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The rolling ball problem on the plane revisited

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Abstract. By a sequence of rollings without slipping or twisting along segments of a straight line of the plane, a spherical ball of unit radius has to be transferred from an initial state to an arbitrary final state taking into account the orientation of the ball. We provide a new proof that with at most 3 moves, we can go from a given initial state to an arbitrary final state. The first proof of this result is due to Hammersley (1983). His proof is more algebraic than ours which is more geometric. We also showed that "generically" no one of the three moves, in any elimination of the spin discrepancy, may have length equal to an integral multiple of 2π.

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1. Introduction and statement of the results

The rollings of a spherical ball B of unit radius over the plane $\mathbb{R}^2$ suggest the consideration of some special kinematic (virtual) motions. A *state* of the ball B is defined as the pair formed by the point of contact between B and $\mathbb{R}^2$ together with the positive orthonormal frame attached to B. So the set of all the states is identified with the manifold $\mathbb{R}^2 \times SO(3)$, where $SO(3)$ denotes the group of all orthogonal 3×3 matrices with positive determinant.

A *move* is a smooth path on $\mathbb{R}^2 \times SO(3)$ corresponding to a rolling of B on $\mathbb{R}^2$ without slipping or twisting along a straight line of the plane $\mathbb{R}^2$. No slipping in the rolling means that at each instant the point of contact between B and $\mathbb{R}^2$ has zero velocity; no twisting means that, at each instant, the axis of rotation must be parallel to the plane $\mathbb{R}^2$.

According to John M. Hammersley, the following problem was proposed by David Kendall in the 1950s: *What is the number N of moves necessary and sufficient to reach any final state of $\mathbb{R}^2 \times SO(3)$ starting at a given initial state?* In an interesting paper written in 1983, Hammersley [4] shows that $N = 3$ after using the theory of quaternions.

The following historical considerations we quote from [4, p. 112]: *The original version of the question set (by David Kendall in the 1950s) for 18-year-old schoolboys, invited candidates to investigate how two moves, each of length π, would change the ball's orientation; and to deduce in the first place that $N \leq 11$, and in the second place that $N \leq 7$. Candidates scored bonus marks for any improvement on 7 moves. When he first set the question, Kendall knew that $N \leq 5$; but, interest being aroused amongst professional mathematicians at Oxford, he and others soon discovered that the answer must be either $N = 3$ or $N = 4$. But in the 1950s nobody could decide between these two possibilities. There was renewed interested in the 1970s, and not only amongst professional mathematicians: for example the President of Trinity (a distinguished biochemist) spent some time rolling a ball around his drawing room floor in search of empirical insight. In 1978, while delivering the opening address to the first Australasian Mathematical Convention, I posed the problem to mathematicians down under; but I have not subsequently received a solution from them. So this is an opportunity to publish the solution.*

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