LIMIT CYCLES OF DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS SEPARATED BY A STRAIGHT LINE AND FORMED BY CUBIC REVERSIBLE ISOCHRONOUS CENTERS HAVING RATIONAL FIRST INTEGRALS

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ABSTRACT. A lot of attention has been paid in recent years to the study of piecewise differential systems, and more especially is studying the maximum number of limit cycles that these systems are able to exhibit. In this paper we consider all classes of discontinuous piecewise differential systems with cubic reversible isochronous centers having rational first integrals separated by the straight line $x = 0$.

First, we solve the extension of the second part of the sixteenth Hilbert problem for each of the three classes of discontinuous piecewise differential systems formed by an arbitrary linear center and one of the three cubic reversible isochronous centers. We establish that, depending on the class presented, the maximum number of limit cycles of these classes varies between one and two. Second, by combining the three types of the cubic reversible isochronous centers, we obtain six classes of discontinuous piecewise differential systems formed by two cubic reversible isochronous centers. So we solve the extended sixteenth Hilbert problem for all these classes and we find the maximum number of limit cycles that such classes can exhibit. Moreover we have reinforced our results by giving examples for each class.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Any *planar* polynomial differential system takes the form $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, where $P(x, y)$ and $Q(x, y)$ are polynomial functions, the *degree* of this system is the maximum degree of these polynomials.

The study of the existence and determination of the upper bound of the maximum number of *limit cycles* of planar polynomial differential systems is an attractive research topic that, until know, is still an unsolved problem in the qualitative theory of differential systems. As one of the 23 problems presented at the international congress of mathematicians in Paris in 1900, this problem is known as the second part of the sixteenth Hilbert problem. For more information, read, for example [17, 19, 21].

In this paper we focus in particular on discontinuous piecewise differential systems of the form

$$(\dot{x}, \dot{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{cases} \mathbf{F}^-(\mathbf{x}, \mathbf{y}) = \left(F_1^-(x, y), F_2^-(x, y)\right)^T & \mathbf{y} \in \Sigma^-, \\ \mathbf{F}^+(\mathbf{x}, \mathbf{y}) = \left(F_1^+(x, y), F_2^+(x, y)\right)^T & \mathbf{y} \in \Sigma^+, \end{cases}$$

where $\Sigma = \{(x, y) : x = 0\}$ is the discontinuity line and

$$\Sigma^- = \{(x, y) : x \leq 0\}, \quad \Sigma^+ = \{(x, y) : x \geq 0\}.$$

If $F_1^+(0, y_0)F_1^-(0, y_0) > 0$ then the point $(0, y_0) \in \Sigma$ is of the *crossing type*. A periodic orbit of a discontinuous piecewise differential system with exactly two crossing points is called a *crossing periodic orbit*. Moreover we call an isolated crossing periodic orbit in the set of all crossing periodic orbits, a *crossing limit cycle*. In what follows for the sake of brevity, we talk on the limit cycle in place of the crossing limit cycle.

The first analyses of discontinuous piecewise linear differential systems in the plane is around 1930's, by Andronov, Vitt and Khaikin [1]. These systems have been intensively studied mainly due to their widely applications in various scientific domains of studies such as engineering, electronics, mechanics, and physics, see for instance [3, 8, 15, 20, 24, 25].

Planar piecewise linear differential systems, which have a straight line as a separating curve, are the simplest type of discontinuous piecewise differential systems. Thus in the literature, many researchers have been interested in solving the extension of the Hilbert sixteenth problem of finding the maximum number of crossing limit cycles that these systems can exhibit.

The simplest kind of discontinuous piecewise differential systems are formed by the linear ones, in which a straight line serves as the separation curve. Since these systems can produce a significant number of crossing limit cycles, many researchers have been interested in identifying them in order to solve the extension of Hilbert's sixteenth problem.

For same classes of discontinuous piecewise linear differential systems where the separation curve is either a straight line, an algebraic conic curve, or a reducible (or irreducible) cubic curve, numerous studies have recently solved the extension second part of Hilbert's sixteenth problem, for example see [2, 4, 5, 9, 10, 13, 14]. Consequently we observe that most of the papers deal with piecewise linear differential systems separated by a straight line, while nonlinear ones are rarely discussed. For instance, in [6, 12] the authors have found an answer to the extension of the Hilbert's 16th problem for some nonlinear discontinuous differential systems of degree 2 or 3.

The first objective of this paper is to solve the second part of the extension of the sixteenth Hilbert problem for the classes of discontinuous piecewise differential systems formed by one of the three cubic reversible isochronous centers with rational first integrals and a linear differential center, separated by the straight line Σ.

Our second objective is to solve the extension of the sixteenth Hilbert problem for the classes of discontinuous piecewise differential systems separated by the straight line Σ and formed only by cubic reversible isochronous centers having rational first integrals in each half-plane.

We will use the following lemma which provides a normal form for an arbitrary linear differential center.

Lemma 1. *After performing a linear change in the variables and a rescaling of the independent variable each linear center can be expressed as*

$$(1) \qquad \dot{x} = -Ax - \left(A^2 + \omega^2\right) y + B, \quad \dot{y} = Ay + C + x,$$

and its first integral is

$$(2) \qquad H(x, y) = (Ay + x)^2 + 2(Cx - By) + \omega^2 y^2.$$

For a proof of Lemma 1 see [23] .

The normal forms of the three cubic reversible isochronous centers with a rational first integral are given in the following theorem.

Theorem 2. *After an affine change of variables and a rescaling of the independent variable the three cubic reversible isochronous centers with rational first integrals can be expressed as one of the following three differential systems.*

$$(\mathcal{C}_1) \quad \dot{x} = y(-1 + 2ax + 2bx^2), \quad \dot{y} = x + a(y^2 - x^2) + 2bxy^2,$$

$$(\mathcal{C}_2) \quad \dot{x} = -y(1 - x)(1 - 2x), \quad \dot{y} = x - 2x^2 + y^2 + 2x^3,$$

$(\mathcal{C}_3)$ $\quad \dot{x} = y\left(-1 + \dfrac{8}{3}x - \dfrac{32}{9}y^2\right), \quad \dot{y} = x - \dfrac{4}{3}y^2.$

For a proof of Theorem 2 see [11].

We denote by $(\tilde{\mathcal{C}}_i)$ with $i = 1, 2, 3$, the three cubic reversible isochronous centers having a rational first integrals after a general affine change of variables. Our first result is the following.

Theorem 3. *The maximum number of crossing limit cycles of the class of planar discontinuous piecewise differential systems separated by the straight line Σ and formed by the linear center (1) and*

- (I) *the cubic reversible isochronous center $(\tilde{\mathcal{C}}_1)$ is one, there are systems with exactly one limit cycle, see Figure 1(a);*
- (II) *the cubic reversible isochronous center $(\tilde{\mathcal{C}}_2)$ is two, there are systems with exactly two limit cycles, see Figure 1(c);*
- (III) *the cubic reversible isochronous center $(\tilde{\mathcal{C}}_3)$ is one, there are systems with exactly one limit cycle, see Figure 1(b).*

Our second result is as follows.

Theorem 4. *The maximum number of crossing limit cycles of the class of planar discontinuous piecewise cubic reversible isochronous centers separated by the straight line Σ and formed by*

- (I) *the cubic reversible isochronous center $(\tilde{\mathcal{C}}_1)$ in each half plane is at most two, this maximum is reached, see Figure 2(a);*
- (II) *the cubic reversible isochronous centers $(\tilde{\mathcal{C}}_1)$ and $(\tilde{\mathcal{C}}_2)$ is at most three, this maximum is reached, see Figure 2(b);*
- (III) *the cubic reversible isochronous centers $(\tilde{\mathcal{C}}_1)$ and $(\tilde{\mathcal{C}}_3)$ is at most three, this maximum is reached, see Figure 2(c);*
- (IV) *the cubic reversible isochronous center $(\tilde{\mathcal{C}}_2)$ in each half plane is at most eight, this maximum is reached, see Figure 3(a);*
- (V) *the cubic reversible isochronous centers $(\tilde{\mathcal{C}}_2)$ and $(\tilde{\mathcal{C}}_3)$ is at most seven, this maximum is reached, see Figure 3(b);*
- (VI) *the cubic reversible isochronous center $(\tilde{\mathcal{C}}_3)$ in each half plane is at most three, this maximum is reached, see Figure 4(a).*

Theorems 3 and 4 are proved in sections 3 and 4, respectively.

2. The cubic reversible isochronous centers after an affine change of variables

The expressions of the cubic reversible isochronous centers $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$ as well as of their first integrals after a general affine change of variables $\{x \to a_1x + b_1y + c_1, y \to \alpha_1x + \beta_1y + \gamma_1\}$, with $a_1\beta_1 - \alpha_1b_1 \neq 0$ are given in this section.

