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A separation theorem for entire transcendental maps

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Abstract

We study the distribution of periodic points for a wide class of maps, namely entire transcendental functions of finite order and with bounded set of singular values, or compositions thereof. Fix $p \in \mathbb{N}$ and assume that all dynamic rays which are invariant under f^p land. An interior p-periodic point is a fixed point of f^p which is not the landing point of any periodic ray invariant under f^p. Points belonging to attracting, Siegel or Cremer cycles are examples of interior periodic points. For functions as above, we show that rays which are invariant under f^p, together with their landing points, separate the plane into finitely many regions, each containing *exactly* one interior p-periodic point or one parabolic immediate basin invariant under f^p. This result generalizes the Goldberg–Milnor Separation Theorem for polynomials, and has several corollaries. It follows, for example, that two periodic Fatou components can always be separated by a pair of periodic rays landing together; that there cannot be Cremer points on the boundary of Siegel disks; that 'hidden components' of a bounded Siegel disk have to be either wandering domains or preperiodic to the Siegel disk itself; or that there are only finitely many non-repelling cycles of any given period, regardless of the number of singular values.

1. *Introduction*

Given a holomorphic map $f : \mathbb{C} \to \mathbb{C}$, we are interested in the dynamical system generated by the iterates of f. In this setup, there is a dynamically meaningful partition of the phase space into two completely invariant subsets: the *Fatou set*, where the dynamics are in some sense stable, and the *Julia set*, where they are chaotic. More precisely, the Fatou set is defined as the open set

$$\mathcal{F}(f) := \{z \in \mathbb{C}; \{f^n\} \text{ is normal in a neighborhood of } z\},$$

and the *Julia set* $J(f)$ as its complement. Another dynamically interesting set is the set of escaping points or *escaping set*

$$I(f) := \{z \in \mathbb{C};\ f^n(z) \to \infty,\ \text{as } n \longrightarrow \infty\}.$$

The relation between them is that, in general, $J(f) = \partial I(f)$, although for some classes of functions $I(f) \subset J(f)$ (see [8]) and hence $J(f) = \overline{I(f)}$.

In this paper, we are mainly concerned with *entire transcendental* maps (abbreviated transcendental maps), that is, those entire maps for which infinity is an essential singularity. There are several important differences between the dynamics of transcendental maps and that of polynomials, coming from the very different behavior of iterates in a neighborhood of infinity. For example, while the Julia set of polynomials is always a compact set disjoint from $I(f)$, its analog for transcendental maps is always unbounded and may contain points of $I(f)$.

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