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Existence of limit cycles in a tritrophic food chain model with Holling functional responses of type II and III

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We are interested in the coexistence of three species forming a tritrophic food chain model. Considering a linear grow for the lowest trophic species, Holling III and Holling II functional response for the predator and the top-predator, respectively. We prove that this model has stable periodic orbits for adequate values of its parameters.

Keywords: periodic orbit; Hopf bifurcation; population dynamics; Lyapunov coefficient

1. Introduction

The dynamic relationship between predators and their prey has long been studied and will continue being one of the dominant topics in both ecology and mathematical ecology due to its universality and importance, see for instance [8, 4]. These problems may appear to be simple mathematically at first sight. In fact they are often very challenging and complicated.

Although the predator–prey theory has seen much progress in the last 40 years, there are still mathematical and ecological problems unsolved. In [3], the authors considered a model for three species that compete for three resources and they proved that the existence of two limit cycles evolves the coexistence equilibrium points. In [4] it is studied a model representing a tritrophic food chain composed of a logistic prey, a Holling type II predator and a Holling type II top-predator. They proved using the averaging theory the existence of a stable periodic orbit contained in the domain of interest. In [2] the authors analyzed a model representing a tritrophic food chain composed by a prey with linear grow and a functional response Holling type III for the predator and the top-predator. They proved using the averaging theory the existence of three-dimensional parameter families for which the model has two equilibrium points of zero-Hopf type contained in the domain of interest and also prove the existence of a simultaneously double zero–Hopf bifurcation.

In this paper we analyze a tritrophic food chain model considering Holling functional response of type III for the predator, Holling type II for the top-predator and linear grow for the prey.

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A tritrophic food chain model with functional response $f(x)$ and $g(y)$ for the middle and the highest trophic species, respectively, has the form

$$\begin{aligned}\dot{x} &= \rho x - f(x)y,\\ \dot{y} &= cyf(x) - g(y)z - dy,\\ \dot{z} &= z\left(eg(y) - d_2\right),\end{aligned} \tag{1}$$

where x represents the prey, that gets eaten by a species of density y (mesopredator) that feeds the species with density z (super-predator). The parameters ρ, c, d, d_2 and e are positive constants.

The functional responses of the mesopredator $f(x)$, and the super-predator $g(x)$ satisfy

$$f(0) = g(0) = 0, \quad \frac{df(x)}{dx} \geq 0 \text{ and } \frac{dg(y)}{dy} \geq 0 \quad \forall x, y \geq 0.$$

There are many functions that satisfy the above conditions, for example the functional responses of predation include the usual functions found in the literature (see, e.g., [5]). In this paper we will consider that f is Holling type III and g is Holling type II, more precisely

$$f(x) = \frac{a_1 x^2}{x^2 + b_1} \text{ and } g(y) = \frac{a_2 y}{y + b_2},$$

where a_1, a_2, b_1 and b_2 are positive constants.

Consequently the tritrophic food chain model that we will study is

$$\begin{aligned}\dot{x} &= x\left(\rho - \frac{a_1 xy}{x^2 + b_1}\right),\\ \dot{y} &= y\left(c\frac{a_1 x^2}{x^2 + b_1} - \frac{a_2 z}{y + b_2} - d\right),\\ \dot{z} &= z\left(e\frac{a_2 y}{y + b_2} - d_2\right).\end{aligned} \tag{2}$$

For ecological considerations the domain of interest Ω is the positive octant of $\mathbb{R}^3$, i.e. Ω is the region $x > 0$, $y > 0$ and $z > 0$.

We give necessary and sufficient conditions on the parameters to guarantee the existence of equilibrium points of the differential system (2) in the region of interest Ω. Our main result shows that system (2) can exhibit Hopf bifurcations in one of their equilibria, and for any convenient values of the parameters the Hopf periodic orbit is stable.

To simply the expressions thought the paper we define

$$k^2 = a_1^2 b_2^2 d_2^2 - 4b_1(d_2 - a_2 e)^2 \rho^2, \quad k > 0. \tag{3}$$

Solving this equation (3) for b_1, we write b_1 in terms of k

$$b_1 = \frac{a_1^2 b_2^2 d_2^2 - k^2}{4(d_2 - a_2 e)^2 \rho^2}. \tag{4}$$

Also we do

$$\begin{aligned}B_1 &= d_2 - a_2 e,\\ B_2 &= a_1 b_2 d_2 - k,\\ B_3 &= a_1 b_2 c d_2 - 2b_2 d d_2 - ck.\end{aligned}$$

Theorem 1 *If the parameters involved in system (2) satisfy the hypothesis*

H1 $a_2 e - d_2 > 0$,

H2 $a_1 b_2 d_2 - k > 0$,

H3 $a_1^2 b_2^2 d_2^2 - 4b_1(d_2 - a_2 e)^2 \rho^2 \geq 0$,

H4 $a_1 b_2 c d_2 - 2b_2 d d_2 - ck > 0$

then we have an equilibrium point p_1 in the positive octant of $\mathbb{R}^3$. If additionally the hypothesis

H5 $-a_2^2 b_2^2 e^2 k B_2^2 (a_1 a_2 b_2 d_2 e - B_1 k) + 4 d_2 B_1^6 \rho^2 \omega^2 > 0$,

is satisfied and the first Lyapunov coefficient $\ell(p_1)$ (see Appendix) at p_1 is not zero, then we have a Hopf bifurcation in p_1.

Theorem 2 *If the parameters involved in system (2) satisfy the hypothesis $H1$, $H3$ and*

H4' $a_1 b_2 c d_2 - 2 b_2 d d_2 + c k > 0$,

then we have an equilibrium point p_2 in the positive octant of $\mathbb{R}^3$. But the standard sufficient conditions which guarantee the existence of a Hopf bifurcation in p_2 do not hold.