The isochronous center $(\tilde{\mathcal{C}}_1)$ is

$$\dot{x} = \frac{-1}{\alpha_1b_1 - a_1\beta_1} \quad (\beta_1(\gamma_1 + \alpha_1x + \beta_1y)(2(a_1x + b_1y + c_1)(a + b(a_1x + b_1y + c_1)) - 1) - b_1(a$$
$$((\gamma_1 + \alpha_1x + \beta_1y)^2 - (a_1x + b_1y + c_1)^2) + 2b(a_1x + b_1y + c_1)(\gamma_1 + \alpha_1x$$
$$+\beta_1y)^2 + a_1x + b_1y + c_1)),$$

$$\dot{y} = \frac{-1}{\alpha_1b_1 - a_1\beta_1} \quad (a_1(a((\gamma_1 + \alpha_1x + \beta_1y)^2 - (a_1x + b_1y + c_1)^2) + 2b(a_1x + b_1y + c_1)(\gamma_1 +$$
$$\alpha_1x + \beta_1y)^2 + a_1x + b_1y + c_1) - \alpha_1(\gamma_1 + \alpha_1x + \beta_1y)(2(a_1x + b_1y + c_1)(a$$
$$+b(a_1x + b_1y + c_1)) - 1)),$$

with the first integral

$$(3) \qquad \tilde{H}_1(x,y) = \frac{1 - 2(a_1x + b_1y + c_1)\big(a + b(a_1x + b_1y + c_1)\big)}{(a_1x + b_1y + c_1)^2 + (\gamma_1 + \alpha_1x + \beta_1y)^2}.$$

The isochronous center $(\tilde{\mathcal{C}}_2)$ is written as

$$\dot{x} = \frac{-1}{\alpha_1b_1 - a_1\beta_1} \quad (\beta_1(-(a_1x + b_1y + c_1 - 1))(2a_1x + 2b_1y + 2c_1 - 1)(\gamma_1 + \alpha_1x + \beta_1y) - b_1$$
$$\big(2(a_1x + b_1y + c_1)^3 - 2(a_1x + b_1y + c_1)^2 + a_1x + b_1y + c_1 + (\gamma_1 + \alpha_1x$$
$$+ \beta_1y)^2)),$$
$$\dot{y} = \frac{-1}{\alpha_1b_1 - a_1\beta_1} \quad (\alpha_1(a_1x + b_1y + c_1 - 1)(2a_1x + 2b_1y + 2c_1 - 1)(\gamma_1 + \alpha_1x + \beta_1y) + a_1\big(2(a_1x$$
$$+ b_1y + c_1)^3 - 2(a_1x + b_1y + c_1)^2 + a_1x + b_1y + c_1 + (\gamma_1 + \alpha_1x + \beta_1y)^2)),$$

with the first integral

$$(4) \qquad \tilde{H}_2(x,y) = \frac{(a_1x + b_1y + c_1 - 1)^2\big((a_1x + b_1y + c_1)^2 + (\gamma_1 + \alpha_1x + \beta_1y)^2\big)}{(2(a_1x + b_1y + c_1) - 1)^2}.$$

The isochronous center $(\tilde{\mathcal{C}}_3)$ is

$$\dot{x} = \frac{-1}{\alpha_1b_1 - a_1\beta_1} \quad \Big(\beta_1(\gamma_1 + \alpha_1x + \beta_1y)\Big(\frac{8}{3}(a_1x + b_1y + c_1) - \frac{32}{9}(\gamma_1 + \alpha_1x + \beta_1y)^2 - 1\Big) - b_1$$
$$\Big(a_1x + b_1y + c_1 - \frac{4}{3}(\gamma_1 + \alpha_1x + \beta_1y)^2\Big)\Big),$$
$$\dot{y} = \frac{-1}{\alpha_1b_1 - a_1\beta_1} \quad \Big(a_1\Big(a_1x + b_1y + c_1 - \frac{4}{3}(\gamma_1 + \alpha_1x + \beta_1y)^2\Big) - \alpha_1(\gamma_1 + \alpha_1x + \beta_1y)\Big(\frac{8}{3}(a_1x$$
$$+ b_1y + c_1) - \frac{32}{9}(\gamma_1 + \alpha_1x + \beta_1y)^2 - 1\Big)\Big),$$

with the first integral

$$(5) \qquad \tilde{H}_3(x,y) = \big(3(a_1x + b_1y + c_1) - 4(\gamma_1 + \alpha_1x + \beta_1y)^2\big)^2 + 9(\gamma_1 + \alpha_1x + \beta_1y)^2.$$
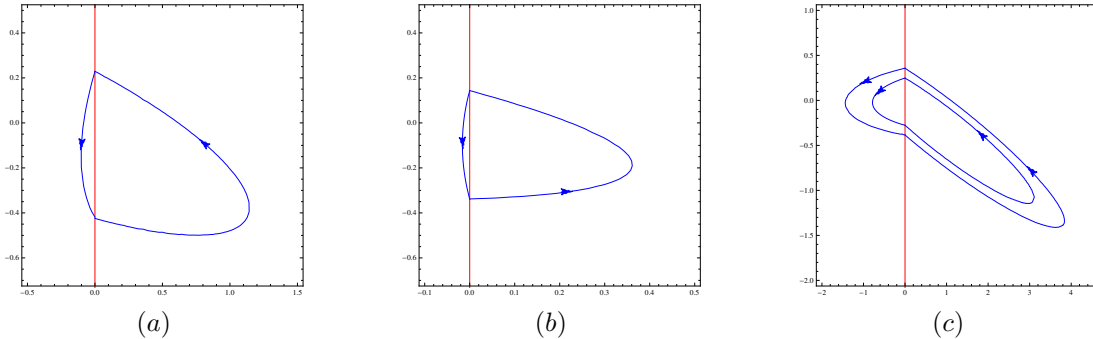

(a) (b) (c)

FIGURE 1. The unique limit cycle of the discontinuous piecewise differential system, (a) for (7)–(8), (b) for (11)–(12), and (c) the two limit cycles of the discontinuous piecewise differential system (9)–(10).

3. Proof of Theorem 3

Now we will prove Theorem 3 for the class of planar discontinuous piecewise differential systems separated by the straight line Σ and formed by a linear center and one of the three cubic reversible isochronous center $(\tilde{\mathcal{C}}_i)$ with $i = 1, 2, 3$.

In the first half-plane Σ^+ we consider the cubic reversible isochronous center $(\tilde{\mathcal{C}}_i)$ with its first integral $\tilde{H}_i(x, y)$ given in either (3), or (4), or (5), and in the second half-plane Σ^- we consider the planar linear differential center (1) with its first integral $H(x, y)$ given in (2).

To prove that the discontinuous piecewise differential systems (1)–$(\tilde{\mathcal{C}}_i)$ has at most one crossing limit cycle that crosses the line of discontinuity Σ at two different points $(0, y)$ and $(0, Y)$, with $y \neq Y$. These two points must satisfy the following system of equations

(6)
$$e_1 = H(0, y) - H(0, Y) = (y - Y)(2B - M(y + Y)) = 0,$$
$$e_2 = \tilde{H}_i(0, y) - \tilde{H}_i(0, Y) = (y - Y)F_i(y, Y) = 0.$$

Where $M = A^2 + \omega^2$.

From $e_1 = 0$ we obtain $Y = -y + \dfrac{2B}{M} = f(y)$ for $i = 1, 2, 3$. Substituting the expression of Y in $F_i(y, Y) = 0$, we get an equation $K_i(y) = F_i(y, f(y)) = 0$ in the variable y, which varies according to the expressions of the first integrals $\tilde{H}_i(x, y)$.

Proof of statement (I) *of Theorem 3.* For $i = 1$ we consider the class of discontinuous piecewise differential systems formed by the linear differential center (1) with the first integral $H(x, y)$ and the cubic reversible isochronous center $(\tilde{\mathcal{C}}_1)$ with its corresponding first integral $\tilde{H}_1(x, y)$. We find that $K_1(y) = 0$ is the quadratic equation in the variable y

$$K_1(y) = \quad b_1^2 \left(M(2ac_1 - 2b\gamma_1(\gamma_1 + 2\beta_1 y) - 1) + 4b\beta_1\gamma_1 y^2 \right) + 2b_1 \left(ac_1^2 + a\left(-\gamma_1^2 + \beta_1^2 My - \beta_1^2 y^2 \right) + \right.$$
$$\left. c_1 \left(2b(-\gamma_1^2 + \beta_1^2 My - \beta_1^2 y^2) - 1 \right) \right) + \beta_1 \left(2ac_1 + 2bc_1^2 - 1 \right)(2\gamma_1 + \beta_1 M) + 2ab_1^3(M - y)y.$$

This equation can have at most two real solutions y_1 and y_2 for the variable y. Thus system (6) has at most two real solutions. Since $(y_1, f(y_1)) = (f(y_2), y_2)$, then the class of the discontinuous piecewise differential systems (1)–$(\tilde{\mathcal{C}}_1)$ has at most one limit cycle.

In order to complete the proof of this statement we give an example with exactly one limit cycle. In the half-plane Σ^- we consider the linear center

(7)
$$\dot{x} = -1 - \frac{5}{2}x - \frac{41}{4}y, \qquad \dot{y} = \frac{1}{2} + x + \frac{5}{2}y,$$

with the first integral

$$H(x, y) = \left(x + \frac{5}{2}y \right)^2 + 2\left(\frac{1}{2}x + y \right) + 4y^2.$$

In the other half-plane Σ^+ we consider the cubic reversible isochronous center

(8)
$$\dot{x} = \quad -\frac{21}{500}x^3 + x^2(0.632258..y + 1.02182..) + x\left(-2.05837..y^2 - 4.49659..y - 2.69497.. \right)$$
$$-8.34146..y^2 - 15.1823..y - 2.29501..,$$
$$\dot{y} = \quad x^2\left(-\frac{21}{500}y - 0.066818.. \right) + x\left(0.632258..y^2 + 2.31636..y + 1.93406.. \right) - 2.05837..y^3$$
$$-10.8976..y^2 - \frac{3461}{200}y - 8.48195..,$$

with the first integral

$$\tilde{H}_1(x,y) = \frac{1 - 2(0.21x - y - 1)(-0.21x + y + 1.5)}{(0.21x - y - 1)^2 + (0.1x - 1.02919..y - 2)^2}.$$

Eventually, system (6) for $i = 1$ has the unique real solution $(-0.42479.., 0.229668..)$ which provides the unique limit cycle of the discontinuous piecewise differential system (7)–(8) shown in Figure 1(a). Thus statement (I) of Theorem 3 is proved. $\square$

