integral

$$
\begin{aligned}
H_2(x,y) =&\ \frac{1}{1500000}(-15(x+120y+422)(x+7y-15)^3-36481(x+7y-15)^2+5730(x+120y+422)\\
&(x+7y-15)-225(x+120y+422)^2).
\end{aligned}
$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$
(42)\quad \begin{aligned}
\dot{x} =&\ x^2(4.49553..-20.3414..y)+x(y(30.4253..-28.7946..y)+76.3734..)-4.63556..x^3\\
&-174.323..+y(y(29.0644..-13.3083..y)+56.4567..),\\
\dot{y} =&\ 3.06778..x^3+x^2(13.9067..y+1.03313..)+y(y(9.59821..y-15.2126..)-76.3734..)\\
&+8.93679..+x(y(20.3414..y-8.99106..)-53.9556..),
\end{aligned}
$$

of type (7) with the first integral

$$
\begin{aligned}
H_4(x,y) =&\ \frac{1}{4}(x^2(y(9.15875..-20.7208..y)+54.962..)-1.5625..x^4+x^3(-9.44405..y-0.7016..)\\
&-786.27..+x(y(y(30.9927..-19.5545..y)+155.596..)-18.2069..)+y(y(y(19.7377..\\
&-6.77827..y)+57.5097..)-355.149..)).
\end{aligned}
$$

The four real solutions of system (24) with $i = 5$ and $j = 7$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (41)–(42) shown in Figure 8(a) are the set $S_{2,4}$ given by

$$S_{2,4} = \{(-2.45273.., -2.05487..), (-2.42253.., -2.0884..), (-2.38468.., -2.12957..), (-2.32697.., -2.19059..)\}.$$

**Four limit cycles for the class formed by system** (5)–(8)**.** In the half-plane $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(43)\quad \begin{aligned} \dot{x} =& \frac{1}{60000}(9(-245(680x+1223)y^2+210x(320x+1023)y-3x(15x(200x+823)-101087)\\ &+137200y^3+16174410-5238281y)),\\ \dot{y} =& \frac{1}{20000}(-315(640x+1023)y^2+270x(300x+823)y-27x(5x(80x+267)-16641)\\ &+1881630+166600y^3-909783y), \end{aligned}$$

of type (5) with the first integral

$$\begin{aligned} H_2(x,y) =& \frac{1}{1500000}((-30(20x-60y-211)(3x-7y+15)^3-36481(3x-7y+15)^2+11460(20x-60y\\ &-211)(3x-7y+15)-900(-20x+60y+211)^2). \end{aligned}$$

In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(44)\quad \begin{aligned} \dot{x} =& x^2(13.6637..-2.68368..y)+x(y(2198.16..-277.103..y)-5181.26..)-0.0123584..x^3\\ &+549164.+y(y(41924.7..y-91560.6..)-177854.),\\ \dot{y} =& -0.000276416..x^3+x^2(0.0370751..y+0.176651..)+x(y(2.68368..y-27.3275..)\\ &-29.9689..)+427.069..+y(y(92.3677..y-1099.08..)+5181.26..), \end{aligned}$$

of type (8) with the first integral

$$\begin{aligned} H_5(x,y) =& \frac{1}{4}(-0.0006..x^4+x^3(0.107302..y+0.511262..)+x^2(y(11.6506..y-118.636..)-130.103..)\\ &+1.3308..\times 10^6+y(y(y(264993.-91003.4..y)+772111.)-4.76815..\times 10^6)+x(y(y\\ &(801.987..y-9542.81..)+44986.6..)+3708.05..). \end{aligned}$$

The four real solutions of system (24) with $i = 5$ and $j = 8$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (43)–(44) shown in Figure 8(b) are the set $S_{2,5}$ given by

$$S_{2,5} = \{(-2.45273.., -2.05487..), (-2.42253.., -2.0884..), (-2.38468.., -2.12957..), (-2.32697.., -2.19059..)\}.$$

**Four limit cycles for the class formed by system** (5)–(9)**.** In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(45)\quad \begin{aligned} \dot{x} =& \frac{1}{3652500}(-22032x^3-360x^2(3311y+2413)+x(1575y(4958-10031y)+59419145)-50(y\\ &(2205y(560y-1223)-5238281)+16174410)),\\ \dot{y} =& \frac{1}{18262500}(-768x^3+720x^2(459y+1733)+10x(180y(3311y+4826)-3991259)+25(y(105y\\ &(10031y-7437)-11883829)+2815815)), \end{aligned}$$

of type (5) with the first integral

$$\begin{aligned} H_2(x,y) =& \frac{1}{187500000}(-182405(4x+35y-75)^2-28650(x-600y-2110)(4x+35y-75)-1125\\ &(x-600y-2110)^2+3(x-600y-2110)(4x+35y-75)^3. \end{aligned}$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(46)\quad \begin{aligned} \dot{x} =& x(y(1093.52..-3239.16..y)+15822.9..)-178.362..x^3+x^2(-2024.74..y-3067.91..)\\ &-18986.8..+y(y(3165.61..-1449.5..y)+6149.1..),\\ \dot{y} =& 45.3362..x^3+x^2(535.086..y+835.977..)+y(y(1079.72..y-546.759..)-15822.9..)\\ &-21574.7..+x(y(2024.74..y+6135.83..)+4536.54..), \end{aligned}$$