2. Equilibrium points in the positive octant of $\mathbb{R}^3$

Doing a rescaling of the time, we get that the differential system (2) is equivalent to the following differential system,

$$\begin{array}{rcl} \dot{x} & = & x(b_2 + y)(-a_1 x y + b_1 \rho + x^2 \rho), \\ \dot{y} & = & -y(b_1 b_2 d - a_1 b_2 c x^2 + b_2 d x^2 + b_1 d y - a_1 c x^2 y + d x^2 y + a_2 b_1 z + a_2 x^2 z), \\ \dot{z} & = & -(b_1 + x^2)(b_2 d_2 + d_2 y - a_2 e y) z. \end{array} \tag{5}$$

The equilibrium points of differential systems (5) in the region of interest Ω, are the solution of the equations

$$\begin{aligned} -a_1 x y + b_1 \rho + x^2 \rho = 0, \\ b_1 b_2 d - a_1 b_2 c x^2 + b_2 d x^2 + b_1 d y - a_1 c x^2 y + d x^2 y + a_2 b_1 z + a_2 x^2 z = 0, \\ b_2 d_2 + d_2 y - a_2 e y = 0, \end{aligned}$$

whose solutions are

$$\begin{aligned} p_1 &= \left(\frac{a_1 b_2 d_2 - k}{2(a_2 e - d_2)\rho}, \frac{b_2 d_2}{a_2 e - d_2}, \frac{e\,(a_1 b_2 c d_2 - 2 b_2 d d_2 - c k)}{2 d_2 (a_2 e - d_2)} \right), \\ p_2 &= \left(\frac{a_1 b_2 d_2 + k}{2(a_2 e - d_2)\rho}, \frac{b_2 d_2}{a_2 e - d_2}, \frac{e\,(a_1 b_2 c d_2 - 2 b_2 d d_2 + c k)}{2 d_2 (a_2 e - d_2)} \right). \end{aligned} \tag{6}$$

So the following result guarantee the existence of two non trivial equilibrium points in Ω.

Lemma 3 *The equilibrium point p_1 of differential system (5) is in the positive octant of $\mathbb{R}^3$ if and only if the parameters satisfy the conditions* **H1**, **H2**, **H3** *and* **H4**. *The equillibrium point p_2 of the differential system (5) is in the positive octant of $\mathbb{R}^3$ if and only if the parameters satisfy the conditions* **H1**, **H2** *and* **H4'**

Proof. The proof is a direct consequence of the expression of p_1 and p_2. □

We note that always that p_1 is in the positive octant of $\mathbb{R}^3$ also p_2 is there.

Now our goal is to determine when the equilibrium points p_1 and p_2 exhibits a Hopf bifurcation. In this sense we show the existence of parameters where the equilibrium points have a pair of pure imaginary eigenvalues and a nonzero real eigenvalue and using a Kuznetsov's Theorem we shall prove the existence of a Hopf bifurcation. The tool provided by the Kuznetsov Theorem is a good theory for studying the Hopf bifurcation, in particular in [1] it is applied in the repressilator equation of dimension six for proving the existence of a supercritical Hopf bifurcation.

3. First Lyapunov coefficient and Hopf bifurcation

3.1. Kuznetsov theorem

When a differential system

$$\dot{x} = F(x, \mu),$$

in $\mathbb{R}^n$ has an equilibrium point p having eigenvalues $\pm\omega i$, it is a candidate to exhibit a Hopf bifurcation, that is a local bifurcation in which an equilibrium point of a differential system loses stability as a pair of complex conjugate eigenvalues of the linearization, around the equilibrium point, cross the imaginary axis of the complex plane.

In order to show that this bifurcation takes place it is necessary to compute the first Lyapunov coefficient $\ell_1(p)$ of the differential system at the equilibrium p. When $\ell_1(p) < 0$ the point p is a weak focus of the differential system restricted to the central surface of p, associated to the pair of complex eigenvalues which cross the imaginary axis, and the limit cycle that emerges from p is stable. In this case the Hopf bifurcation is called *supercritical*. When $\ell_1(p_1) > 0$ the point p is also a weak focus of the differential system restricted to the central surface of p but the limit cycle that born from p is unstable. In this second case we say that the Hopf bifurcation is *subcritical*. For more details on the Hopf bifurcation see for instance the book of Kuznetsov [7].

In this work we use the following result presented on page 180 of the book [7] for computing the first Lyapunov constant $\ell_1(p_1)$ at the equilibrium p_1.

Theorem 4 (Kuznetsov, 2004) *Let $\dot{x} = F(x)$ be a differential system having p_0 as an equilibrium point. Consider the third order Taylor approximation of F around p_0 given by*

$$F(x) = Ax + \frac{1}{2!}B(x,x) + \frac{1}{3!}C(x,x,x) + O(|x|^4).$$

Assume that A has a pair of purely imaginary eigenvalues $\pm\omega i$. Let q be the eigenvector of A corresponding to the eigenvalue ωi, normalized so that $q\bar{q} = 1$, where $\bar{q}$ is the conjugate vector of q. Let p be the adjoint eigenvector such that $A^T p = -\omega i p$ and $\bar{p} \cdot q = 1$. If I denotes the identity matrix, then the first Liapunov constant $l_1(p_0)$ of the system $\dot{x} = F(x)$ at the equilibrium point p_0 is

$$\frac{1}{2\omega}\mathrm{Re}(\bar{p} \cdot C(q,q,\bar{q}) - 2\bar{p} \cdot B(q, A^{-1}B(q,\bar{q})) + \bar{p} \cdot B(\bar{q}, (2\omega i I - A)^{-1}B(q,q))).$$

3.2. Hopf bifurcation at equilibrium point p_1

In the next two results we characterize when the equilibrium points p_1 and p_2 have complex eigenvalues with real part zero. Hence system (5) will be a candidate to exhibit a Hopf bifurcation.