Proof of statement (II) of Theorem 3. For $i = 2$ we consider the class of discontinuous piecewise differential systems formed by the linear differential center (1) with the first integral $H(x,y)$ and the cubic reversible isochronous center $(\tilde{\mathcal{C}}_2)$ with its first integral $\tilde{H}_2(x,y)$. We obtain that $K_2(y) = 0$ is the quartic equation in the variable y

$$
\begin{aligned}
K_2(y) = \quad & 4b_1^6 My^2(M-y)^2 + 4b_1^5(2c_1-1)y\left(M^3 - 2My^2 + y^3\right) + b_1^4\big((1-2c_1)^2 M(M^2 + 6My \\
& -6y^2) + 4\beta_1^2 My^2(M-y)^2 + 8\beta_1\gamma_1 y^2(M-y)^2\big) + 2b_1^3\big(M^2(2c_1(4c_1^2 - 6c_1 - 2\beta_1^2 y^2 + 3) \\
& +4\beta_1(2c_1-1)\gamma_1 y - 1) + My\big(2\left(\gamma_1^2 + c_1\left(4c_1^2 - 6c_1 + 3\right) - 4\beta_1 c_1\gamma_1 y\right) + 4\beta_1 y(\gamma_1 + \beta_1 y) \\
& -1\big) - y^2\left(2\gamma_1^2 + 2c_1\left(4c_1^2 - 6c_1 + 3\right) + 2\beta_1^2 y^2 - 1\right) + 2\beta_1^2(2c_1-1)M^3 y\big) + b_1^2\big(M\big(20c_1^4 \\
& -40c_1^3 + 30c_1^2 + \beta_1^2(2c_1-1)\left((2c_1-1)M^2 + 2(2c_1-3)My + 2(3-2c_1)y^2\right) - 10c_1 + 1\big) \\
& +2\beta_1\gamma_1\left((1-2c_1)^2 M^2 + (4c_1(2c_1-3) + 3)My + (4(3-2c_1)c_1 - 3)y^2\right) + (4c_1-3)\gamma_1^2 M\big) \\
& +2b_1(c_1-1)(2c_1-1)\big(2c_1^3 - 2c_1^2 + \gamma_1^2 + c_1(2\beta_1 M(2\gamma_1 + \beta_1 M) + 1) - \beta_1^2\left(M^2 + My - y^2\right) \\
& -2\beta_1\gamma_1 M\big) + \beta_1\left(2c_1^2 - 3c_1 + 1\right)^2(2\gamma_1 + \beta_1 M).
\end{aligned}
$$

This equation can have at most four real solutions, and due to the symmetry stated in the proof of statement (I) we know that system (6) has at most two different real solutions. As a result the class of discontinuous piecewise differential systems (1)–($\tilde{\mathcal{C}}_2$) has at most two limit cycles.

Now we prove that this result is reached by giving an example with exactly two limit cycles. So in Σ^- we consider the linear center

$$(9) \qquad \dot{x} = -\frac{1}{10} - \frac{5}{2}x - \frac{29}{4}y, \qquad \dot{y} = -\frac{1}{10} + x + \frac{5}{2}y,$$

its first integral is

$$H(x,y) = \left(x + \frac{5}{2}y\right)^2 + 2\left(\frac{1}{10}y - \frac{1}{10}x\right) + y^2.$$

In Σ^+ we consider the cubic reversible isochronous center

$$
\begin{aligned}
(10) \qquad \dot{x} = \quad & \frac{2}{10}x^3 + x^2(-16y - 1.65628..) + x(114.847y + 4.14776..) - 205.093y - 2.82886.., \\
\dot{y} = \quad & \frac{101}{400}x^3 + x^2\left(-\frac{2}{10}y - 2.39869..\right) + x(1.23559..y + 7.63844..) + 8y^2 - 2.34297.. \\
& y - 8.12409..,
\end{aligned}
$$

which has the first integral

$$\tilde{H}_2(x,y) = \frac{1}{(2(x-2.83898..)-1)^2}(x - 3.83898..)^2\left(\left(-\frac{1}{10}x + 8y + 0.110345..\right)^2 + (x - 2.83898..)^2\right).$$

The discontinuous piecewise differential system (9)–(10) has two limit cycles because system (6) when $i = 2$ has the two real solutions $(-0.276768.., 0.249182..)$ and $(-0.38544.., 0.357854..)$, these two limit cycles are drawn in Figure 1(c). Hence this statement is proved. $\square$

Proof of statement (III) of Theorem 3. For $i = 3$ we consider the class of discontinuous piecewise differential systems composed by the linear differential center (1) with the first integral $H(x,y)$ and the

cubic reversible isochronous center $(\tilde{\mathcal{C}}_3)$ with its first integral $\tilde{H}_3(x,y)$. We realize that $K_3(y) = 0$ is the quadratic equation in the variable y

$$K_3(y) = \quad 9b_1^2 M + 6b_1\left(-4\gamma_1^2 + 3c_1 - 4\beta_1^2\left(M^2 - My + y^2\right) - 8\beta_1\gamma_1 M\right) + \beta_1(2\gamma_1 + \beta_1 M)\left(-24c_1\right.$$
$$+16\left(2\gamma_1^2 + \beta_1^2\left(M^2 - 2My + 2y^2\right) + 2\beta_1\gamma_1 M\right) + 9\right).$$

Therefore system (6) has at most one distinct real solution. Consequently the discontinuous piecewise differential systems (1)–$(\tilde{\mathcal{C}}_3)$ has at most one limit cycle.

Now we will build an example with exactly one limit cycle to prove that this maximum is reached.

In Σ^- we consider the linear center

(11)
$$\dot{x} = -1 - \frac{5}{2}x - \frac{11}{4}y, \qquad \dot{y} = 1 + x + \frac{5}{2}y,$$

which has the first integral

$$H(x,y) = \left(x + \frac{5}{2}y\right)^2 + 2(x + y) + 4y^2.$$

In Σ^+ we consider the cubic reversible isochronous center

(12)
$$\dot{x} = \quad x^2(0.0257383.. - 0.00119261..y) - 0.00002265..x^3 + x(-0.0209315..y^2 + 0.435441..y$$
$$-0.783717..) - 0.122457..y^3 - 0.285973..y^2 - 20.1494..y - 1.95833..,$$
$$\dot{y} = \quad 1.290533111466.10^{-6}x^3 + x^2(0.00006795..y - 0.0022262..) + x(0.00119261..y^2$$
$$-0.0514766..y + 0.918295..) + 0.00697717..y^3 - 0.21772..y^2 + 0.783717..y - 35.9736..,$$

with the first integral

$$\tilde{H}_3(x,y) = \left(3\left(\frac{1}{10}x - y - 1\right) - 4\left(\frac{1}{100}x + 0.175511..y - 2\right)^2\right)^2 + 9\left(\frac{1}{100}x + 0.175511..y - 2\right)^2.$$

The pair $(-0.339012.., 0.14389..)$ is the unique real solution of system (6) for $i = 3$, therefore the discontinuous piecewise differential system (11)–(12) has the unique limit cycle shown in Figure 1(b). With this example we complete the proof of Theorem 3. $\qquad\square$
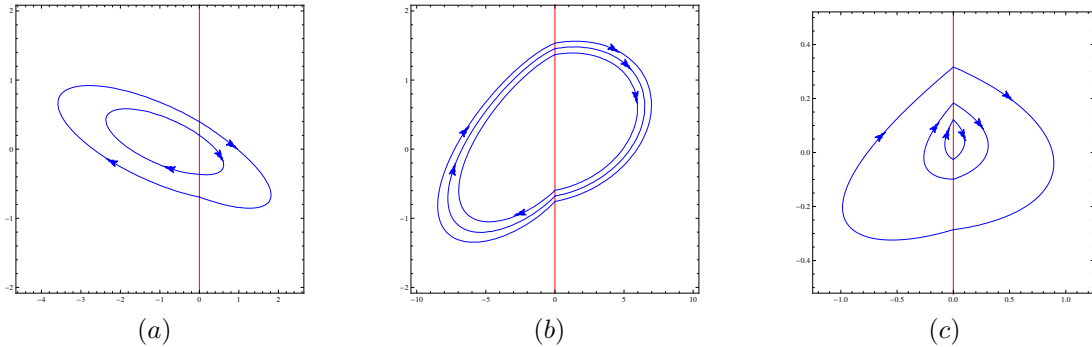

FIGURE 2. (a) The two limit cycles of the discontinuous piecewise differential system (15)–(16), the three limit cycles of the discontinuous piecewise differential system (b) for (17)–(18), and (c) for (19)–(20).
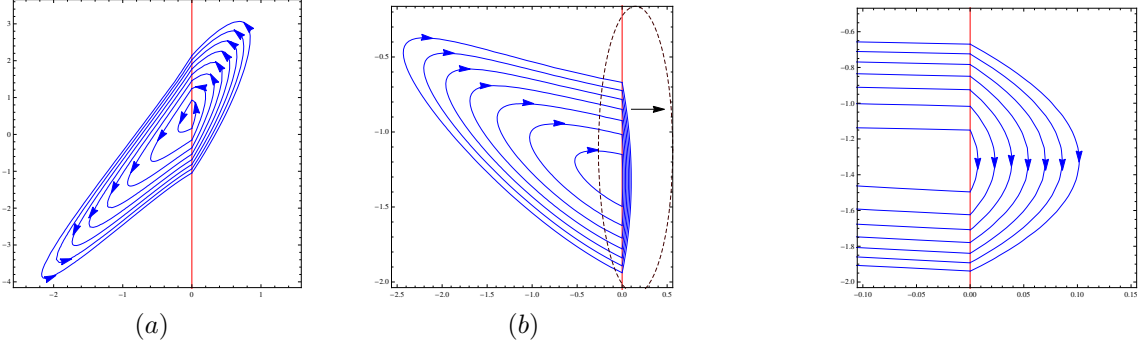
FIGURE 3. (a) The eight limit cycles of the discontinuous piecewise differential system (21)–(22), and (b) the seven limit cycles of the discontinuous piecewise differential system (23)–(24).
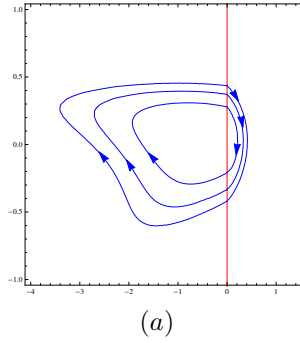

FIGURE 4. (a) The three limit cycles of the discontinuous piecewise differential system (25)–(26).