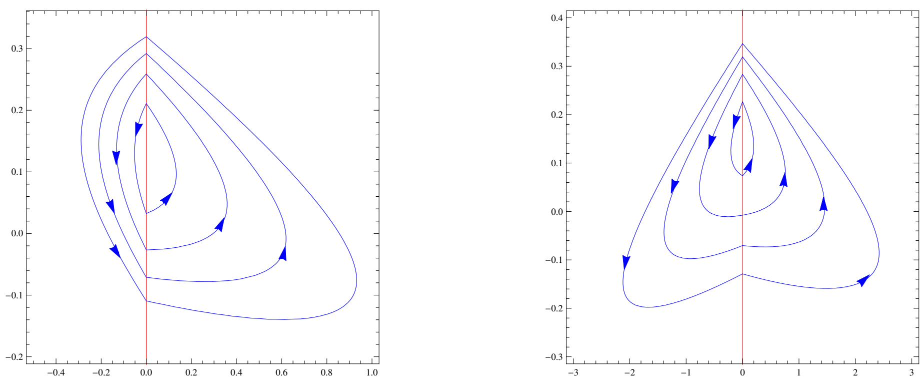
of type (9) with the first integral

$$
\begin{aligned}
H_6(x,y) \quad = \quad & -531.5..(x^4 + x^3(15.7368..y + 24.586..) + x^2(y(89.3211..y + 270.681..) + 200.129..) \\
& +3375.61.. + x(y(y(95.2634..y - 48.2404..) - 1396.05..) - 1903.53..) + y(y(y(31.9723.. \\
& y - 93.1004..) - 271.267..) + 1675.2..)).
\end{aligned}
$$

The four real solutions of system (24) with $i = 5$ and $j = 9$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (45)–(46) shown in Figure 9(a) are the set $S_{2,6}$ given by

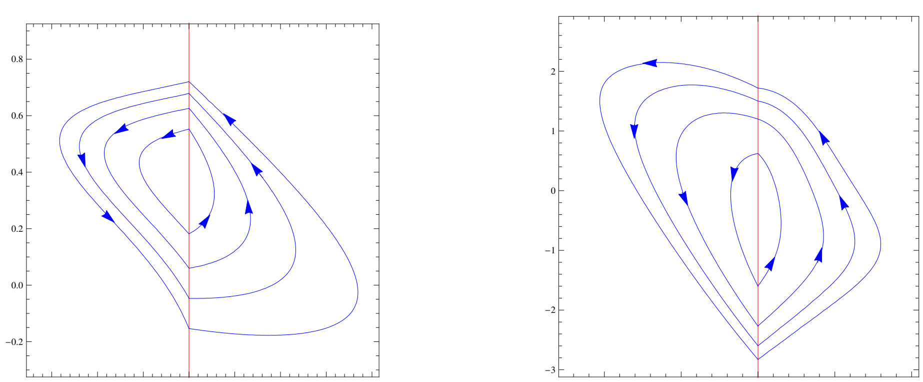
$$
S_{2,6} = \{(-2.45273.., -2.05487..), (-2.42253.., -2.0884..), (-2.38468.., -2.12957..), (-2.32697.., -2.19059..)\}.
$$

The proof of Theorem 5 is done.



$(a)$ $(b)$

FIGURE 10. $(a)$ The four limit cycle of the discontinuous piecewise differential system (6)–(7), $(b)$ is the four limit cycle of the discontinuous piecewise differential system (6)–(8).



$(a)$ $(b)$

FIGURE 11. $(a)$ The four limit cycle of the discontinuous piecewise differential system (6)–(9), $(b)$ is the four limit cycle of the discontinuous piecewise differential system (7)–($\widetilde{7}$).

## 6. Proof of Theorem 6

**Four limit cycles for the class formed by system** (6)–$(\widetilde{6})$**.** In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(47)\quad \begin{aligned} \dot{x} = &\ \frac{1}{2300}(-x(240y(765y+1634)+154871)-10(y(1350y(94y+173)+69677)-21159)+231x^3 \\ &+x^2(600y-5493)), \\ \dot{y} = &\ \frac{1}{23000}(x(60y(1831-100y)+174083)+10(y(240y(255y+817)+154871)+268)+215x^3 \\ &-42x^2(165y+307)), \end{aligned}$$

with its first integral

$$\begin{aligned} H_3(x,y) \ = \ & \frac{1}{4}(\Big(-\frac{x}{10}+3y+4\Big)^4-4\Big(-\frac{x}{10}+3y+4\Big)^2-\frac{1}{25}(x-30y-40)(6x+50y-1)-\frac{1}{100}(6x+ \\ & 50y-1)^2-\frac{3}{5000}(x-30y-40)^2(6x+50y-1)^2). \end{aligned}$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(48)\quad \begin{aligned} \dot{x} = &\ -0.00706076..x^3+x^2(0.366337..y+0.347189..)+y((-17.2644..y-31.7738..)y-9.47936..) \\ &+2.87862..+x((-0.320759y-4.23999)y-3.06582), \\ \dot{y} = &\ 0.0028674..x^3+x^2(0.0211823..y-0.031237..)+x((-0.366337..y-0.694378..)y+0.40340..) \\ &-1.70916..+y((0.10692y+2.11999)y+3.06582), \end{aligned}$$

which has the first integral

$$\begin{aligned} \widetilde{H}_3\ (x,y \ = \ & x^3(0.00181564..-0.00123121..y)-0.000125..x^4+x^2((0.0319398..y+0.0605407..) \\ & y-0.0351717..)-0.894789..+x(y((-0.018644..y-0.369671..)y-0.534599..) \\ & +0.298033..)+y(y((-0.752614..y-1.84684..)y-0.826476..)+0.501956..). \end{aligned}$$