Proposition 5 *If the parameters satisfy the hypothesis H1,H2, H3, H4 and H5, then there exists a parameter set such that the equilibrium points p_1 is in the positive octant of $\mathbb{R}^3$ and have a pair of pure imaginary eigenvalues and the other eigenvalue is a non zero real.*

*Proof.*The characteristic polynomial of the linear approximation of the tritropic system (5) at the equilibrium point p_1 is

$$p(\lambda) = -\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0,$$

where,

$$\begin{aligned}
A_0 &= -\frac{a_1^2 a_2^2 b_2^4 d_2^2 e^2 B_2^3 k B_3}{(16B_1^8 \rho^5}, \\
A_1 &= \frac{a_1 a_2 b_2^2 e B_2^2 \left(a_1 b_2 d_2^2 B_1 B_3 + \left(-a_1^2 a_2 b_2^2 c d_2^2 e + b_2(a_1 c - 2d) d_2^2 k - c B_1 k^2\right)\rho\right)}{8B_1^6 \rho^4}, \\
A_2 &= \frac{b_2 B_2 (a_1 d_2 B_3 - 2a_2 e k \rho)}{4B_1^3 \rho^2}.
\end{aligned}$$

Now imposing the condition that $p(\lambda) = -(\lambda - \alpha)(\lambda + \omega i)(\lambda - \omega i)$, we obtain a system of three polynomial equations, that correspond to the coefficients of the terms of degree 0, 1 and 2 in λ. Since the variables c, d and α are of degree one in this system, then we have a linear system in this variables whose solution is

$$
\begin{aligned}
c &= -\frac{B_1^5\rho^4\omega^2\left(a_2^2b_2^2e^2k^2B_2^2 + 4B_1^6\rho^2\omega^2\right)}{a_1a_2^2b_2^2e^2B_2^3(a_1b_2d_2 + k)\left(a_1a_2^2b_2^3d_2e^2kB_2^2 - 4B_1^5\rho^3\omega^2\right)},\\
d &= -\frac{4B_1^5\rho^4\omega^2\left(-a_2^2b_2^2e^2kB_2^2(a_1a_2b_2d_2e - B_1k) + 4d_2B_1^6\rho^2\omega^2\right)}{a_1a_2^2b_2^3d_2^2e^2B_2^2(a_1b_2d_2 + k)\left(a_1a_2^2b_2^3d_2e^2kB_2^2 - 4B_1^5\rho^3\omega^2\right)},\\
\alpha &= -\frac{a_1a_2^3b_2^4d_2e^3B_2^3k^2}{2a_1a_2^2b_2^3d_2e^2(-d_2 + a_2e)^3kB_2^2\rho + 8B_1^8\rho^4\omega^2}.
\end{aligned}
\tag{7}
$$

From the hypotheses of the proposition, we have that $B_1 < 0$, $B_2 > 0$ and $B_3 > 0$. Thus c is positive. On the other hand, the denominator of d is positive, so, we only verify the sign of numerator to know the sign of d. As $B_1 < 0$ then $B_1^5 < 0$, and so $-4B_1^5\rho^4\omega^2 > 0$ in the expression of d, therefore the sign of d is determined by the sign of the expression in the parenthesis of numerator,

$$
-a_2^2b_2^2e^2kB_2^2(a_1a_2b_2d_2e - B_1k) + 4d_2B_1^6\rho^2\omega^2,
$$

that is positive by hypothesis **H5**. Hypothesis **H5** is realizable because this is a polynomial equation of degree two in the variable ω of the form $t_0 + t_1\omega^2$ with $t_0 < 0$ and $t_1 > 0$.

With these values of the parameters the equilibrium point p_1 takes the form

$$
p_1 = \left(-\frac{B_2}{2B_1\rho}, -\frac{b_2d_2}{B_1}, \frac{4a_2e^2B_1^4k\rho^4\omega^2}{a_1d_2^2\left(a_1a_2^2b_2^3d_2e^2kB_2^2 - 4B_1^5\rho^3\omega^2\right)}\right),
\tag{8}
$$

and the linear approximation at this point has a pair of pure imaginary eigenvalues ωi, $-\omega i$ and one real, α. It is clear that p_1 is in the positive octant. $\square$

4. Proof of main results

4.1. Proof of Theorem 1

Lemma 3 guarantees the existence of parameter families for which the equilibrium points p_1 and p_2 are in the positive octant of $\mathbb{R}^3$ and Proposition 5 establishes that also there is a family of parameters for which the equilibrium point p_1 has a pair of pure imaginary eigenvalues and one real eigenvalues. Theorem 1 is our main result, which guarantees the existence of Hopf bifurcation at the point p_1 and for convenient values of the parameters it will be stable. Example 6 shows that these last claims are satisfied for some values of the parameters.

In order that this bifurcation takes place we compute the first Lyapunov coefficient $\ell_1(p_1)$ of the differential system at the equilibrium p_1. This calculation is done using the formula of the Kuznetsov's Theorem.

Let p_1 be as in (8) and consider a linear change of variable to translate p_1 to at the origin of coordinates, after that change we obtain a differential system

$$
\dot{X} = F(X),
\tag{9}
$$

with $X = (x, y, z)$ and $F(0) = 0$. Denote the vector field

$$
F(x, y, z) = (f_1(x, y, z), f_1(x, y, z), f_1(x, y, z)),
$$

associated to this differential system. Then we compute the linear part A, the bilinear B and trilinear C forms of the Taylor expanssion of the function F.

The linear part of system (9) at 0 is

$$A = \begin{pmatrix} \frac{a_2 b_2 e B_2 k}{2B_1^3\rho} & \frac{a_1 a_2 b_2 e b_2^2}{4B_1^3\rho^2} & 0 \\ \frac{4B_1^2\rho^3\omega^2\left(a_2^2 b_2^2 e^2 k^2 b_2^2 + 4B_1^6\rho^2\omega^2\right)}{a_1 a_2 b_2 e B_2^2 B_4} & \frac{2a_2 b_2 e B_1^2 B_2 k\rho^2\omega^2}{B_4} & \frac{a_1 a_2 b_2^2 d_2^2 B_2}{2B_1^3\rho^2} \\ 0 & \frac{2a_2 b_2 e^2 B_1^3 B_2 k\rho^2\omega^2}{-d_2 B_4} & 0 \end{pmatrix},$$

where

$$B_4 = a_1 a_2^2 b_2^3 d_2 e^2 k B_2^2 - 4B_1^5\rho^3\omega^2.$$

The eigenvalues of A are

$$-i\omega, \quad i\omega, \quad -\frac{a_1 a_2^3 b_2^4 d_2 e^3 B_2^3 k^2}{8B_1^8\rho^4\omega^2 - 2a_1 a_2^2 b_2^3 d_2 e^2 B_1^3 k B_2 2\rho}.$$

The eigenvector $q = (q_{11}, q_{12}, q_{13})$ of A corresponding to eigenvalue $i\omega$ normalized such that $q \cdot \bar{q} = 1$, where $\bar{q}$ is the conjugate vector of q is presented in the Appendix. The normalized adjoint eigenvector $p = (p_{11}, p_{12}, p_{13})$ of transpose matrix A corresponding to the eigenvalue $-i\omega$, with the property that $\bar{p} \cdot q = 1$ is also in the Appendix.