4. PROOF OF THEOREMS 4

For the proof of this theorem we start by the class of the discontinuous piecewise differential system created by systems $(\tilde{\mathcal{C}}_i)$– $(\tilde{\tilde{\mathcal{C}}}_i)$ with $i = 1, 2, 3$. In Σ^+ we consider the cubic reversible isochronous center $(\tilde{\mathcal{C}}_i)$ with its corresponding first integral $\tilde{H}_i(x, y)$ given in either (3), or (4), or (5), and in Σ^- we consider the second differential cubic reversible isochronous center $(\tilde{\tilde{\mathcal{C}}}_i)$ with its first integral $\tilde{\tilde{H}}_i(x, y)$ but with the parameters $(\alpha, \beta, a_2, b_2, c_2, \alpha_2, \beta_2, \gamma_2)$ instead of the parameters $(a, b, a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1)$. The following system of equations must be satisfied at the points $(0, y)$ and $(0, Y)$ with $y < Y$, in order to prove that the discontinuous piecewise differential system created by the systems $(\tilde{\mathcal{C}}_i)$– $(\tilde{\tilde{\mathcal{C}}}_i)$ for $i = 1, 2, 3$ has a limit cycle intersecting the line of discontinuity Σ in these two different points

$$
\begin{aligned}
e_1 &= \tilde{H}_i(0, y) - \tilde{H}_i(0, Y) = 0, \\
e_2 &= \tilde{\tilde{H}}_i(0, y) - \tilde{\tilde{H}}_i(0, Y) = 0.
\end{aligned}
\tag{13}
$$

Later on we shall give the proof of Theorem 4 for the class of the discontinuous piecewise differential system formed by systems $(\tilde{\mathcal{C}}_i)$– $(\tilde{\mathcal{C}}_j)$, for $i \neq j$ and $i, j = 1, 2, 3$. In the first half-plane we consider the cubic reversible isochronous center $(\tilde{\mathcal{C}}_i)$ with its corresponding first integral $\tilde{H}_i(x, y)$. In the second one we consider the second differential cubic reversible isochronous center $(\tilde{\mathcal{C}}_j)$ with its corresponding first integral $\tilde{H}_j(x, y)$, but with the parameters $(\alpha, \beta, a_2, b_2, c_2, \alpha_2, \beta_2, \gamma_2)$ instead of the parameters $(a, b, a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1)$. Thus if we assume the existence of a limit cycle of the previous discontinuous

piecewise differential system, which must intersect the discontinuity line Σ in the two points $(0, y)$ and $(0, Y)$ with $y \neq Y$, then these points are the solutions of the following system of equations

(14)
$$
\begin{aligned}
e_1 &= \tilde{H}_i(0, y) - \tilde{H}_i(0, Y) = 0, \\
e_2 &= \tilde{H}_j(0, y) - \tilde{H}_j(0, Y) = 0.
\end{aligned}
$$

Proof of statement (I) of Theorem 4. We consider the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_1)$–$(\tilde{\tilde{\mathcal{C}}}_1)$, we obtain that system (13) when $i = 1$ is equivalent to

$$
\begin{aligned}
e_1 = \ & 2ab_1^3 yY + 2ab_1^2 c_1 y + 2ab_1^2 c_1 Y + 2ab_1 c_1^2 - 2ab_1\gamma_1^2 + 2a\beta_1^2 b_1 yY + 4a\beta_1 c_1\gamma_1 + 2a\beta_1^2 c_1 y + 2a\beta_1^2 c_1 Y \\
& -2bb_1^2\gamma_1^2 y - 4b\beta_1 b_1^2\gamma_1 yY - 2bb_1^2\gamma_1^2 Y - 4bb_1 c_1\gamma_1^2 + 4b\beta_1^2 b_1 c_1 yY + 4b\beta_1 c_1^2\gamma_1 + 2b\beta_1^2 c_1^2 y + 2b\beta_1^2 c_1^2 Y \\
& -b_1^2 y - b_1^2 Y - 2b_1 c_1 - 2\beta_1\gamma_1 - \beta_1^2 y - \beta_1^2 Y = 0, \\
e_2 = \ & 2\alpha b_2^3 yY + 2\alpha b_2^2 c_2 y + 2\alpha b_2^2 c_2 Y - 2\beta b_2^2\gamma_2^2 y - 4\beta\beta_2 b_2^2\gamma_2 yY - b_2^2 y - 2\beta b_2^2\gamma_2^2 Y - b_2^2 Y - 2\alpha b_2\gamma_2^2 \\
& +2\alpha b_2 c_2^2 - 4\beta b_2 c_2\gamma_2^2 + 4\beta\beta_2^2 b_2 c_2 yY - 2b_2 c_2 - 2\beta_2\gamma_2 + 2\alpha\beta_2^2 b_2 yY + 4\beta\beta_2 c_2^2\gamma_2 + 2\beta\beta_2^2 c_2^2 y \\
& +2\beta\beta_2^2 c_2^2 Y + 4\alpha\beta_2 c_2\gamma_2 + 2\alpha\beta_2^2 c_2 y + 2\alpha\beta_2^2 c_2 Y - \beta_2^2 y - \beta_2^2 Y = 0.
\end{aligned}
$$

Now by doing the change of variable $yY \to z$ in e_1 and e_2, and solving $e_1 = 0$ with respect to the variable z and by substituting the value of z in $e_2 = 0$ we get an equation $P(y, Y) = 0$ in the variables y and Y. Due to the big expression of $P(y, Y)$ we omit it.

From $P(y, Y) = 0$ we obtain $Y = h(y)$, and by substituting it in $e_1 = 0$ we get the quadratic equation $E(y) = 0$ in the variable y

$$
\begin{aligned}
E(y) = \ & \frac{1}{M}(2b_1 y^2(a(b_1^2 + \beta_1^2) + 2b\beta_1(c_1\beta_1 - b_1\gamma_1))(((2b\alpha\gamma_1^2 + \alpha)b_2^3 - 2(\beta\beta_2\gamma_2 + b\gamma_1(2\beta\beta_2\gamma_1\gamma_2 + \beta_1 \\
& (-2\beta\gamma_2^2 + 2c_2\alpha - 1)))b_2^2 + (\alpha + 2c_2\beta)\beta_2^2(2b\gamma_1^2 + 1)b_2 - 2b((2\beta c_2^2 + 2\alpha c_2 - 1)\beta_1\beta_2^2\gamma_1)b_1^2 \\
& +2bc_1\beta_1^2((-2\beta\gamma_2^2 + 2c_2\alpha - 1)b_2^2 + ((2\beta c_2^2 + 2\alpha c_2 - 1)\beta_2^2))b_1 - b_2 c(2bc_1^2 - 1)\beta_1^2(\alpha b_2^2 - 2\beta\beta_2\gamma_2 b_2 \\
& +(\alpha + 2c_2\beta)\beta_2^2) + a(b_1^2 + \beta_1^2)(b_1((-2\beta\gamma_2^2 + 2c_2\alpha - 1)b_2^2 + (2\beta c_2^2 + 2\alpha c_2 - 1)\beta_2^2) - 2b_2 c_1(\alpha b_2^2 \\
& -2\beta\beta_2\gamma_2 b_2 + (\alpha + 2c_2\beta)\beta_2^2)))y^2 + 4b_1(a(b_1^2 + \beta_1^2) + 2b\beta_1(c_1\beta_1 - b_1\gamma_1))(2b\beta_1\gamma_1((-2\beta c_2^2 \\
& -2\alpha c_2 + 1)\beta_2\gamma_2 + b_2(-\alpha c_2^2 + 2\beta\gamma_2^2 c_2 + c_2 + \alpha\gamma_2^2))b_1^2 + c_1((2b\alpha\gamma_1^2 + \alpha)b_2^3 - 2\beta\beta_2(2b\gamma_1^2 + 1) \\
& \gamma_2 b_2^2 + ((\alpha + 2c_2\beta)\beta_2^2 + 2b(-\alpha\gamma_2^2\beta_1^2 - 2c_2\beta\gamma_2^2\beta_1^2 - c_2\beta_1^2 + c_2^2\alpha\beta_1^2 + \alpha\beta_2^2\gamma_1^2 + 2c_2\beta\beta_2^2\gamma_1^2)) \\
& b_2 + 2b(2\beta c_2^2 + 2\alpha c_2 - 1)\beta_1^2\beta_2\gamma_2)b_1 - b_2(2bc_1^2 - 1)\beta_1\gamma_1(\alpha b_2^2 - 2\beta\beta_2\gamma_2 b_2 + (\alpha + 2c_2\beta)\beta_2^2) \\
& +a(((2\beta c_2^2 + 2\alpha c_2 - 1)\beta_2\gamma_2 + b_2(\alpha c_2^2 - 2\beta\gamma_2^2 c_2 - c_2 - \alpha\gamma_2^2))b_1^3 - (\alpha(c_1^2 - \gamma_1^2)b_2^3 + 2\beta\beta_2(\gamma_1^2 \\
& -c_1^2)\gamma_2 b_2^2 + (-c_2^2\alpha\beta_1^2 + \alpha(c_1^2\beta_2^2 - \gamma_1^2\beta_2^2 + \beta_1^2\gamma_2^2) + c_2((2\beta\gamma_2^2 + 1)\beta_1^2 + 2\beta\beta_2^2(c_1^2 - \gamma_1^2)))b_2 \\
& +((-2\beta c_2^2 - 2\alpha c_2 + 1)\beta_1^2\beta_2\gamma_2)b_1 - 2b_2 c_1\beta_1\gamma_1((\alpha b_2^2 - 2\beta\beta_2\gamma_2 b_2 + (\alpha + 2c_2\beta)\beta_2^2)))y \\
& +2b_1((a((b_1^2 + \beta_1^2) + 2b\beta_1(c_1\beta_1 - b_1\gamma_1))((-2b\gamma_1^2 + 2ac_1 - 1)((2\beta c_2^2 + 2\alpha c_2 - 1)\beta_2\gamma_2 \\
& +b_2(\alpha c_2^2 - ((2\beta\gamma_2^2 + 1)c_2 - \alpha\gamma_2^2))b_1^2 - (2bc_1^2 + 2ac_1 - 1)\beta_1((\gamma_1(-2\beta\gamma_2^2 + 2c_2\alpha - 1)b_2^2 \\
& +\beta_1(-\alpha c_2^2 + 2\beta\gamma_2^2 c_2 + c_2 + \alpha\gamma_2^2)b_2 + (2\beta c_2^2 + 2\alpha c_2 - 1)\beta_2(\beta_2\gamma_1 - \beta_1\gamma_2)) - b_1((ac_1^2 - ((2b\gamma_1^2 \\
& +1)c_1 - a\gamma_1^2 t)((-2\beta\gamma_2^2 + 2c_2\alpha - 1)b_2^2 + (2\beta c_2^2 + 2\alpha c_2 - 1)\beta_2^2))),
\end{aligned}
$$

with

$$
\begin{aligned}
M = \ & a(b_1^2 + \beta_1^2)(b_1(b_2^2(-2\beta\gamma_2^2 + 2\alpha c_2 - 1) + \beta_2^2(2c_2(\alpha + \beta c_2) - 1)) - 2b_2 c_1(\alpha b_2^2 - 2\beta\beta_2 b_2\gamma_2 + \beta_2^2(\alpha \\
& +2\beta c_2))) + b_1^2(b_2^3(\alpha + 2\alpha b\gamma_1^2) - 2b_2^2(\beta\beta_2\gamma_2 + b\gamma_1(2\beta\beta_2\gamma_1\gamma_2 + \beta_1(-2\beta\gamma_2^2 + 2\alpha c_2 - 1))) + \beta_2^2 b_2 \\
& (2b\gamma_1^2 + 1)(\alpha + 2\beta c_2) - 2b\beta_1\beta_2^2\gamma_1(2c_2(\alpha + \beta c_2) - 1)) + 2b\beta_1^2 b_1 c_1(b_2^2(-2\beta\gamma_2^2 + 2\alpha c_2 - 1) + \beta_2^2 \\
& (2c_2(\alpha + \beta c_2) - 1)) - \beta_1^2 b_2(2bc_1^2 - 1)(\alpha b_2^2 - 2\beta\beta_2 b_2\gamma_2 + \beta_2^2(\alpha + 2\beta c_2)).
\end{aligned}
$$