The four real solutions of system (23) with $i=6$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (47)–(48) shown in Figure 9(b) are the set $S_{3,3}$ given by

$$S_{3,3}=\{(-0.066374..,0.37132..),(-0.024393..,0.34618..),(0.022844..,0.31494..),(0.082568..,0.27046..)\}.$$

**Four limit cycles for the class formed by system** (6)—(7)**.** In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(49)\quad \begin{aligned} \dot{x} = &\ \frac{1}{1575000}(178200x^3+43875x^2(188y+145)-x(54000y(435y+2741)+124969919)-20(y \\ &(4500y(2920y+8511)+23084027)-3511546)), \\ \dot{y} = &\ \frac{1}{31500000}(4787100x^3-267300x^2(40y+353)+x(426674243-1755000y(94y+145))+20(y \\ &(27000y(290y+2741)+124969919)+3839138)), \end{aligned}$$

of type (6) with the first integral

$$\begin{aligned} H_3(x,y) \ = \ & \frac{1}{4}\left(-\frac{3x}{10}+2y+4\right)^4-\frac{9}{10}\left(-\frac{3x}{10}+2y+4\right)^2+2\left(x+5y-\frac{1}{100}\right)(-\frac{3x}{10}+2y+4)-\frac{3}{2} \\ & \left(-\frac{3x}{10}+2y+4\right)^2\left(x+5y-\frac{1}{100}\right)^2-\frac{10}{9}\left(x+5y-\frac{1}{100}\right)^2. \end{aligned}$$

In the half-plane $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(50)\quad \begin{aligned} \dot{x} = &\ -0.0202932..x^3+x^2(0.385691..y+1.29997..)+y((-91.4913..y-266.672..)y-160.73..) \\ &+24.4502..+x(y(4.25192..y-6.95693..)-20.8016..), \\ \dot{y} = &\ y(y(3.47847..-1.41731..y)+20.8016..)+0.00535288..x^3+x^2(0.0608796..y-0.227359..) \\ &-20.3172..+x((-0.385691..y-2.59993..)y+3.33596..), \end{aligned}$$

of type (7) with the first integral

$$H_4(x,y) = \frac{1}{4}(-0.0001..(-1.x - 15.y + 6.19385..)^4 - 0.4..(-1.x + 8.35192..y + 22.7859..)^2 - 0.0024..(-x - 15.y + 6.19385..)^2(-1.x + 8.35192..y + 22.7859..)^2).$$

The four real solutions of system (24) with $i = 6$ and $j = 7$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (49)–(50) shown in Figure 10(a) are the set $S_{3,4}$ given by

$$S_{3,4} = \{(-0.10950.., 0.31947..), (-0.071226.., 0.29253..), (-0.026675.., 0.25905..), (0.032266.., 0.21093..)\}.$$

**Four limit cycles for the class formed by system (6)–(8).** In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$\begin{aligned}\dot{x} &= \frac{1}{65000}(5150x^3 + 15x^2(3000y - 7217) - 3x(1600y(1675y + 2749) + 1332199) - 20(2y(200y \\ &\quad (10000y + 13773) + 570303) - 323519)),\\ \dot{y} &= \frac{1}{2600000}(14900x^3 - 150x^2(4120y + 5923) + x(1200y(7217 - 1500y) + 11294003) + 40(y \\ &\quad (800y(3350y + 8247) + 3996597) + 154122)),\end{aligned} \tag{51}$$

of type (6) with the first integral

$$H_3(x,y) = \frac{1}{4}(\left(-\frac{x}{10} + 4y + 4\right)^4 - 6\left(\frac{x}{2} + 6y - \frac{1}{100}\right)^2\left(-\frac{x}{10} + 4y + 4\right)^2 + 2\left(\frac{x}{2} + 6y - \frac{1}{100}\right)^2).$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$\begin{aligned}\dot{x} &= 130.423..x^3 + x^2(-4227.15..y - 2560.18..) + y((-141687.y - 195146.)y - 40402.3..) \\ &\quad +11459.6.. + x(y(43422.8..y + 46247.9..) + 7549.92..),\\ \dot{y} &= 11.3396..x^3 + x^2(-391.268..y - 270.532..) + y((-14474.3..y - 23124.)y - 7549.92..) \\ &\quad +202.374.. + x(y(4227.15..y + 5120.37..) + 1034.18..),\end{aligned} \tag{52}$$

of type (8) with the first integral

$$\begin{aligned}H_5(x,y) &= -128.627..x^4 + x^3(5917.67..y + 4091.62..) + x^2((-95899.2..y - 116163.)y - 23461.8..) \\ &\quad +229911. + x(y(y(656741.y + 1.0492.. \times 10^6) + 342563.) - 9182.31..) + y(y((-1.60719.. \\ &\quad \times 10^6 y - 2.95145.. \times 10^6)y - 916588.) + 519958.).\end{aligned}$$