The calculus of the bilinear B and trilinar C forms, are provided in the Appendix because they have large expressions. So we have all the elements to compute the first Lyapunov coefficient $\ell(p_1)$ at p_1. The expression of $\ell(p_1)$ is given in the Appendix. This conclude, the proof of the theorem.

4.2. Proof of Theorem 2

To proof this theorem we do an equivalent procedure as the one done in the proof of Theorem 1. From the expression of p_2 establish in (6) we have that it is in the positive octant of $\mathbb{R}^3$. The characteristic polynomial of the linear approximation of system (5) at the equilibrium p_2 is

$$p(\lambda) = -\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0,$$

where,

$$\begin{aligned} C_0 &= -\frac{a_1^2 a_2^2 b_2^4 d_2^2 e^2 k D_1^3 D_3}{16B_1^8\rho^5}, \\ C_1 &= \frac{a_1 a_2 b_2^2 e D_1^2\left(-a_1 b_2 d_2^2 B_1 D_2 + \left(a_1^2 a_2 b_2^2 c d_2^2 e + b_2(a_1 c - 2d)d_2^2 k + cB_1 k^2\right)\rho\right)}{8B_1^6\rho^4}, \\ C_2 &= \frac{b_2 D_1(a_1 d_2 D_2 + 2a_2 e k\rho)}{4B_1^3\rho^2}, \\ D_1 &= a_1 b_2 d_2 + k, \\ D_2 &= a_1 b_2 c d_2 - 2b_2 d d_2 + ck. \end{aligned}$$

Similarly imposing the condition that $p(\lambda) = -(\lambda - \alpha)(\lambda + \omega I)(\lambda - \omega I)$, we obtain a system of three polynomial equations, that are linear in the parameters c, d and α. The solution of this system in terms of these variables is

$$\begin{aligned} c &= \frac{8B_1^5\rho^4\omega^2\left(a_2^2 b_2^2 e^2 k^2(a_1 b_2 d_2 + k)^2 + 4B_1^6\rho^2\omega^2\right)}{a_1 a_2^2 b_2^2 e^2 B_2 D_1^3\left(a_1 a_2^2 b_2^3 d_2 e^2 k D_1^2 + 4B_1^5\rho^3\omega^2\right)}, \\ d &= \frac{4B_1^5\rho^4\omega^2\left(a_2^2 b_2^2 e^2 k D_1^2(a_1 a_2 b_2 d_2 e + B_1 k) + 4d_2 B_1^6\rho^2\omega^2\right)}{a_1 a_2^2 b_2^3 d_2^2 e^2 B_2 D_1^2\left(a_1 a_2^2 b_2^3 d_2 e^2 k D_1^2 + 4B_1^5\rho^3\omega^2\right)}, \\ \alpha &= -\frac{a_1 a_2^3 b_2^4 d_2 e^3 k^2 D_1^3}{2a_1 a_2^2 b_2^3 d_2 e^2 B_1^3 k D_1^2\rho + 8B_1^8\rho^4\omega^2}. \end{aligned} \tag{10}$$

The equilibrium point p_2 with the values of parameters given in (10) takes the form

$$p_2 = \left(-\frac{D_1}{2B_1\rho}, -\frac{b_2 d_2}{B_1}, \frac{4a_2 e^2 B_1^4 k\rho^4\omega^2}{a_1 d_2^2\left(a_1 a_2^2 b_2^3 d_2 e^2 k D_1^2 + 4B_1^5\rho^3\omega^2\right)}\right).$$

The linear approximation at this point has a pair of pure imaginary eigenvalues ωi, $-\omega i$ and one real, α as we need for apply the Kuznetsov Theorem. With these values of the parameters c and d, the equillibrium point p_2 remains in the positive octant of $\mathbb{R}^3$ if the condition

$$a_1 a_2^2 b_2^3 d_2 e^2 k D_1^2 + 4B_1^5 \rho^3 \omega^2 > 0$$

is satisfied. But in this case the parameter c is negative and this is not possible because in our model all parameter are positive. This complete the proof of the theorem.

4.3. *Examples*

Theorem 1 is our main result, which guarantees the existence of Hopf bifurcation at the point p_1. Example 6 shows that this theorem is not empty.

Example 6 *We show a concrete example where Theorem 1 holds and where the Hopf periodic orbit is stable. Consider the parameters values $a_1 = 1$, $a_2 = 2$, $b_2 = 2$, $d_2 = 1$, $e = 2$, $k = 1$. Then, from (4) and (7) we have*

$$\begin{aligned}
b_1 =& \frac{1}{12\rho^2},\\
c =& \frac{81\rho^4\omega^2\left(64 + 2916\rho^2\omega^2\right)}{8\left(128 + 972\rho^3\omega^2\right)},\\
d =& \frac{81\rho^4\omega^2\left(-704 + 2916\rho^2\omega^2\right)}{32\left(128 + 972\rho^3\omega^2\right)},\\
\alpha =& -\frac{1024}{6912\rho + 52488\rho^4\omega^2}.
\end{aligned}$$

Note that b_1 and c are always positive reals numbers and if

$$\rho\omega > \frac{4}{27}\sqrt{11}, \tag{11}$$

then d is positive too. The eigenvalue α is always negative.

All the hypothesis of Theorem 1 hold for these values of parameters, indeed

H1 $a_2 e - d_2 = 3 > 0$,

H2 $a_1 b_2 d_2 - k = 1 > 0$,

H3 $a_1^2 b_2^2 d_2^2 - 4b_1(d_2 - a_2 e)^2 \rho^2 = 1 \geq 0$,

H4 $a_1 b_2 c d_2 - 2b_2 d d_2 - ck = 8\rho\left(-1 + \dfrac{32}{32 + 243\rho^3\omega^2}\right) > 0$,

H5 $-a_2^2 b_2^2 e^2 k B_2^2 (a_1 a_2 b_2 d_2 e - B_1 k) + 4d_2 B_1^6 \rho^2 \omega^2 = -704 + 2916\rho^2\omega^2 > 0$.