Using Descartes Theorem, we know that $E(y) = 0$ can have at most two positive real solutions y_1 and y_2. Therefore system (13) when $i = 1$ has at most two distinct real solutions (y_1, Y_1) and (y_2, Y_2). Consequently the discontinuous piecewise differential systems $(\tilde{\mathcal{C}}_1)$–$(\tilde{\tilde{\mathcal{C}}}_1)$ can have at most two limit cycles.

In order to complete the proof of this statement we build an example with exactly two limit cycles. In the first half-plane Σ^- we consider the cubic reversible isochronous differential center $(\tilde{\mathcal{C}}_1)$

$$
\begin{aligned}
(15) \qquad \dot{x} = \;& (0.0967917.. - 0.292708..y)x^2 + 0.0390278..x^3 + (-0.975694..y^2 - 0.644167..y \\
& +0.955083..)x - 1.29306..y^2 + \frac{779}{200}y + 0.222722.., \\
\dot{y} = \;& (0.0390278..y + 0.0161167..)x^2 + (-0.292708..y^2 + 0.236694..y - 0.406908..)x \\
& -0.975694..y^3 + 0.122083..y^2 - 0.718972..y - 0.233283..,
\end{aligned}
$$

with the first integral

$$
\tilde{H}_1(x,y) = -\frac{0.750534..\,\left(x^2 + 5xy + 1.51246..x + 6.25y^2 + 3.78114..y - 4.61833..\right)}{x^2 + 4.03846..xy + 1.11538..x + 9.85577..y^2 + 0.865385..y + 0.394231..}.
$$

In the other half-plane Σ^+ we consider the cubic reversible isochronous center $(\tilde{\tilde{\mathcal{C}}}_1)$

$$
\begin{aligned}
(16) \qquad \dot{x} = \;& \frac{1}{100}x^3 + \left(-\frac{3}{40}y - \frac{31}{1000}\right)x^2 + x\left(-\frac{1}{4}y^2 - \frac{22}{25}y + \frac{433}{500}\right) - \frac{49}{40}y^2 + \frac{369}{100}y \\
& +\frac{211}{1000}, \\
\dot{y} = \;& \left(\frac{1}{100}y + \frac{101}{2500}\right)x^2 + \left(-\frac{3}{40}y^2 + \frac{121}{500}y - \frac{1887}{5000}\right)x - \frac{1}{4}y^3 + \frac{1}{25}y^2 - 0.666..y \\
& -0.2264..,
\end{aligned}
$$

its first integral is

$$
\tilde{\tilde{H}}_1(x,y) = -\frac{5\left(4x^2 + 20xy + 20x + 25y^2 + 50y - 71\right)}{104x^2 + 420xy + 116x + 1025y^2 + 90y + 41}.
$$

For $i = 1$ the two real solutions $(0.2, -0.3625..)$ and $(0.395217.., -0.690869..)$ of system (13) provide the two limit cycles of discontinuous piecewise differential system (15)–(16) shown in Figure 2(a). This example completes the proof of statement (I). $\qquad\square$

Proof of statement (II) of Theorem 4. Now we consider the class of discontinuous piecewise differential systems created by the cubic reversible isochronous center $(\tilde{\mathcal{C}}_1)$ with the first integral $\tilde{H}_1(x,y)$ and the cubic reversible isochronous differential center $(\tilde{\mathcal{C}}_2)$ with its first integral $\tilde{H}_2(x,y)$, we obtain that system (14) when $i = 1$ and $j = 2$ is

$$
\begin{aligned}
e_1 = \;& 2ab_1^3yY + 2ab_1^2c_1y + 2ab_1^2c_1Y + 2ab_1c_1^2 - 2ab_1\gamma_1^2 + 2a\beta_1^2b_1yY + 4a\beta_1c_1\gamma_1 + 2a\beta_1^2c_1y + 2a\beta_1^2c_1Y \\
& -2bb_1^2\gamma_1^2y - 4b\beta_1b_1^2\gamma_1yY - 2bb_1^2\gamma_1^2Y - 4bb_1c_1\gamma_1^2 + 4b\beta_1^2b_1c_1yY + 4b\beta_1c_1^2\gamma_1 + 2b\beta_1^2c_1^2y + 2b\beta_1^2c_1^2Y \\
& -b_1^2y - b_1^2Y - 2b_1c_1 - 2\beta_1\gamma_1 - \beta_1^2y - \beta_1^2Y = 0,
\end{aligned}
$$

$$
\begin{aligned}
e_2 = \;& 4b_2^6y^2Y^2(y+Y) + 4b_2^5(2c_2-1)yY\left(y^2 + 3yY + Y^2\right) + b_2^4\Big((y+Y)\big((1-2c_2)^2\left(y^2 + 8yY + Y^2\right) \\
& +4\beta_2^2y^2Y^2\big) + 8\beta_2\gamma_2y^2Y^2\Big) + 2b_2^3\big(y^2\big(8c_2^3 - 12c_2^2 + c_2(8\beta_2Y(\gamma_2 + \beta_2Y) + 6) - 2\beta_2Y(2\gamma_2 + 3\beta_2Y) \\
& -1\big) + 2\beta_2^2(2c_2-1)y^3Y + yY\left(2\gamma_2^2 + 2\beta_2^2(2c_2-1)Y^2 + 4\beta_2\gamma_2(2c_2-1)Y + 3(2c_2-1)^3\right) + (2c_2-1)^3 \\
& Y^2\big) + b_2^2\big((y+Y)\big(20c_2^4 - 40c_2^3 + 30c_2^2 + \beta_2^2(2c_2-1)\left((2c_2-1)y^2 + 8(c_2-1)yY + (2c_2-1)Y^2\right) - 10 \\
& c_2 + 1\big) + 2\beta_2\gamma_2\left((1-2c_2)^2y^2 + (4c_2(4c_2-5)+5)yY + (1-2c_2)^2Y^2\right) + (4c_2-3)\gamma_2^2(y+Y)\big) + 2b_2 \\
& (c_2-1)(2c_2-1)\big(2c_2^3 - 2c_2^2 + \gamma_2^2 + c_2(2\beta_2(y+Y)(2\gamma_2 + \beta_2(y+Y)) + 1) - \beta_2^2\left(y^2 + 3yY + Y^2\right)(y \\
& -2\beta_2\gamma_2 + Y)\big) + \beta_2\left(2c_2^2 - 3c_2 + 1\right)^2\left(2\gamma_2 + \beta_2(y+Y)\right) = 0.
\end{aligned}
$$

From $e_1 = 0$ we obtain $Y = f(y)$, substituting it in $e_2 = 0$ we find an equation of the variable y of degree six. Therefore system (14) when $i = 1$ and $j = 2$ has at most six real solutions namely (y_i, Y_i)

with $i \in \{1, \ldots, 6\}$. Since $(y_i, Y_i) = (y_j, Y_j)$ with $i \in \{1, 3, 5\}$ and $j \in \{2, 4, 6\}$, these solutions provide at most three limit cycles for the discontinuous piecewise differential systems $(\tilde{\mathcal{C}}_1)$– $(\tilde{\mathcal{C}}_2)$.

Now we prove that this result is reached by giving an example with exactly three limit cycles.

In Σ^- we consider the cubic reversible isochronous center of type $(\tilde{\mathcal{C}}_2)$

(17)
$$
\begin{aligned}
\dot{x} = \ & -0.514085..x^3 + (3.75423..y + 2.2093..)x^2 + \left(-0.735423..y^2 - 25.5419..y + 3.44704..\right)x \\
& +0.0365141..y^3 + 2.17709..y^2 + 43.1293..y - 16.5836.., \\
\dot{y} = \ & -0.140845..x^3 + (0.542254..y + 0.792958..)x^2 + (-0.104225..y^2 - 2.65859..y \\
& -1.42958..)x + 0.00514085..y^3 - 3.29207..y^2 + 8.88296..y - 1.43575..,
\end{aligned}
$$

with the first integral

$$
\tilde{H}_2(x, y) = \frac{(15x - 14)^2 \left(12325x^2 - 12100xy + 5580x + 3025y^2 - 3300y + 2056\right)}{14400(5x - 8)^2}.
$$

In Σ^+ we consider the cubic reversible isochronous center of type $(\tilde{\mathcal{C}}_1)$

(18)
$$
\begin{aligned}
\dot{x} = \ & -0.00141019..x^2 + (0.000171292..y - 0.217157..)x + 0.0496747..y^2 + 6.04893..y \\
& -2.29839.., \\
\dot{y} = \ & -2.42082291791.10^{-6}x^2 + x(-0.00280815..y - 0.170976..) + 0.00362918..y^2 \\
& +0.214343..y - 0.0817112..,
\end{aligned}
$$

its corresponding first integral is

$$
\tilde{H}_1(x, y) = \frac{1}{(-10x + 13.2y - 5)^2 + (0.1x + 58y + 7084.68)^2}\left(1 - 0.000281426..\left(\frac{1}{10}x + 58y + \frac{177117}{25}\right)\right).
$$