The four real solutions of system (24) with $i = 6$ and $j = 8$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (51)–(52) shown in Figure 10(b) are the set $S_{3,5}$ given by

$$S_{3,5} = \{(-0.1288.., 0.34684..), (-0.070429.., 0.31926..), (-0.0076186.., 0.28364..), (0.073805.., 0.22667..)\}.$$

**Four limit cycles for the class formed by system (6)–(9).** In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$\begin{aligned}\dot{x} &= \frac{1}{4300}(-6x(5y(9675y + 19508) + 35837) - 20(y(135y(430y + 719) + 8137) - 20774) + 516x^3 \\ &\quad -15x^2(301y + 1021)),\\ \dot{y} &= \frac{1}{21500}(3x(25y(301y + 2042) + 54561) + 10(3y(5y(3225y + 9754) + 35837) + 3628) + 215x^3 \\ &\quad -6x^2(1290y + 2149)),\end{aligned} \tag{53}$$

of type (6) with the first integral

$$H_3(x,y) = \frac{1}{4}\left(-\frac{x}{10} + 3y + 4\right)^4 - \frac{3}{2}\left(\frac{3x}{5} + \frac{7y}{2} - \frac{1}{10}\right)^2\left(-\frac{x}{10} + 3y + 4\right)^2 + \left(\frac{3x}{5} + \frac{7y}{2} - \frac{1}{10}\right)^2.$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(54)\quad \begin{aligned} \dot{x} = \;& x^2(815.182.. - 828.334..y) - 108.667..x^3 + y((-3368.19..y - 5631.92..)y - 472.126..) \\ & +1205.35.. + x(y(3908.52..y + 1656.62..) - 1781.38..), \\ \dot{y} = \;& 38.3667..x^3 + x^2(326.y - 294.748..) + y(y(-1302.84..y - 828.31..) + 1781.38..) - 617.22.. \\ & +x(y(828.334..y - 1630.36..) + 750.82..), \end{aligned}$$

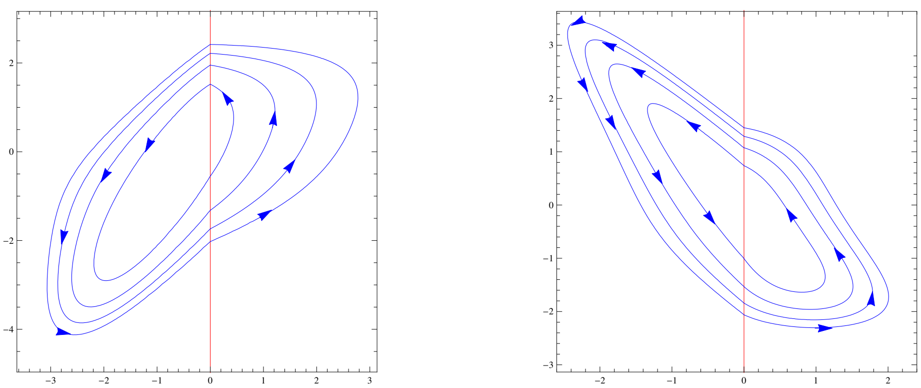
of type (9) with the first integral

$$\begin{aligned} H_6(x,y) \;=\; & -156.25..(x - 1.08253..y - 1.75156..)^4 - 2.5(x - 1.08253..y - 1.75156..)^2 - 375.(x \\ & -1.08253..y - 1.75156..)^2(x + 10.y - 4.04242..)^2 - 0.25..(x + 10.y - 4.04242..)^4. \end{aligned}$$

The four real solutions of system (24) with $i = 6$ and $j = 9$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (53)–(54) shown in Figure 11(a) are the set $S_{3,6}$ given by

$$S_{3,6} = \{(-0.153565.., 0.720851..), (-0.0467576.., 0.678862..), (0.0599189.., 0.626584..), (0.18123.., 0.552656..)\}.$$

The proof of Theorem 6 is done.



$(a)$ $(b)$

FIGURE 12. $(a)$ The four limit cycle of the discontinuous piecewise differential system (7)–(8), $(b)$ is the four limit cycle of the discontinuous piecewise differential system (7)–(9).

## 7. Proof of Theorem 7

**Four limit cycles for the class formed by system** (7)–($\widetilde{7}$)**.** In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(55)\quad \begin{aligned} \dot{x} = \;& \frac{1}{8100}(100x(39y(17y - 2) - 368) - 2000(y(y(22y + 39) + 44) + 12) - 4744x^3 - 15x^2 \\ & (6097y + 6574)), \\ \dot{y} = \;& \frac{1}{40500}(25x(3y(6097y + 13148) + 53180) - 500(y(13y(17y - 3) - 368) - 1308) + 2401x^3 \\ & +60x^2(1186y + 1231)), \end{aligned}$$