And the first Lyapunov coefficient $\ell(p_1)$ is

$$\ell(p_1) = \frac{l_a}{l_b}$$

Where

$$\begin{aligned}
l_a =& -\left(8192(-1356 + 22295\rho) + 512\rho^3(4076159 + 4054698\rho)\omega^2 + 51018336\rho^6(285 + 101\rho)\omega^4 + 3486784401\rho^9\omega^6\right).\\
l_b =& 648\omega\left(4096 + \rho^2\left(16384 + 243\rho\omega^2\left(256\left(1 + 4\rho\left(3 + \rho + 3\rho^2\right)\right) + 972\rho^3(1 + 4\rho(12 + 37\rho))\omega^2 + 177147\rho^7\omega^4\right)\right)\right).
\end{aligned}$$

We have that $l_b > 0$ for any ρ and ω. Note that $l_a < 0$ if

$$\rho > \frac{1356}{22295}.$$

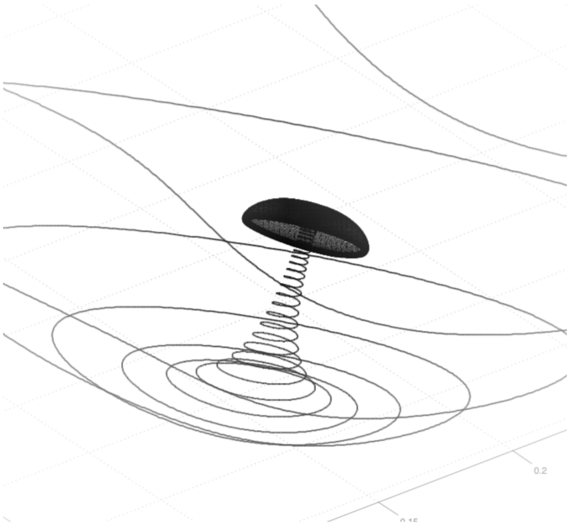

Figure 1. We show an orbit tending to the local attractor definided by a stable Hopf periodic orbit.

Then condition (11) *holds if* $\omega > \dfrac{22295\sqrt{11}}{9153}$, *and then we obtain that the first Lyapunov coefficient* $\ell(p_1)$ *is negative for all the previous values of* ρ *and* ω. *In the particular case that* $\rho = \omega = 1$ *the value of* $\ell(p_1)$ *is*

$$\ell(p_1) = -\frac{27514393169}{59228395560}.$$

Recall that when $\ell(p_1)$ *is negative we have a subcritical Hopf bifurcation and then the periodic orbit obtained of the bifurcation is stable in the planar central surface, and since the real eigenvalue* α *is negative, then the limit cycle living in the central surface is a local attractor.*

In Figure 1 we exhibit the local attractor defined by the stable limit cycle of differential systems (2) *with parameters values* $a_1 = 1$, $a_2 = 2$, $b_2 = 2$, $d_2 = 1$, $e = 2$, $k = 1$, $\rho = 1$, $\omega = 1$ *and* $\epsilon = 0.1$. *The singular points are* $p_1 = (1/6, 2/3, 648/275)$ *and* $p_2 = (1/2, 2/3, 22707/1100)$. *The eigenvalues of linear approximation of system at* p_2 *are* $\{1.8292 + 3.82836i, 1.8292 - 3.82836i, 0.226503\}$ *then it is a repellor.*

Appendix

The bilinear function B at vector (x, y, z) is given by

$$B((x, y, z), (x, y, z)) = (B_{11}, B_{12}, B_{13}),$$

where

$$\begin{aligned}
B_{11} &= -\frac{a_1 B_2^2 y^2 + 2B_2(2a_1 a_2 b_2 e + k)xy\rho - 2a_2 b_2 e(a_1 b_2 d_2 - 3k)x^2\rho^2}{2B_1^2\rho^2};\\
B_{12} &= -\frac{s_1}{s_2};\\
B_{13} &= -\frac{B_2 y\left(a_1^3 a_2^2 b_2^4 d_2^4 e^2 k B_2^2 z - 4B_1^5\rho^3\left(a_1^2 b_2 d_2^3 z + 2a_2 e^2 kx\rho^2\right)\omega^2\right)}{a_1 d_2^2 B_1\rho^2\left(a_1 a_2^2 b_2^3 d_2 e^2 k B_2^2 - 4B_1^5\rho^3\omega^2\right)};
\end{aligned}$$

$$
\begin{aligned}
s_1 =& a_1^9 a_2^5 b_2^{12} d_2^9 e^4 kyz - 6a_1^8 a_2^5 b_2^{11} d_2^8 e^4 k^2 yz + 15a_1^7 a_2^5 b_2^{10} d_2^7 e^4 k^3 yz \\
&- 20a_1^6 a_2^5 b_2^9 d_2^6 e^4 k^4 yz + 15a_1^5 a_2^5 b_2^8 d_2^5 e^4 k^5 yz - 6a_1^4 a_2^5 b_2^7 d_2^4 e^4 k^6 yz \\
&+ a_1^3 a_2^5 b_2^6 d_2^3 e^4 k^7 yz + 2a_1^8 a_2^5 b_2^{12} d_2^9 e^4 kxz\rho - 12a_1^7 a_2^5 b_2^{11} d_2^8 e^4 k^2 xz\rho \\
&+ 30a_1^6 a_2^5 b_2^{10} d_2^7 e^4 k^3 xz\rho - 40a_1^5 a_2^5 b_2^9 d_2^6 e^4 k^4 xz\rho + 30a_1^4 a_2^5 b_2^8 d_2^5 e^4 k^5 xz\rho \\
&- 12a_1^3 a_2^5 b_2^7 d_2^4 e^4 k^6 xz\rho + 2a_1^2 a_2^5 b_2^6 d_2^3 e^4 k^7 xz\rho - 4a_2^2 b_2^2 e^2 B_1^5 B_2^2 \rho^3 s_3 \omega^2 \\
&+ 32B_1^{11} x\rho^7 ((d_2 + a_2 e)B_2 y + a_2 b_2 d_2 ex\rho)\omega^4; \\
s_2 =& a_1 a_2^2 b_2^2 d_2 e^2 B_1^2 B_2^3 \rho^2 \left(a_1 a_2^2 b_2^3 d_2 e^2 k B_2^2 - 4B_1^5 \rho^3 \omega^2\right); \\
s_3 =& a_1^2 a_2 b_2 d_2^2 B_2^2 yz + a_1 a_2 B_2^2 \left(-eky^2 + 2b_2 d_2^2 xz\right)\rho - 2d_2 B_2 k D_1 xy\rho^2 \\
&- 2a_2 b_2 d_2 ek^2 x^2 \rho^3.
\end{aligned}
$$

