Hurwitz systems with periodic orbits. Counterexamples to Kalman and Markus-Yamabe conjectures

by

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Abstract. We construct a system that is globally Hurwitz with a periodic orbit. With this system we build a counterexemple to Kalman conjecture in dimension 4, and consequently to Markus-Yamabe conjecture. Then we extend these counterexamples to dimension larger than 4.

1 Introduction

A matrix is *Hurwitz* if all its eigenvalues have negative real part. A system is *Hurwitz* if its Jacobian is Hurwitz in every point of $\mathbb{R}^n$.

Consider the system of ordinary differential equations $x'(t) = F(x(t))$, where $F(x)$ is C^1 and $x = 0$ is an equilibrium point. If $DF(0)$ is Hurwitz then by Hartman-Grobman Theorem [9] the origin is a local asymptotically stable solution. The question is what hypothesis we have to add to $F(x)$ to assure that the origin is a global attractor. In 1960 Markus and Yamabe [16] conjectured that if we have a system $x' = F(x)$ with $F(x) \in C^1$, Hurwitz in all $\mathbb{R}^n$, with the origin a critical point then the origin is a global asymptotically stable solution.

It is easy to prove the Markus-Yamabe conjecture for $n = 1$. For $n = 2$ several authors proved that the Markus-Yamabe conjecture is true adding additional assumptions. Thus, for instance; Markus and Yamabe in [16] proved the conjecture if $F = (F_1, F_2)$ and $\frac{\partial F_i}{\partial x_j} = 0$ for $i \neq j$, $i, j \in \{1, 2\}$. Olech in [17] also proved the conjecture if $\int_0^\infty \min_{||x||=r} ||F(x)|| dr = \infty$.

In [7] Gasull, Llibre and Sotomayor summarize a list with more than 10 different additional sufficient conditions which force that Markus-Yamabe conjecture holds for $n = 2$.

In 1988 Meistres and Olech [15] proved the Markus-Yamabe conjecture in the plane when F_1, F_2 are polynomials. The last results have been given by Gutierrez [10] and Fessler [5] that independently proved the Markus-Yamabe conjecture for $n = 2$ in 1993. A more simple proof has been done by Glutsyuk [8].

If $n > 2$ there are also some additional conditions forcing the Markus-Yamabe conjecture. For example if $DF(x)$ is negative definite for all $x \in \mathbb{R}^n$ the conjecture was proved by Hartman in [11]. Other additional sufficient conditions were given by Hartamn and Olech in [12]. But the open question is: Without additional assumptions is it true the Markus-Yamabe conjecture for $n > 2$?

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This paper is a short version of [4] where appear all the proofs and details.