Journal of Difference Equations and Applications
Vol. 16, Nos. 5–6, May–June 2010, 523–553

Taylor & Francis
Taylor & Francis Group

An entire transcendental family with a persistent Siegel disc

Rubén Berenguel[1] and Núria Fagella*

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, Barcelona 08007, Spain

(*Received 4 August 2009; final version received 6 August 2009*)

Dedicated to Robert Devaney on the occasion of his 60th birthday

Downloaded By: [Consorci de Biblioteques Universitaries de Catalunya] At: 17:12 11 June 2010

We study the class of entire transcendental maps of finite order with one critical point and one asymptotic value, which has exactly one finite pre-image, and having a persistent Siegel disc. After normalization, this is a one parameter family f_a with $a \in \mathbb{C}^*$ which includes the semi-standard map $\lambda z e^z$ at $a = 1$, approaches the exponential map when $a \to 0$ and a quadratic polynomial when $a \to \infty$. We investigate the stable components of the parameter plane (capture components and semi-hyperbolic components) and also some topological properties of the Siegel disc in terms of the parameter.

Keywords: Siegel disc; transcendental maps; Riemann; dynamical systems

1. Introduction

Given a holomorphic endomorphism $f : S \to S$ on a Riemann surface S we consider the dynamical system generated by the iterates of f, denoted by $f^n = f \circ \overset{n)}{\cdots} \circ f$. The *orbit* of an initial condition $z_0 \in S$ is the sequence $\mathcal{O}^+(z_0) = \{f^n(z_0)\}_{n \in \mathbb{N}}$ and we are interested in classifying the initial conditions in the *phase space* or *dynamical plane S*, according to the asymptotic behaviour of their orbits when n tends to infinity.

There is a dynamically natural partition of the phase space S into the *Fatou set* $\mathcal{F}(f)$ (open) where the iterates of f form a normal family and the *Julia set* $\mathcal{J}(f) = S \setminus \mathcal{F}(f)$ which is its complement (closed).

If $S = \hat{\mathbb{C}} = \mathbb{C} \cup \infty$, then f is a rational map. If $S = \mathbb{C}$ and f does not extend to the point at infinity, then f is an *entire transcendental map*, that is, infinity is an essential singularity. Entire transcendental functions present many differences with respect to rational maps.

One of them concerns the singularities of the inverse function. For a rational map, all branches of the inverse function are well defined except at a finite number of points called the *critical values*, points $w = f(c)$ where $f'(c) = 0$. The point c is then called a *critical point*. If f is an entire transcendental map, there is another possible obstruction for a branch of the inverse to be well defined, namely its *asymptotic values*. A point $v \in \mathbb{C}$ is called an asymptotic value if there exists a path $\gamma(t) \to \infty$ when $t \to \infty$, such that $f(\gamma(t)) \to v$ as $t \to \infty$. An example is $v = 0$ for $f(z) = e^z$, where $\gamma(t)$ can be chosen to be the negative real axis.

In any case, the set of singularities of the inverse function also called *singular values*, plays a very important role in the theory of iteration of holomorphic functions. This statement is motivated by the non-trivial fact that most connected components of the Fatou

*Corresponding author. Email: fagella@maia.ub.es

ISSN 1023-6198 print/ISSN 1563-5120 online
© 2010 Taylor & Francis
DOI: 10.1080/10236190903251738
http://www.informaworld.com