with its first integral

$$H_4(x,y) \;=\; -\frac{1}{4}\left(\frac{x}{10} + 2y + 2\right)^4 - \frac{5}{2}(-4x + y - 2)^2 - \frac{3}{800}(-4x + y - 2)^2(x + 20(y + 1))^2.$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(56)\quad \begin{aligned} \dot{x} = \;& -4.13647..x^3 + x^2(-9.24004..y - 2.27664..) + x((-7.40128..y - 5.43808..)y - 3.06562..) \\ & -1.2533.. + y((-2.29771..y - 4.07322..)y - 4.59542..), \\ \dot{y} = \;& 5.63647..x^3 + x^2(12.4094..y + 2.746..) + y(y(2.46709..y + 2.71904..) + 3.06562..) - 4.77918.. \\ & +x(y(9.24004..y + 4.55327..) + 6.59598..), \end{aligned}$$

wich has the first integral

$$\widetilde{H}_4\,(x,y) \quad = \quad -0.375(x+y+0.685221..)^2(-1.x-0.11292..y+1.05753..)^2-1.25..(-1.x-0.11292..y+1.05753..)^2-0.25..(x+y+0.685221..)^4.$$

The four real solutions of system (23) with $i=7$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (55)–(56) shown in Figure 11(b) are the set $S_{4,4}$ given by

$$S_{4,4}=\{(-2.82875..,1.71996..),(-2.59836..,1.50325..),(-2.26931..,1.20032..),(-1.60171..,0.62322..)\}\,.$$

**Four limit cycles for the class formed by system** (7)—(8)**.** In the half-plane $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(57)\qquad \begin{aligned}\dot{x} = {} & \frac{1}{18600}(x(225y(1017y+236)+343573)-5(y(2025y(11y+2)+41881)-36946)-10449x^3\\ & -45x^2(3133y+2482)),\\ \dot{y} = {} & \frac{1}{93000}(x(225y(3133y+4964)+4421809)-5(y(225y(339y+118)+343573)-425818)\\ & +7203x^3+45x^2(3483y+2522)),\end{aligned}$$

of type (7) with the first integral

$$H_4(x,y) \quad = \quad \frac{1}{3}(-50)(-2x+y-1)^2+\frac{1}{10}(2x-y+1)(x+15y+10)-\frac{3}{20000}(x+15y+10)^2-\frac{1}{40000}(x+15y+10)^4-\frac{3}{200}(-2x+y-1)^2(x+15y+10)^2.$$

In the half-plane $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(58)\qquad \begin{aligned}\dot{x} = {} & 87.0946..x^3+x^2(-1306.36..y-88.2943..)+y((-10886.y-1979.26..)y-20467.6..)\\ & +18055.8..+x(y(6531.61..y+837.747..)+4025.15..),\\ \dot{y} = {} & 17.42..x^3+x^2(-261.284..y-18.3513..)+y((-2177.2..y-418.874..)y-4025.15..)\\ & +49010.8..+x(y(1306.36..y+176.589..)+628.569..),\end{aligned}$$

of type (8) with the first integral

$$H_5(x,y) \quad = \quad 4.00058..x^4+x^3(-80.0064..y-5.61926..)+x^2(y(600.02..y+81.1085..)+288.706..)-8.723..\times10^6+x(y((-2000.01..y-384.784..)y-3697.57..)+45022.1..)+y(y(y(2500.y+606.061..)+9400.91..)-16586.3..).$$

The four real solutions of system (24) with $i=7$ and $j=8$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (57)–(58) shown in Figure 12(a) are the set $S_{4,5}$ given by

$$S_{4,5}=\{(-2.02489..,2.41933..),(-1.7375..,2.21959..),(-1.32018..,1.9543..),(-0.552275..,1.52342..)\}\,.$$

**Four limit cycles for the class formed by system** (7)–(9)**.** In the half-plane $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$(59)\qquad \begin{aligned}\dot{x} = {} & \frac{1}{3700000}(-3691300x^3-2100x^2(26267y+22148)-y(9100y(4771y+7074)+99840207)\\ & -398268-3x(3100y(10387y+12554)+47561113)),\\ \dot{y} = {} & \frac{1}{3700000}(540100x^3+300x^2(36913y+32421)+y(3100y(10387y+18831)+142683339)\\ & -2692564+x(2100y(26267y+44296)+218366903)),\end{aligned}$$

of type (7) with the first integral

$$H_4(x,y) \quad = \quad \frac{1}{2000000}(-50(x+13y+12)^4-500(x+13y+12)^2+2000(300x+200y+1)(x+13y+12)-3(x+13y+12)^2(300x+200y+1)^2-2000(300x+200y+1)^2\big)\,.$$

In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(60)\quad \begin{aligned} \dot{x} = &\ 0.0759897..x^3 + x^2(1.21879..y + 0.430192..) + y((-1.6343..y - 2.42318..)y - 3.75825..) \\ &-0.0149919.. + x(y(2.19373..y + 1.18861..) - 5.21479..), \\ \dot{y} = &\ 0.010401..x^3 + x^2(-0.227969..y - 0.0969069..) + x(y(-1.21879..y - 0.860385..) \\ &+10.5872..) - 0.284961.. + y((-0.731243..y - 0.594303..)y + 5.21479..), \end{aligned}$$

of type (9) with the first integral

$$\begin{aligned} H_6(x,y) = &\ -0.0007..x^4 + x^3(0.0204567..y + 0.00869591..) + x^2((0.164052..y + 0.11581..)y - 1.42505..) \\ &-0.00515174.. + x(y((0.196854..y + 0.159989..)y - 1.40384..) + 0.0767126..) + y(y(( \\ &-0.10999..y - 0.217444..)y - 0.505868..) - 0.00403587..). \end{aligned}$$

The four real solutions of system (24) with $i = 7$ and $j = 9$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (59)–(60) shown in Figure 12(b) are the set $S_{4,6}$ given by

$$S_{4,6} = \{(-2.06178.., 1.45127..), (-1.84186.., 1.28924..), (-1.5337.., 1.07579..), (-1.00683.., 0.739019..)\}.$$

The proof of Theorem 7 is done.