For the discontinuous piecewise differential system (17)–(18) system (14) when $i = 1$ and $j = 2$ has the three real solutions $(-0.598492.., 1.36883..)$, $(-0.682118.., 1.45531..)$ and $(-0.759451.., 1.53549..)$. These solutions provide the three limit cycles of system (17)–(18) drawn in Figure 2(b). Then this statement is proved. $\qquad\square$

Proof of statement (III) of Theorem 4. We consider the class of discontinuous piecewise differential systems composed by the cubic reversible isochronous center $(\tilde{\mathcal{C}}_1)$ with the first integral $\tilde{H}_1(x, y)$ and the cubic reversible isochronous differential center $(\tilde{\mathcal{C}}_3)$ with its first integral $\tilde{H}_3(x, y)$, so system (14) when $i = 1$ and $j = 3$ is

$$
\begin{aligned}
e_1 = \ & 2ab_1^3yY + 2ab_1^2c_1y + 2ab_1^2c_1Y + 2ab_1c_1^2 - 2ab_1\gamma_1^2 + 2a\beta_1^2b_1yY + 4a\beta_1c_1\gamma_1 + 2a\beta_1^2c_1y + 2a\beta_1^2c_1Y \\
& -2bb_1^2\gamma_1^2y - 4b\beta_1b_1^2\gamma_1yY - 2bb_1^2\gamma_1^2Y - 4bb_1c_1\gamma_1^2 + 4b\beta_1^2b_1c_1yY + 4b\beta_1c_1^2\gamma_1 + 2b\beta_1^2c_1^2y + 2b\beta_1^2c_1^2Y \\
& -b_1^2y - b_1^2Y - 2b_1c_1 - 2\beta_1\gamma_1 - \beta_1^2y - \beta_1^2Y = 0, \\
e_2 = \ & 9b_2^2(y + Y) + 6b_2\left(3c_2 - 4\left(\beta_2^2y^2 + \beta_2y(2\gamma_2 + \beta_2Y) + (\gamma_2 + \beta_2Y)^2\right)\right) + \beta_2(2\gamma_2 + \beta_2(y + Y))\big(-24c_2 \\
& +16\left(2\gamma_2^2 + \beta_2^2\left(y^2 + Y^2\right) + 2\beta_2\gamma_2(y + Y)\right) + 9\big) = 0.
\end{aligned}
$$

From $e_1 = 0$ we obtain $Y = f(y)$, substituting it in $e_2 = 0$ we find an equation of the variable y of degree six. Thus system (14) when $i = 1$ and $j = 3$ has at most six real solutions, due to the symmetry of the solutions of this system, we conclude that the discontinuous piecewise differential systems $(\tilde{\mathcal{C}}_1)$– $(\tilde{\mathcal{C}}_3)$ has at most three limit cycles.

In what follows we give a class of discontinuous piecewise differential system which has exactly three limit cycles. So in Σ^- we consider the cubic reversible isochronous center of type $(\tilde{\mathcal{C}}_3)$

$$(19) \quad \begin{aligned} \dot{x} = & \ -0.013446..x^3 + x^2(0.305282..y + 0.45729..) + x\left(-2.31041..y^2 - 4.90347..y - 2.46581..\right) \\ & +5.82847..y^3 + 10.9181..y^2 + 9.12441..y - 0.481199.., \\ \dot{y} = & \ -0.00177667..x^3 + x^2(0.040338..y + 0.0780411..) + x(-0.305282..y^2 - 0.914579..y \\ & -0.84173..) + 0.770136..y^3 + 2.45173..y^2 + 2.46581..y + 0.233753.., \end{aligned}$$

with the first integral

$$\tilde{H}_3(x,y) = \ \left(3\left(\frac{1}{2}x - 1.7828..y\right) - 4\left(\frac{1}{10}x - 0.75681..y + 0.410816..\right)^2\right)^2 + 9\left(\frac{1}{10}x - 0.75681..y\right. $$
$$\left. +0.410816..\right)^2.$$

In Σ^+ we consider the cubic reversible isochronous center of type $(\tilde{\mathcal{C}}_1)$

$$(20) \quad \begin{aligned} \dot{x} = & \ \left(\frac{7}{200} - \frac{9}{40}y\right)x^2 + \frac{1}{100}x^3 + \left(-\frac{5}{8}y^2 - \frac{27}{20}y + 0.405909..\right)x - \frac{15}{8}y^2 + 5.07727..y \\ & -0.247727.., \\ \dot{y} = & \ \left(\frac{1}{100}y + \frac{9}{500}\right)x^2 + x\left(-\frac{9}{40}y^2 + \frac{3}{20}y - 0.173364..\right) - \frac{5}{8}y^3 - \frac{17}{40}y^2 - 0.105909..y \\ & -0.113909.., \end{aligned}$$

its corresponding first integral is

$$\tilde{H}_1(x,y) = -\frac{5\left(4x^2 + 20xy + 20x + 25y^2 + 50y - 71\right)}{104x^2 + 300xy + 124x + 3125y^2 - 350y + 61}.$$

The pairs $(0.316546.., -0.285777..)$, $(0.183095.., -0.0988845..)$ and $(0.12139.., -0.0261521..)$ are the distinct three real solution of system (14) when $i = 1$ and $j = 3$. Then the discontinuous piecewise differential system (19)–(20) has three limit cycles shown in Figure 2(c). This example completes the proof of statement (III) of Theorem 4. $\qquad\square$

Proof of statement (IV) of Theorem 4. We consider the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_2)$–$(\tilde{\mathcal{C}}_2)$, we obtain that system (13) when $i = 2$ is written as

$$e_1 = \ (y - Y)E_Y = 0, \ \ e_2 = (y - Y)E_y = 0.$$

We denote by E_y and E_Y the polynomials of variables y and Y where

$$\begin{aligned} E_Y = & \ \left(4b_1^6 y^2 Y^2 (y + Y) + 4b_1^5(2c_1 - 1)yY\left(y^2 + 3yY + Y^2\right) + b_1^4\left((y + Y)((1 - 2c_1)^2\left(y^2 + 8yY + Y^2\right)\right.\right. \\ & +4\beta_1^2 y^2 Y^2) + 8\beta_1\gamma_1 y^2 Y^2\right) + 2b_1^3\left(y^2\left(8c_1^3 - 12c_1^2 + c_1(8\beta_1 Y(\gamma_1 + \beta_1 Y) + 6) - 2\beta_1 Y(2\gamma_1 + 3\beta_1 Y)\right.\right. \\ & \left.-1) + 2\beta_1^2(2c_1 - 1)y^3 Y + yY\left(2\gamma_1^2 + 2\beta_1^2(2c_1 - 1)Y^2 + 4\beta_1\gamma_1(2c_1 - 1)Y + 3(2c_1 - 1)^3\right) + (2c_1 - 1)^3 \right. \\ & \left. Y^2\right) + b_1^2\left((y + Y)(20c_1^4 - 40c_1^3 + 30c_1^2 + \beta_1^2(2c_1 - 1)((2c_1 - 1)y^2 + 8(c_1 - 1)yY + (2c_1 - 1)Y^2)\right. \\ & -10c_1 + 1) + 2\beta_1\gamma_1\left((1 - 2c_1)^2 y^2 + (4c_1(4c_1 - 5) + 5)yY + (1 - 2c_1)^2 Y^2\right) + (4c_1 - 3)\gamma_1^2(y + Y)\right) \\ & +2b_1(c_1 - 1)(2c_1 - 1)\left(2c_1^3 - 2c_1^2 + \gamma_1^2 + c_1(2\beta_1(y + Y)(2\gamma_1 + \beta_1(y + Y)) + 1) - \beta_1^2(y^2 + 3yY \right. \\ & \left. +Y^2) - 2\beta_1\gamma_1(y + Y)\right) + \beta_1\left(2c_1^2 - 3c_1 + 1\right)^2 (2\gamma_1 + \beta_1(y + Y))\right), \end{aligned}$$

$$E_y = \Big(4b_2^6 y^2 Y^2 (y+Y) + 4b_2^5 (2c_2 - 1)yY\left(y^2 + 3yY + Y^2\right) + b_2^4\big((y+Y)\big((1-2c_2)^2\left(y^2 + 8yY + Y^2\right)$$
$$+ 4\beta_2^2 y^2 Y^2\big) + 8\beta_2\gamma_2 y^2 Y^2\big) + 2b_2^3\big(y^2\big(8c_2^3 - 12c_2^2 + c_2(8\beta_2 Y(\gamma_2 + \beta_2 Y) + 6) - 2\beta_2 Y(2\gamma_2 + 3\beta_2 Y)$$
$$-1\big) + 2\beta_2^2 (2c_2 - 1)y^3 Y + yY\left(2\gamma_2^2 + 2\beta_2^2 (2c_2 - 1)Y^2 + 4\beta_2\gamma_2(2c_2 - 1)Y + 3(2c_2 - 1)^3\right) + (2c_2 - 1)^3$$
$$Y^2\big) + b_2^2\big((y+Y)\big(20c_2^4 - 40c_2^3 + 30c_2^2 + \beta_2^2(2c_2 - 1)\big((2c_2 - 1)y^2 + 8(c_2 - 1)yY + (2c_2 - 1)Y^2\big) - 10$$
$$c_2 + 1\big) + 2\beta_2\gamma_2\big((1-2c_2)^2 y^2 + (4c_2(4c_2 - 5) + 5)yY + (1-2c_2)^2 Y^2\big) + (4c_2 - 3)\gamma_2^2(y+Y)\big) + 2b_2$$
$$(c_2 - 1)(2c_2 - 1)\big(2c_2^3 - 2c_2^2 + \gamma_2^2 + c_2(2\beta_2(y+Y)(2\gamma_2 + \beta_2(y+Y)) + 1) - \beta_2^2\left(y^2 + 3yY + Y^2\right)(y$$
$$-2\beta_2\gamma_2 + Y)\big) + \beta_2\left(2c_2^2 - 3c_2 + 1\right)^2\left(2\gamma_2 + \beta_2(y+Y)\right)\Big).$$

The number of the common zeros (y, Y) of E_y and E_Y show the existence and the number of limit cycles of the discontinuous piecewise differential systems $(\tilde{\mathcal{C}}_2)$– $(\tilde{\tilde{\mathcal{C}}}_2)$. To find this number, we calculate the resultants, Resultant $[E_y, E_Y, y]$ and Resultant $[E_y, E_Y, Y]$ of E_y and E_Y (or simply R_y and R_Y, respectively) with respect to y and Y, respectively. By the symmetry of E_y and E_Y with respect to y and Y, we know that the resultant R_y and R_Y have the same expression. So we only need to calculate one of them, and in this case we consider R_y which is a polynomial of degree sixteen in the variable Y, and because of the big expression of R_y we omit it. Consequently the maximum number of solutions of system (13) when $i = 2$ is at most sixteen. Due to the symmetry of these solutions it results that the discontinuous piecewise differential systems $(\tilde{\mathcal{C}}_2)$– $(\tilde{\tilde{\mathcal{C}}}_2)$ can have at most eight limit cycles.