The trilinear function C at vector (x, y, z) is given by

$$
C((x, y, z), (x, y, z), (x, y, z)) = (C_{11}, C_{12}, C_{13}),
$$

where

$$
\begin{aligned}
C_{11} =& -\frac{3x\left(2a_1(-B_2)y^2 + (a_1 b_2(d_2 - 2a_2 e) - 3k)xy\rho + 2a_2 b_2 ex^2\rho^2\right)}{B_1\rho}; \\
C_{12} =& -\frac{s_4}{s_5}; \\
C_{13} =& \frac{6xy\left(a_1^2 a_2^2 b_2^3 d_2^3 e^2 B_2^3 kz - 4B_1^5\rho^3\left(a_1 d_2^2 B_2 z + a_2 e^2 kx\rho^2\right)\omega^2\right)}{a_1 d_2^2 \rho\left(a_1 a_2^2 b_2^3 d_2 e^2 k B_2^2 - 4B_1^5\rho^3\omega^2\right)};
\end{aligned}
$$

$$
\begin{aligned}
s_4 =& 6xs_7; \\
s_5 =& \left(a_1 a_2^2 b_2^3 d_2^2 e^2 B_1 B_2^3 \rho\left(-a_1 a_2^2 b_2^3 d_2 e^2 k B_2^2 + 4B_1^5\rho^3\omega^2\right)\right); \\
s_6 =& a_1 a_2 b_2 d_2^2 B_2^2 yz + B_2\left(k(-a_1 a_2 b_2 d_2 e - d_2 k + a_2 ek)y^2 + a_1 a_2 b_2^2 d_2^3 xz\right)\rho \\
&- b_2 d_2^2 k(a_1 a_2 b_2 e + k)xy\rho^2; \\
s_7 =& a_1^8 a_2^5 b_2^{12} d_2^9 e^4 kyz - 6a_1^7 a_2^5 b_2^{11} d_2^8 e^4 k^2 yz + 15a_1^6 a_2^5 b_2^{10} d_2^7 e^4 k^3 yz \\
&- 20a_1^5 a_2^5 b_2^9 d_2^6 e^4 k^4 yz + 15a_1^4 a_2^5 b_2^8 d_2^5 e^4 k^5 yz - 6a_1^3 a_2^5 b_2^7 d_2^4 e^4 k^6 yz \\
&+ a_1^2 a_2^5 b_2^6 d_2^3 e^4 k^7 yz + a_1^7 a_2^5 b_2^{12} d_2^9 e^4 kxz\rho - 5a_1^6 a_2^5 b_2^{11} d_2^8 e^4 k^2 xz\rho \\
&+ 10a_1^5 a_2^5 b_2^{10} d_2^7 e^4 k^3 xz\rho - 10a_1^4 a_2^5 b_2^9 d_2^6 e^4 k^4 xz\rho + 5a_1^3 a_2^5 b_2^8 d_2^5 e^4 k^5 xz\rho \\
&- a_1^2 a_2^5 b_2^7 d_2^4 e^4 k^6 xz\rho + 4a_2^2 b_2^2 e^2(-B_1)^5 B_2^2 \rho^3 s_6 \omega^2 \\
&+ 16d_2 B_1^{11} y\rho^6(a_1 b_2 d_2 y - ky + b_2(d_2 + a_2 e)x\rho)\omega^4.
\end{aligned}
$$

The eigenvector $q = (q_{11}, q_{12}, q_{13})$ of A corresponding to eigenvalue $i\omega$ have the coordinates

$$
\begin{aligned}
q_{11} =& \frac{a_1 d_2 B_2 r_3}{eB_1^3 k\rho^3\omega r_0}\sqrt{\frac{r_2}{r_1}}; \\
q_{12} =& -\frac{2id_2 r_3}{a_2 b_2 e^2 B_1^3 B_2 k\rho^2\omega}\sqrt{\frac{r_2}{r_1}}; \\
q_{13} =& 4\sqrt{\frac{r_2}{r_1}};
\end{aligned}
$$

$$
\begin{aligned}
r_1 =& a_1^{12}a_2^6b_2^{16}d_2^{12}e^6k^2 - 8a_1^{11}a_2^6b_2^{15}d_2^{11}e^6k^3 + 28a_1^{10}a_2^6b_2^{14}d_2^{10}e^6k^4 - 56a_1^9a_2^6b_2^{13}d_2^9e^6k^5 \\
&+ 70a_1^8a_2^6b_2^{12}d_2^8e^6k^6 - 56a_1^7a_2^6b_2^{11}d_2^7e^6k^7 + 28a_1^6a_2^6b_2^{10}d_2^6e^6k^8 - 8a_1^5a_2^6b_2^9d_2^5e^6k^9 \\
&+ a_1^4a_2^6b_2^8d_2^4e^6k^{10} + 4a_1^8a_2^6b_2^{14}d_2^{10}e^6k^4\rho^2 - 24a_1^7a_2^6b_2^{13}d_2^9e^6k^5\rho^2 + 60a_1^6a_2^6b_2^{12}d_2^8e^6k^6\rho^2 \\
&- 80a_1^5a_2^6b_2^{11}d_2^7e^6k^7\rho^2 + 60a_1^4a_2^6b_2^{10}d_2^6e^6k^8\rho^2 - 24a_1^3a_2^6b_2^9d_2^5e^6k^9\rho^2 \\
&+ 4a_1^2a_2^6b_2^8d_2^4e^6k^{10}\rho^2 + 8a_2^4b_2^4e^4(-B_1)^5kB_2^4\rho^3r_{11};
\end{aligned}
$$