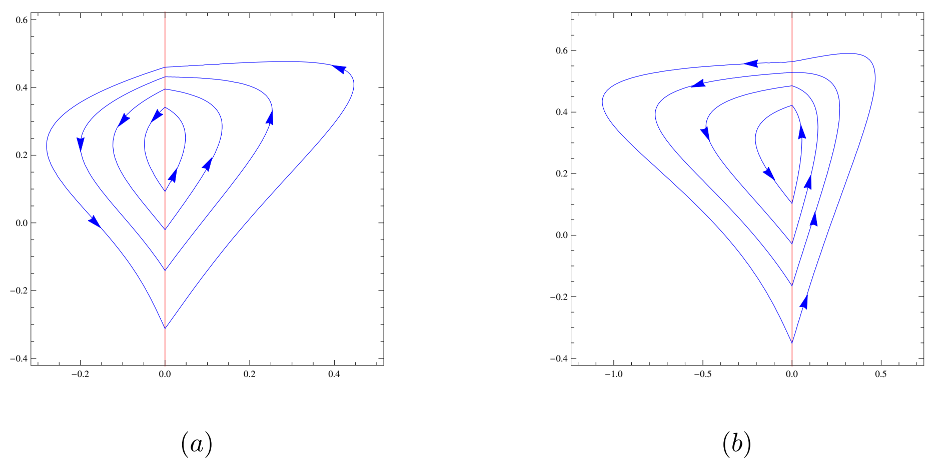


FIGURE 13. $(a)$ The four limit cycle of the discontinuous piecewise differential system (8)–$(\widetilde{8})$, $(b)$ is the four limit cycle of the discontinuous piecewise differential system (8)–(9).

## 8. Proof of Theorem 8

**Four limit cycles for the class formed by system** (8)–$(\widetilde{8})$**.** In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$(61)\quad \begin{aligned} \dot{x} = &\ \frac{1}{15000}(-135x^2(5885y + 431) + 9x(5y(30775y + 8606) - 6498) - 75y^2(10375y + 4819) \\ &+128925x^3 + 4974y + 28448), \\ \dot{y} = &\ \frac{1}{5000}(135x^2(321 - 955y) + 3x(15y(5885y + 862) - 18014) + y(19494 - 5y(30775y \\ &+12909)) + 16528 - 2025x), \end{aligned}$$

with its first integral

$$\begin{aligned} H_5(x,y) = &\ \frac{1}{5000}(6075x^4 + 540x^3(955y - 321) - 18x^2(15y(5885y + 862) - 18014) + 39280 + 12x(y(5y \\ &(30775y + 12909) - 19494) - 16528) + y(y(9948 - 25y(31125y + 19276)) + 113792)). \end{aligned}$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$
(62)\quad \begin{aligned}
\dot{x} = &\ x^2(0.267014.. - 1.32335..y) + 0.0626179..x^3 + y(y(-38.0898..y - 17.692..) + 0.243481..) \\
&+1.39255.. + x(y(11.0206..y - 0.669421..) - 0.110061..), \\
\dot{y} = &\ y(y(0.334711.. - 3.67353..y) + 0.110061..) + 0.00478179..x^3 + x^2(-0.187854..y \\
&-0.0586411..) - 2.13677.. + x(y(1.32335..y - 0.534028..) - 0.892372..),
\end{aligned}
$$

wich has the first integral

$$
\begin{aligned}
\widetilde{H}_5(x,y) = &\ x^2(y(0.112573.. - 0.278961..y) + 0.188112..) - 0.000504..x^4 + x^3(0.0263997..y + 0.00824102..) \\
&+1.9306.. + x(y(y(1.54876..y - 0.141114..) - 0.0464015..) + 0.900861..) + y(y(y(-4.01466..y \\
&-2.48632..) + 0.0513258..) + 0.587099..).
\end{aligned}
$$

The four real solutions of system (23) with $i = 8$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (61)–(62) shown in Figure 13(a) are the set $S_{5,5}$ given by

$$
S_{5,5} = \{(-0.31212.., 0.45996..), (-0.14074.., 0.43160..), (-0.020105.., 0.39546..), (0.092971.., 0.34164..)\}.
$$

**Four limit cycles for the class formed by system** (8)—(9)**.** In the half-plane $R_1$, we consider the Hamiltonian cubic system with nilpotent center

$$
(63)\quad \begin{aligned}
\dot{x} = &\ \frac{1}{900000}(8059500x^3 - 540x^2(74835y + 7883) - y(120y(212420y + 138733) + 159917) \\
&+1963206 + 3x(60y(314700y + 123439) - 1238443)), \\
\dot{y} = &\ \frac{1}{300000}(-16200x^3 + 270x^2(11087 - 29850y) + 3x(60y(74835y + 15766) - 1360603) \\
&+1308666 + y(1238443 - 30y(209800y + 123439))),
\end{aligned}
$$

of type (8) with the first integral

$$
\begin{aligned}
H_5(x,y) = &\ \frac{1}{200000}(24300x^4 + 540x^3(29850y - 11087) - 9x^2(60y(74835y + 15766) - 1360603) + 1609167 \\
&-6x(y(30y(209800y + 123439) - 1238443) - 1308666) + y(y(80y(159315y + 138733) + \\
&159917) - 3926412).
\end{aligned}
$$