In what follows we construct a class of discontinuous piecewise differential system which has exactly eight limit cycles.

In the first half-plane Σ^- we consider the cubic reversible isochronous center $(\tilde{\mathcal{C}}_2)$

$$\dot{x} = -1.33333..\left(\frac{3}{4}x - \frac{7}{10}\right)\left(\frac{24}{10} - \frac{1}{2}x\right)\left(\frac{11}{2}x - \frac{11}{4}y + \frac{3}{2}\right),$$

(21) $\quad \dot{y} = 0.484848..\left(\frac{11}{2}\left(\frac{24}{10} - \frac{3}{2}x\right)\left(\frac{7}{10} - \frac{3}{4}x\right)\left(\frac{11}{2}x - \frac{11}{4}y + \frac{3}{2}\right) - \frac{3}{4}\left(2\left(\frac{17}{10} - \frac{3}{4}x\right)^3 - 2\left(\frac{17}{10}\right)\right.$

$$\left. - \frac{3}{4}x\right)^2 + \left(\frac{11}{2}x - \frac{11}{4}y + \frac{3}{2}\right)^2 - \frac{3}{4}x + \frac{17}{10}\right),$$

with the first integral

$$\tilde{H}_2(x, y) = \frac{(15x - 14)^2\left(12325x^2 - 12100xy + 5580x + 3025y^2 - 3300y + 2056\right)}{14400(5x - 8)^2}.$$

In Σ^+ we consider the cubic reversible isochronous center $(\tilde{\tilde{\mathcal{C}}}_2)$

$$\dot{x} = x^2(4.19651.. - 0.367605..y) + \frac{9}{10}x^3 + x((-1.08819.. * 10^{-15}y - 1.63216..)y + 5.10083..)$$

(22) $\quad + y((-8.05325.. * 10^{-31}y - 1.87169.. * 10^{-15})y - 1.7198..) + 0.938074..,$

$$\dot{y} = x^2\left(10.8999.. - \frac{9}{10}y\right) + 2.88353..x^3 + x((-2.66421.. * 10^{-15}y - 2.196..)y + 15.8829..)$$

$$+ y((-1.97166.. * 10^{-30}y - 0.367605..)y - 3.80954..) + 4.12556..,$$

its corresponding first integral is

$$\tilde{\tilde{H}}_2(x, y) = \frac{1}{(2(-0.5x - 7.40057.. * 10^{-16}y - 0.36) - 1)^2}\left(\left(\left(-\frac{1}{2}x - 7.40057.. * 10^{-16}y - \frac{36}{100}\right)^2\right.\right.$$

$$\left.\left. + (\frac{9}{10}x - 0.367605..y + 0.200512..)^2\right)\left(-\frac{1}{2}x - 7.40057.. * 10^{-16}y - \frac{36}{100}\right)^2\right).$$

The eight pairs $(0.162904.., 0.928005..)$, $(0.162904.., 0.928005..)$ $(-0.365742.., 1.45665..)$, $(-0.537248..,$ $1.62816..)$, $(-0.685078.., 1.77599..)$, $(-0.816962.., 1.90787..)$, $(-0.93716.., 2.02807..)$ and $(-1.04832..,$

2.13923..) are solutions of system (13) when $i = 2$. Consequently the discontinuous piecewise differential system (21)–(22) has eight limit cycles, see Figure 3(a). Thus, the proof of statement (IV) holds.   □

Proof of statement (V) of Theorem 4. We consider the class of discontinuous piecewise differential systems formed by the cubic reversible isochronous center $(\tilde{C}_2)$ with the first integral $\tilde{H}_2(x, y)$ and the cubic reversible isochronous differential center $(\tilde{C}_3)$ with its first integral $\tilde{H}_3(x, y)$, so system (14) when $i = 2$ and $j = 3$ becomes

$$
\begin{aligned}
e_1 = \ & 4b_2^6 y^2 Y^2 (y + Y) + 4b_2^5 (2c_2 - 1)yY \left(y^2 + 3yY + Y^2\right) + b_2^4 \Big((y + Y)\big((1 - 2c_2)^2 \left(y^2 + 8yY + Y^2\right) \\
& + 4\beta_2^2 y^2 Y^2\big) + 8\beta_2 \gamma_2 y^2 Y^2\Big) + 2b_2^3 \big(y^2 \big(8c_2^3 - 12c_2^2 + c_2(8\beta_2 Y(\gamma_2 + \beta_2 Y) + 6) - 2\beta_2 Y(2\gamma_2 + 3\beta_2 Y) \\
& - 1\big) + 2\beta_2^2 (2c_2 - 1)y^3 Y + yY \left(2\gamma_2^2 + 2\beta_2^2 (2c_2 - 1)Y^2 + 4\beta_2 \gamma_2 (2c_2 - 1)Y + 3(2c_2 - 1)^3\right) + (2c_2 - 1)^3 \\
& Y^2\big) + b_2^2 \big((y + Y)\big(20c_2^4 - 40c_2^3 + 30c_2^2 + \beta_2^2 (2c_2 - 1)\left((2c_2 - 1)y^2 + 8(c_2 - 1)yY + (2c_2 - 1)Y^2\right) - 10 \\
& c_2 + 1\big) + 2\beta_2 \gamma_2 \left((1 - 2c_2)^2 y^2 + (4c_2(4c_2 - 5) + 5)yY + (1 - 2c_2)^2 Y^2\right) + (4c_2 - 3)\gamma_2^2 (y + Y)\big) + 2b_2 \\
& (c_2 - 1)(2c_2 - 1)\big(2c_2^3 - 2c_2^2 + \gamma_2^2 + c_2(2\beta_2(y + Y)(2\gamma_2 + \beta_2(y + Y)) + 1) - \beta_2^2 \left(y^2 + 3yY + Y^2\right)(y \\
& - 2\beta_2 \gamma_2 + Y)\big) + \beta_2 \left(2c_2^2 - 3c_2 + 1\right)^2 (2\gamma_2 + \beta_2(y + Y)) = 0, \\
e_2 = \ & 9b_1^2(y + Y) + 6b_1 \left(3c_1 - 4 \left(\beta_1^2 y^2 + \beta_1 y(2\gamma_1 + \beta_1 Y) + (\gamma_1 + \beta_1 Y)^2\right)\right) + \beta_1(2\gamma_1 + \beta_1 y + \beta_1 Y)(32\gamma_1^2 \\
& - 24c_1 + 16\beta_1^2 y^2 + 32\beta_1 \gamma_1 y + 16\beta_1^2 Y^2 + 32\beta_1 \gamma_1 Y + 9) = 0.
\end{aligned}
$$

Using Bezout theorem we know that the maximum number of solutions of system (14) when $i = 2$ and $j = 3$ is at most fifteen. As the solutions of this system are symmetric, therefore we conclude that the maximum number of solutions of the system (14) is at most seven. Hence the maximum number of limit cycles of the discontinuous piecewise differential systems $(\tilde{C}_2)$– $(\tilde{C}_3)$ is at most seven.

Now we give a class of discontinuous piecewise differential system which has exactly seven limit cycles. In Σ^+ we consider the cubic reversible isochronous center of type $(\tilde{C}_2)$

(23)
$$
\begin{aligned}
\dot{x} = \ & 0.0000870489..x^3 + x^2(-0.00792718..y - 0.079887..) + x(0.235678..y^2 + 5.98489..y \\
& - 39.2858..) - 2.29883..y^3 - 103.885..y^2 + 862.313..y + 1319.27.., \\
\dot{y} = \ & 2.963419321.. \times 10^{-6} x^3 + x^2(-0.000241147..y - 0.00641714..) + x(0.00652981..y^2 \\
& + 0.351331..y + 4.42621..) - 0.0588451..y^3 - 4.88435..y^2 - 100.652..y - 2352.32..,
\end{aligned}
$$

its first integral is

$$
\begin{aligned}
\tilde{H}_2(x, y) = \ & \frac{1}{(-x + 26y + 831.968..)^2} (6.5 \times 10^{-6} x^4 + x^3(-0.000719868..y - 0.0160353..) + x^2((0.0317101..y \\
& + 0.864893..)y + 44.7714..) + x(y((-0.646011..y - 15.2721..)y - 1692.73..) - 49965..) + y(y \\
& (y(5.04228..y + 94.2432..) + 14776.6) + 1.05795... \times 10^6) + 1.7008 \times 10^7).
\end{aligned}
$$

In Σ^- we consider the cubic reversible isochronous center of type $(\tilde{C}_3)$

(24)
$$
\begin{aligned}
\dot{x} = \ & \frac{1}{9590625} (531250x^3 + 1875x^2(578y - 1535) + 75x \left(9826y^2 - 110160y - 65075\right) + 167042y^3 \\
& - 4287315y^2 + 21974850y + 37022000), \\
\dot{y} = \ & \frac{1}{383625} (-31250x^3 - 3750x^2(17y + 5) - 75x \left(578y^2 - 3070y - 2825\right) - 9826y^3 + 165240y^2 \\
& + 195225y - 41500),
\end{aligned}
$$

which has the first integral

$$
\tilde{H}_3(x, y) = 9\left(\frac{1}{4}x + \frac{17}{100}y + \frac{1}{5}\right)^2 + \left(3\left(\frac{1}{10}x + \frac{3}{4}y + 1\right) - 4\left(\frac{1}{4}x + \frac{17}{100}y + \frac{1}{5}\right)^2\right)^2.
$$

