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POLYNOMIAL DIFFERENTIAL SYSTEMS WITH EXPLICIT NON-ALGEBRAIC LIMIT CYCLES

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Abstract. Up to now all the examples of polynomial differential systems for which non-algebraic limit cycles are known explicitly have degree at most 5. Here we show that already there are polynomial differential systems of degree at least exhibiting explicit non-algebraic limit cycles. It is well known that polynomial differential systems of degree 1 (i.e. linear differential systems) has no limit cycles. It remains the open question to determine if the polynomial differential systems of degree 2 can exhibit explicit non-algebraic limit cycles.

1. Introduction and statement of the main results

Probably the existence of limit cycles is one of the more difficult objects to study in the qualitative theory of differential equations in the plane. There is a huge literature dedicated to this topic, see for instance the book of Ye et al [12], or the famous Hilbert 16th problem [6] and [7]. Publications more closely related to the problem in this article are [4, 5, 1, 2, 8, 9].

A *polynomial differential system* is a system of the form

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y. The *degree* of the system is the maximum of the degrees of the polynomials P and Q. As usual the dot denotes derivative with respect to the independent variable t.

A *limit cycle* of system (1.1) is an isolated periodic solution in the set of all periodic solutions of system (1.1). If a limit cycle is contained in an algebraic curve of the plane, then we say that it is *algebraic*, otherwise it is called *non-algebraic*. In other words a limit cycle is algebraic if there exists a real polynomial $f(x,y)$ such that the algebraic curve $f(x,y)=0$ contains the limit cycle. In general, the orbits of a polynomial differential system (1.1) are contained in analytic curves which are not algebraic.

To distinguish when a limit cycle is algebraic or not, usually, it is not easy. Thus, the well-known limit cycle of the van der Pol differential system exhibited in 1926 (see [11]) was not proved until 1995 by Odani [10] that it was non-algebraic.

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