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Hyperbolic entire functions with bounded Fatou components

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Abstract. We show that an invariant Fatou component of a hyperbolic transcendental entire function is a Jordan domain (in fact, a quasidisc) if and only if it contains only finitely many critical points and no asymptotic curves. We use this theorem to prove criteria for the boundedness of Fatou components and local connectivity of Julia sets for hyperbolic entire functions, and give examples that demonstrate that our results are optimal. A particularly strong dichotomy is obtained in the case of a function with precisely two critical values.

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1. Introduction

Dynamical systems that are *hyperbolic* (or "Axiom A" in Smale's terminology) exhibit, in a certain sense, the simplest possible behaviour. (For the formal definition of hyperbolicity in our context, see Definition 1.1 below.) In any given setting, understanding hyperbolic systems is the first step on the way to studying more general types of behaviour. Furthermore, in many one-dimensional situations, hyperbolic dynamics is either known or believed to be topologically generic (see e.g. [28, 30, 31, 48]), and hence many systems are indeed hyperbolic.

In the iteration of complex polynomials $p\colon \mathbb{C} \to \mathbb{C}$, the dynamics of hyperbolic functions has been essentially completely understood since the seminal work of Douady, Hubbard and Thurston in the 1980s. In particular, these can be classified — in a variety of ways — using finite combinatorial objects such as "Hubbard trees". Typically, any qualitative question about the iterative behaviour of the map under consideration can be answered from this encoding.

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