System (14) when $i = 2$ and $j = 3$ has the seven solutions $(-1.9409.., -0.669484..)$, $(-1.89267.., -0.72376..)$, $(-1.83955.., -0.782914..)$, $(-1.77978.., -0.848708..)$, $(-1.71019.., -0.924315..)$, $(-1.62382.., -1.0167..)$ and $(-1.49637.., -1.15015..)$. Therefore the discontinuous piecewise differential system (23)–(24) has seven limit cycles shown in Figure 3(b). Thus, the proof of this statement is done. □

Proof of statement (VI) *of Theorem 4.* For the discontinuous piecewise differential systems $(\tilde{\mathcal{C}}_3)$– $(\tilde{\tilde{\mathcal{C}}}_3)$ we obtain that the system (13) when $i = 3$ is given by

$$e_1 = (y - Y)E_Y = 0, \quad e_2 = \quad (y - Y)E_y = 0.$$

Where

$$E_Y = \left(9b_1^2(y + Y) + 6b_1\left(3c_1 - 4\left(\beta_1^2 y^2 + \beta_1 y(2\gamma_1 + \beta_1 Y) + (\gamma_1 + \beta_1 Y)^2\right)\right) + \beta_1(2\gamma_1 + \beta_1 y + \beta_1 Y) \right.$$
$$\left. \left(32\gamma_1^2 - 24c_1 + 16\beta_1^2 y^2 + 32\beta_1\gamma_1 y + 16\beta_1^2 Y^2 + 32\beta_1\gamma_1 Y + 9\right) \right),$$

$$E_y = \left(9b_2^2(y + Y) + 6b_2\left(3c_2 - 4\left(\beta_2^2 y^2 + \beta_2 y(2\gamma_2 + \beta_2 Y) + (\gamma_2 + \beta_2 Y)^2\right)\right) + \beta_2(2\gamma_2 + \beta_2 y + \beta_2 Y) \right.$$
$$\left. \left(32\gamma_2^2 - 24c_2 + 16\beta_2^2 y^2 + 32\beta_2\gamma_2 y + 16\beta_2^2 Y^2 + 32\beta_2\gamma_2 Y + 9\right) \right).$$

As in statement (IV) and by computing the resultants, Resultant $[E_y, E_Y, y]$ and Resultant $[E_y, E_Y, Y]$ of E_y and E_Y with respect to y and Y, respectively. Due to the symmetry of E_y and E_Y with respect to y and Y, we obtain that the resultant R_y is a polynomial of degree six. Consequently the maximum number of solutions of system (13) is at most six. Since their solutions are symmetric we know that the discontinuous piecewise differential systems $(\tilde{\mathcal{C}}_3)$– $(\tilde{\tilde{\mathcal{C}}}_3)$ has at most three limit cycles.

To prove that our result is reached we give an example of discontinuous piecewise differential system with exactly three limit cycles.

In the first half-plane Σ^- we consider the cubic reversible isochronous center $(\tilde{\mathcal{C}}_3)$

(25)
$$\dot{x} = -0.00343827..x^3 + x^2(0.0908048..y - 0.258721..) + x(-0.799387..y^2 + 2.20767..y$$
$$+0.10374..) + 2.34576..y^3 + 0.615762..y^2 + 0.895054..y - 0.0485279..,$$
$$\dot{y} = -0.000390563..x^3 + x^2(0.0103148..y - 0.0445347..) + x(-0.0908048..y^2 + 0.517442..y$$
$$-1.13956..) + 0.266462..y^3 - 1.10383..y^2 - 0.10374..y - 0.00943529..,$$

with the first integral

$$\tilde{H}_3(x, y) = \left(3(-x - 0.300305..y) - 4\left(-\frac{1}{10}x + 0.880335..y - 0.0508927..\right)^2\right)^2 + 9\left(-\frac{1}{10}x + 0.880335..y \right.$$
$$\left. -0.0508927..\right)^2.$$

In Σ^+ we consider the cubic reversible isochronous center $(\tilde{\tilde{\mathcal{C}}}_3)$

(26)
$$\dot{x} = -0.0118519..x^3 + x^2(0.355556..y - 0.235556..) + x(-3.55556..y^2 + 2.04444..y$$
$$-0.218889..) + 11.8519..y^3 + 3.11111..y^2 + 4.52222..y - 0.245185..,$$
$$\dot{y} = -0.00118519..x^3 + x^2(0.0355556..y - 0.0368889..) + x(-0.355556..y^2 + 0.471111..y$$
$$-0.298556..) + 1.18519..y^3 - 1.02222..y^2 + 0.218889..y - 0.137852..,$$

its corresponding first integral is

$$\tilde{\tilde{H}}_3(x, y) = 9\left(-\frac{1}{10}x + y - \frac{1}{10}\right)^2 + \left(3\left(-\frac{1}{4}x - \frac{1}{2}y - \frac{1}{10}\right) - 4\left(-\frac{1}{10}x + y - \frac{1}{10}\right)^2\right)^2.$$

The discontinuous piecewise differential system (25)–(26) has three limit cycles because system (13) when $i = 3$ has the three real solutions $(-0.333962.., 0.372192..)$, $(-0.209785.., 0.278363..)$ and $(-0.420112.., 0.436464..)$. These three limit cycles are shown in Figure 4(a). Thus, the proof of Theorem 4 is done. $\square$

References

[1] A. ANDRONOV, A. VITT AND S. KHAIKIN, Theory of Oscillations. Pergamon Press, Oxford. 1966.

[2] J.C. ARTÉS, J. LLIBRE, J.C. MEDRADO AND M.A. TEIXEIRA, Piecewise linear differential systems with two real saddles. Math.Comput. Simul. **95**, 13–22, 2013.

[3] S. BANERJEE AND G. VERGHESE, Nonlinear Phenomena in Power Electronics: Attractors, Bifurcations, Chaos, and Nonlinear. Control, Wiley-IEEE Press, New York. 2001.

[4] R. BENTERKI, L. DAMENE AND J. LLIBRE, The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves II. Differ. Equ. Dyn. Syst. 2021. https://doi.org/10.1007/s12591-021-00564-w.

[5] R. BENTERKI AND J. LLIBRE, The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves I. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis. **28**, 153-192, 2021.

[6] R. BENTERKI AND J. LLIBRE, The solution of the second part of the 16th Hilbert problem for nine families of discontinuous piecewise differential systems. Nonlinear Dyn. **102**, 2453–2466, 2020.

[7] I.S. BEREZIN AND N.P. ZHIDKOV, Computing Methods. Pergamon Press, Oxford, Volume II, 1964.

[8] M. DI BERNARDO, C. J. BUDD, A. R. CHAMPNEYS AND P. KOWALCZYK, Piecewise-Smooth Dynamical Systems. Theory and Applications, Appl. Math. Sci. Series 163, Springer-Verlag, London. 2008.

[9] D.C. BRAGA AND L.F. MELLO, Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane. Nonlinear Dyn. **73**, 1283–1288, 2013.

[10] J. CASTILLO, J. LLIBRE AND F. VERDUZCO, The pseudo-Hopf bifurcation for planar discontinuous piecewise linear differential systems. Nonlinear Dyn. **90**, 1829–1840, 2017.

[11] J. CHAVARRIGA AND M. SABATINI, A survey on isochronous centers, Qual. Theory Dyn. Syst. **1** 1–70, 1999.

[12] M. ESTEBAN, C. VALLS AND J. LLIBRE, The 16th Hilbert problem for discontinuous piecewise isochronous centers of degree one or two separated by a straight line. Chaos. **31**, no. 4, Paper No.043112, 18 pp, 2021.

[13] R.D. EUZÉBIO AND J. LLIBRE, On the number of limit cycles in discontinuous piecewise linear differential systems with two pieces separated by a straight line. J. Math. Anal. Appl. **424**, 475–486, 2015.

[14] E. FREIRE, E. PONCE, F. RODRIGO AND F. TORRES, Bifurcation sets of continuous piecewise linear systems with two zones. Int.J. Bifurc. Chaos. **8**, 2073–2097, 1998.

[15] A.F. FILIPPOV, Differential Equation with Discontinuous Right-Hand Sides. Kluwer Academic. Amsterdam. 1988.

[16] M. HAN AND W. ZHANG, On hopf bifurcation in non-smooth planar systems. J. Differential Equations. **284**, 2399–2416, 2010.

[17] D. HILBERT, Mathematische Probleme. Lecture, in Second Internat. Congr. Math., Paris, Nachr. Ges. Wiss. Goett. Math.-Phys. Kl. 1900.

[18] S.M. HUAN AND X.S. YANG, On the number of limit cycles in general planar piecewise linear systems. Disc. Cont. Dyn. Syst. **32**(6), 2147–2164, 2012.

[19] YU. ILYASHENKO, Centennial history of Hilbert's 16th problem. Bull. Amer. Math. Soc. (N.S). **39**, 301–354, 2002.

[20] M. KUKUCKA, Non-Smooth Dynamical Systems. Springer-Verlag, Berlin, Heidelberg. 2000.

[21] J. LI, Hilbert's 16th problem and bifurcations of planar polynomial vector fields. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13**, 47–106, 2003.

[22] J. LLIBRE AND E. PONCE, Three nested limit cycles in discontinuous piecewise linear differential systems with two zones. Dyn. Cont. Disc. Impul. Syst. Series B. **19**, 325–335, 2012.

[23] J. LLIBRE AND M.A. TEIXEIRA, Piecewise linear differential systems with only centers can create limit cycles?. Nonlinear Dynam. **91**, 249–255, 2018.

[24] O. MAKARENKOV AND J.S.W. LAMB, Dynamics and bifurcations of non-smooth systems. a survey, Phys. D. **241**, 1826–1844, 2012.

[25] D.J.W. SIMPSON, Bifurcations in piecewise smooth continuous systems. World Scientific Series on Nonlinear Science A. **69**, 2010.

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