In the half-plane $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$
(64)\quad \begin{aligned}
\dot{x} = &\ -3.56089..x^3 + x^2(116.091..y + 85.2029..) + y((-11417.2..y - 7456.68..)y - 71.6274..) \\
&+879.327.. + x(y(1210.81..y - 2367.7..) - 442.536..), \\
\dot{y} = &\ y(y(1183.85.. - 403.605..y) + 442.536..) + 0.699665..x^3 + x^2(10.6827..y - 20.2353..) \\
&-622.065.. + x(y(-116.091..y - 170.406..) + 197.017..),
\end{aligned}
$$

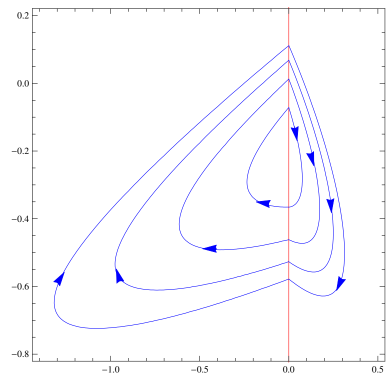
of type (9) with the first integral

$$
\begin{aligned}
H_6(x,y) = &\ x^3(688.421.. - 363.433..y) - 17.8524..x^4 + x^2(y(5924.27..y + 8696.03..) - 10054.) - 148006. \\
&+x(y(y(41193.y - 120827.) - 45166.4..) + 63489.6..) + y(y((-291318.y - 253683.)y \\
&-3655.24..) + 89746.4..).
\end{aligned}
$$

The four real solutions of system (24) with $i = 8$ and $j = 9$ satisfying $y_1 < y_2$, which provide four limit cycles for the discontinuous differential piecewise system (63)–(64) shown in Figure 13(b) are the set $S_{5,6}$ given by

$$
S_{5,6} = \{(-0.3509.., 0.563625..), (-0.164808.., 0.529183..), (-0.0283422.., 0.485655..), (0.102906.., 0.422227..)\}.
$$

The proof of Theorem 8 is done.




$(a)$

FIGURE 14. The four limit cycles of the discontinuous piecewise differential system (9)–$(\widetilde{9})$.

## 9. PROOF OF THEOREM 9

**Four limit cycles for the class formed by system** (9)–$(\widetilde{9})$**.** In the region $R_1$ we consider the Hamiltonian cubic system with nilpotent center

$$
(65)\quad
\begin{aligned}
\dot{x} = & \; x\left(77y^2+\frac{1381y}{25}+\frac{102681}{5500}\right)+\frac{1}{49500}(y(220y(6905y+6072)+945963)+156374)+24x^3\\
& +\frac{3}{25}x^2(590y+257),\\
\dot{y} = & \; \frac{1}{33000}(-660(3540x+1381)y^2-7920x(300x+257)y-847000y^3-616086y-864x(55x(20x\\
& +29)+932)-148411,
\end{aligned}
$$

with its first integral

$$
\begin{aligned}
H_6(x,y) \;=\; & \frac{1}{440000}(-12509920y-2471451+40(-54x(220x(200x+257)+34227)y-4840(525x+184)y^3\\
& -3x(288x(55x(15x+29)+1398)+148411)-9(220x(1770x+1381)+105107)y^2\\
& -759550y^4)).
\end{aligned}
$$

In the region $R_2$ we consider the Hamiltonian cubic system with nilpotent center

$$
(66)\quad
\begin{aligned}
\dot{x} = & \; -7.88317..x^3+x^2(71.1577..y+23.9206..)+x((-214.918..y-135.413..)y-46.5053..)\\
& +22.3614..+y(y(217.23..y+191.024..)+135.272..),\\
\dot{y} = & \; -2.63012..x^3+x^2(23.6495..y+8.4588..)+x((-71.1577..y-47.8411..)y-16.5638..)\\
& +11.4369..+y(y(71.6392..y+67.7066..)+46.5053..),
\end{aligned}
$$

wich has the first integral

$$
\begin{aligned}
\widetilde{H}_6\,(x,y) \;=\; & -0.250036..x^4+x^3(2.99769..y+1.07219..)+x^2((-13.5294y-9.09615..)y-3.14931..)\\
& -8.26225..+x(y(y(27.2419..y+25.7465..)+17.6843..)+4.34905..)\\
& +y(y((-20.6513..y-24.2133..)y-25.7196..)-8.50325..),
\end{aligned}
$$

The four real solutions of system (23) with $i=9$ satisfying $y_1<y_2$, which provide four limit cycles for the discontinuous differential piecewise system (65)–(66) shown in Figure 14 are the set $S_{6,6}$ given by

$$S_{6,6}=\{(-0.57830..,0.11190..),(-0.52717..,0.068607..),(-0.46202..,0.01295..),(-0.3659..,-0.07126..)\}\,.$$

The proof of Theorem 9 is done.

