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A GENERALIZED VERSION OF THE MCMULLEN DOMAIN

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We study the family of complex maps given by $F_\lambda(z) = z^n + \lambda/z^n + c$ where $n \geq 3$ is an integer, λ is an arbitrarily small complex parameter, and c is chosen to be the center of a hyperbolic component of the corresponding Multibrot set. We focus on the structure of the Julia set for a map of this form generalizing a result of McMullen. We prove that it consists of a countable collection of Cantor sets of closed curves and an uncountable number of point components.

Keywords: Complex dynamics; Julia set; Mandelbrot set; Cantor set of circles; McMullen domain; symbolic dynamics.

1. Introduction

In this paper we consider complex rational maps of the form

$$F_\lambda(z) = z^n + \frac{\lambda}{z^n} + c \tag{1}$$

where $\lambda, c \in \mathbb{C}$ are parameters. For this family of maps, we fix c to be the center of a hyperbolic component of the corresponding Multibrot set, i.e. a parameter such that, for the map

$$F_0(z) = z^n + c, \tag{2}$$

0 lies in a superattracting cycle. For reasons explained below, we shall assume that $n \geq 3$ and that c is chosen so that this superattracting cycle has period $k > 1$. The maps F_λ are obtained by replacing the only finite critical point of F_0 with a pole of order n. For these maps, ∞ is always a superattracting fixed point, so there is an immediate basin of ∞ that we denote by B_λ. Since 0 is the only pole of F_λ when $\lambda \neq 0$, there is a preimage of B_λ that contains 0. If this preimage of B_λ is disjoint from B_λ, this set is called the trap door and denoted T_λ. In this paper, we shall only consider cases where B_λ and T_λ are disjoint, which always happens when $|\lambda|$ is sufficiently small.

For the special case $c = 0$, the origin is a superattracting fixed point of F_0. The maps F_λ in this case have been extensively studied; see [Devaney *et al.*, 2007b; Devaney *et al.*, 2005; Devaney & Look, 2006]. When $\lambda = 0$, the Julia set for this family is the unit circle. When $\lambda \neq 0$, there are $2n$ "free" critical points for F_λ, i.e. critical points not equal to 0 or ∞. Each of the free critical points maps to one of two critical values. When n is even, these critical values then map to the same point so there is really only one free critical orbit in this case.

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