$$
\begin{aligned}
r_{11} =& \left(a_1^3b_2d_2^3B_2^2 - 2a_1^2b_2^2d_2^4B_1k\rho + 4a_1b_2d_2^3k^2\rho^2 + 2e^2(-B_1)k^3\rho^3\right)\omega^2 \\
&+ 16a_2^2b_2^2e^2B_1^{10}B_2^2\rho^6\left(a_1^2d_2^2B_2^2 - 8a_1b_2d_2^3B_1k\rho + 4\left(-2a_2d_2e^3 + a_2^2e^4\right.\right. \\
&\left.\left.+d_2^2\left(1+e^2\right)\right)k^2\rho^2\right)\omega^4 + 256d_2^2B_1^{16}\rho^{10}\omega^6; \\
r_0 =& -ia_1a_2b_2^2d_2ek - 2d_2^3\rho\omega - 6a_2^2d_2e^2\rho\omega + 2a_2^3e^3\rho\omega + a_2e\left(ib_2k^2 + 6d_2^2\rho\omega\right); \\
r_2 =& a_2^2b_2^2e^4B_1^6k^2B_2^2\rho^6\omega^2\left(a_2^2b_2^2e^2k^2B_2^2 + 4B_1^6\rho^2\omega^2\right); \\
r_3 =& a_1^3a_2^2b_2^5d_2^3e^2k - 2a_1^2a_2^2b_2^4d_2^2e^2k^2 + a_1a_2^2b_2^3d_2e^2k^3 - 4B_1^5\rho^3\omega^2;
\end{aligned}
$$

The adjoint eigenvector $p = (p_{11}, p_{12}, p_{13})$ of transpose matrix A corresponding to the eigenvalue $-i\omega$ have the coordinates

$$
\begin{aligned}
p_{11} =& \frac{16eB_1^8kB_2^2\rho^6\omega^3g_0\left(ia_2b_2eB_2k + 2B_1^3\rho\omega\right)}{a_1d_2B_2^3\left(a_1a_2^2b_2^3d_2e^2kB_2^2 - 4B_1^5\rho^3\omega^2\right)g_3}\sqrt{\frac{g_1}{g_2}}, \\
p_{12} =& -\frac{2ia_2b_2e^2B_1^3kB_2^2\rho^2\omega g_0}{d_2B_2g_3}\sqrt{\frac{g_1}{g_2}}, \\
p_13 =& \frac{a_1a_2^2b_2^3d_2e^2kB_2^2g_0}{g_3}\sqrt{\frac{g_1}{g_2}};
\end{aligned}
$$

$$
\begin{aligned}
g_0 =& ia_2b_2eB_2k - 2(B_1)^3\rho\omega; \\
g_1 =& a_1^{12}a_2^6b_2^{16}d_2^{12}e^6k^2 - 8a_1^{11}a_2^6b_2^{15}d_2^{11}e^6k^3 + 28a_1^{10}a_2^6b_2^{14}d_2^{10}e^6k^4 \\
&- 56a_1^9a_2^6b_2^{13}d_2^9e^6k^5 + 70a_1^8a_2^6b_2^{12}d_2^8e^6k^6 - 56a_1^7a_2^6b_2^{11}d_2^7e^6k^7 \\
&+ 28a_1^6a_2^6b_2^{10}d_2^6e^6k^8 - 8a_1^5a_2^6b_2^9d_2^5e^6k^9 + a_1^4a_2^6b_2^8d_2^4e^6k^{10} \\
&+ 4a_1^8a_2^6b_2^{14}d_2^{10}e^6k^4\rho^2 - 24a_1^7a_2^6b_2^{13}d_2^9e^6k^5\rho^2 + 60a_1^6a_2^6b_2^{12}d_2^8e^6k^6\rho^2 \\
&- 80a_1^5a_2^6b_2^{11}d_2^7e^6k^7\rho^2 + 60a_1^4a_2^6b_2^{10}d_2^6e^6k^8\rho^2 - 24a_1^3a_2^6b_2^9d_2^5e^6k^9\rho^2 \\
&+ 4a_1^2a_2^6b_2^8d_2^4e^6k^{10}\rho^2 + 8a_2^4b_2^4e^4(-B_1)^5kB_2^4\rho^3\left(a_1^3b_2d_2^3B_2^2\right. \\
&\left.-2a_1^2b_2^2d_2^4(B_1)k\rho + 4a_1b_2d_2^3k^2\rho^2 + 2e^2(-B_1)k^3\rho^3\right)\omega^2 \\
&+ 16a_2^2b_2^2e^2(B_1)^{10}(-B_2)^2\rho^6\left(a_1^2d_2^2(-B_2)^2 - 8a_1b_2d_2^3(B_1)k\rho\right. \\
&\left.+4\left(-2a_2d_2e^3 + a_2^2e^4 + d_2^2\left(1+e^2\right)\right)k^2\rho^2\right)\omega^4 \\
&+ 256d_2^2(B_1)^{16}\rho^{10}\omega^6; \\
g_2 =& a_2^2b_2^2e^4(B_1)^6k^2(-B_2)^2\rho^6\omega^2\left(a_2^2b_2^2e^2k^2(-B_2)^2 + 4(B_1)^6\rho^2\omega^2\right); \\
g_3 =& 8ia_1a_2^3b_2^4d_2e^3B_2^3k^2 + 16a_1a_2^2b_2^3d_2e^2(-B_1)^3k(-B_2)^2\rho\omega \\
&+ 64(B_1)^8\rho^4\omega^3.
\end{aligned}
$$

The expression of the first Lyapunov coefficient $\ell(p_1)$, at the equillibrium point p_1.