## Conflict of interest

This paper has no conflict of interest.

## References


[1] A. Andronov, A. Vitt, S. Khaikin, Theory of Oscillations, Pergamon Press, Oxford, 1996.
[2] J.C. Artés, J. Llibre, J.C. Medrado and M.A. Teixeira, Piecewise linear differential systems with two real saddles, Math. Comput. Simul. 95, 13–22 (2013).
[3] B.H. Belousov, A periodic reaction and its mechanism, A Collection of Short hapers on Radiation Medicine for 1958. Moscow: Med. Publ. 1959 [in Russian].
[4] R.Benterki and J. Llibre, The solution of the second part of the 16th Hilbert problem for nine families of discontinuous piecewise differential systems, Nonlinear Dyn 102, 2453–2466 (2020).
[5] Di Bernardo, M., Budd, C. J., Champneys, A. R. and Kowalczyk, h., piecewise-Smooth Dynamical Systems: Theory and Applications, Springer-Verlag, London, 2008.
[6] D.C. Braga and L.F. Mello, Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane, Nonlinear Dynam. 73, 1283–1288 (2013).
[7] J. Castillo, J. Llibre and F. Verduzco, The pseudo–Hopf bifurcation for planar discontinuous piecewise linear differential systems, Nonlinear Dynam. 90, 1829–1840 (2017).
[8] I.E. Colak, J. Llibre, C. Valls, Bifurcation diagrams for Hamiltonian nilpotent centers of linear plus cubic homogeneous polynomial vector fields, J. Differential Equations, 262, 5518–5533 (2017).
[9] I.E. Colak, J. Llibre, C. Valls, Hamiltonian nilpotent centers of linear plus cubic homogeneous polynomial vector fields, Adv. Math. 259, 655–687 (2014).
[10] S. Coombes, Neuronal networks with gap junctions: A study of piecewise linear planar neuron models, SIAM J. Appl. Dyn. Syst. 7, 1101–1129 (2008).
[11] M. Esteban, C. Valls and J. Llibre, The 16th Hilbert problem for discontinuous piecewise isochronous centers of degree one or two separated by a straight line, Chaos 31, no. 4, Paper No.043112, 18 pp (2021).
[12] A.F. Filippov, Differential equations with discontinuous right–hand sides, translated from Russian. Mathematics and its Applications (Soviet Series), vol. 18, Kluwer Academic Publishers Group, Dordrecht, 1988.
[13] E. Freire, E. Ponce, F. Rodrigo and F. Torres, Bifurcation sets of continuous piecewise linear systems with two zones, Int. J. Bifurc. Chaos, 8, 2073–2097 (1998).
[14] W. Fulton, Algebraic Curves. Mathematics Lecture Note Series. W.A. Benjamin, 1974.
[15] H. Glendinning and M.R. Jeffrey, An Introduction to piecewise Smooth Dynamics, Springer, Cham, 2019.
[16] M.R.A. Gouveia, J. Llibre and D. D. Novaes, On limit cycles bifurcating from the infinity in discontinuous piecewise linear differential systems, Appl. Math. Comput. 271, 365–374 (2015).
[17] D. Hilbert, Mathematische Probleme, Lecture, Second Internat. Congr. Math. (Paris, 1900), Nachr. Ges. Wiss. G"ottingen Math. Phys. KL., 253–297 (1900); English transl., Bull. Amer. Math. Soc. 8, 437–479 (1902); Bull. (New Series) Amer. Math. Soc. 37, 407–436 (2000).
[18] J. Llibre and M. A. Teixeira, Piecewise linear differential systems with only centers can create limit cycles?, Nonlinear Dynam. 91, 249–255 (2018).
[19] S.M. Huan, X.S. Yang, On the number of limit cycles in general planar piecewise linear systems of node-node types, J. Math. Anal. Appl. 411, 340–353 (2014).
[20] Yu. Ilyashenko, Centennial history of Hilbert's 16th problem, Bull. (New Series) Amer. Math. Soc. 39, 301–354 (2002).
[21] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13, 47–106 (2003).
[22] L. Li, Three crossing limit cycles in planar piecewise linear systems with saddle-focus type, Electron. J. Qual. Theory Differ. Equ. 2014, No. 70, 14 pp (2014).
[23] J. Llibre, M.A. Teixeira and J. Torregrosa, Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation, Int. J. Bifurcation and Chaos 23, No. 1350066, 10 pp (2013).
[24] J. Llibre and M.A. Teixeira , Piecewise linear differential systems without equilibria produce limit cycles?, Nonlinear Dyn. 88, 157–164 (2017).
[25] J. Llibre and X. Zhang, Limit cycles for discontinuous planar piecewise linear differential systems separated by one straight line and having a center, J. Math. Anal. Appl. 467, 537–549 (2018).
[26] B. Van Der Pol, A theory of the amplitude of free and forced triode vibrations, Radio Review (later Wireless World), 1, 701–710 (1920).
[27] B. Van Der Pol, On relaxation-oscillations, The London, Edinburgh and Dublin ppil. Mag. and J. of Sci. 2(7), 978–992 (1926).