$$
\ell(p_1) = \frac{l_a}{l_b}. \tag{12}
$$

The expression of numerator of $\ell(p_1)$, is l_a

$$
\begin{aligned}
l_a =& a_1 d_2 \left(l_{a1} + a_1^8 a_2^5 b_2^{12} d_2^8 e^5 k l_{a2} + a_1^6 a_2^3 b_2^{10} d_2^6 e^3 k^3 l_{a3} + a_1^7 a_2^4 b_2^{11} d_2^7 e^4 k^2 l_{a4} \right.\\
&+ 24(d_2 - a_2 e)^7 k \rho^4 \omega^2 l_{a5} + a_1^2 a_2^2 b_2^4 d_2^2 e^2 k^3 l_{a6} + a_1^3 a_2^2 b_2^5 d_2^3 e^2 k^2 l_{a7}\\
&\left. - a_1^5 a_2^3 b_2^7 d_2^5 e^3 l_{a8} + a_1^4 a_2^2 b_2^6 d_2^4 e^2 k l_{a9} + l_{a10} \right).
\end{aligned}
$$

The expression of denominator of $\ell(p_1)$, is l_b

$$
\begin{aligned}
l_b =& 6(d_2 - a_2 e)^3 \omega \left(l_{b1} + 8 a_2^4 b_2^4 e^4 (-d_2 + a_2 e)^5 k (-a_1 b_2 d_2 + k)^4 \rho^3 l_{b2} \omega^2 \right.\\
&+ 16 a_2^2 b_2^2 e^2 (d_2 - a_2 e)^{10} (-a_1 b_2 d_2 + k)^2 \rho^6 l_{b3} \omega^4\\
&\left. + 256 d_2^2 (d_2 - a_2 e)^{16} \rho^{10} \omega^6 \right).
\end{aligned}
$$

Where,

$$
\begin{aligned}
l_{a1} =& a_1^{10} a_2^6 b_2^{16} d_2^{10} e^6 (d_2 - a_2 e)^2 k^3 (10 d_2 (d_2 - a_2 e) - 3 a_2 e \rho)\\
&+ a_1^9 a_2^6 b_2^{15} d_2^9 e^6 (d_2 - a_2 e)^2 k^4 (-80 d_2 (d_2 - a_2 e) + (d_2 + 6 a_2 e) \rho),\\
l_{a2} =& b_2^2 k^4 \left(a_2 d_2^3 e (280 d_2 - 13\rho) + 3 a_2^3 d_2 e^3 (280 d_2 - \rho) + 2 d_2^4 \rho \right.\\
&\left. - a_2^4 e^4 (280 d_2 + 3\rho) + a_2^2 d_2^2 e^2 (-876 d_2 + 5\rho) \right) + 36 (d_2 - a_2 e)^7 \rho^4 \omega^2,\\
l_{a3} =& a_2^2 b_2^2 e^2 k^4 \left(7 a_2 d_2^3 e (100 d_2 - 23\rho) + 42 d_2^4 \rho - 5 a_2^4 e^4 (140 d_2 + 93\rho) \right.\\
&\left. - a_2^2 d_2^2 e^2 (2418 d_2 + 457\rho) + a_2^3 d_2 e^3 (2100 d_2 + 853\rho) \right) + 4\rho^3 \left(2 d_2^{10} \right.\\
&+ 51 a_2 d_2^9 e - 444 a_2^2 d_2^8 e^2 + 1476 a_2^3 d_2^7 e^3 - 3024 a_2^4 d_2^6 e^4\\
&+ 2562 a_2^5 d_2^5 e^5 - 1176 a_2^6 d_2^4 e^6 - 156 a_2^7 d_2^3 e^7 + 442 a_2^8 d_2^2 e^8\\
&\left. - 221 a_2^9 d_2 e^9 + 36 a_2^{10} e^{10} + 3 a_2 e (d_2 - a_2 e)^7 (10 d_2 + 9 a_2 e) \rho \right) \omega^2\\
l_{a4} =& a_2 b_2^2 e k^4 \left(-14 d_2^4 \rho + 7 a_2 d_2^3 e (-80 d_2 + 9\rho) + 10 a_2^4 e^4 (56 d_2 + 15\rho) \right.\\
&\left. + 6 a_2^2 d_2^2 e^2 (308 d_2 + 23\rho) - 5 a_2^3 d_2 e^3 (336 d_2 + 53\rho) \right)\\
&- 4\rho^3 \left(11 d_2^9 - 82 a_2 d_2^8 e + 260 a_2^2 d_2^7 e^2 - 504 a_2^3 d_2^6 e^3 + 434 a_2^4 d_2^5 e^4 \right.\\
&- 196 a_2^5 d_2^4 e^5 - 28 a_2^6 d_2^3 e^6 + 72 a_2^7 d_2^2 e^7 - 37 a_2^8 d_2 e^8\\
&\left. + 6 a_2^9 e^9 + 3 (d_2 - a_2 e)^7 (3 d_2 + 10 a_2 e) \rho \right) \omega^2\\
l_{a5} =& - a_2^5 b_2^4 e^5 k^8 + 4 a_2^2 b_2^2 e^2 (d_2 - a_2 e)^5 k^4 \rho^2 \left(-a_2^2 e^2 + a_2 e (d_2 - 8\rho) \right.\\
&\left. + 5 d_2 \rho \right) \omega^2 + 8 (d_2 - a_2 e)^{11} (d_2 + a_2 e) \rho^5 \omega^4,
\end{aligned}
$$

$$
\begin{aligned}
l_{a6} =& a_2^3 b_2^4 e^3 k^8 \left(a_2 d_2^3 e (10 d_2 - 19\rho) + 6 d_2^4 \rho - 3 a_2^2 d_2^2 e^2 (12 d_2 + 17\rho) \right.\\
&\left. + 5 a_2^3 d_2 e^3 (6 d_2 + 19\rho) - a_2^4 e^4 (10 d_2 + 51\rho) \right) + 4 a_2 b_2^2 e k^4 \rho^3 \left(2 d_2^{10} \right.\\
&+ 55 a_2 d_2^9 e - 476 a_2^2 d_2^8 e^2 + 1588 a_2^3 d_2^7 e^3 - 3024 a_2^4 d_2^6 e^4\\
&+ 2842 a_2^5 d_2^5 e^5 - 1400 a_2^6 d_2^4 e^6 - 44 a_2^7 d_2^3 e^7 + 442 a_2^8 d_2^2 e^8\\
&\left. - 217 a_2^9 d_2 e^9 + 36 a_2^{10} e^{10} + 3 a_2 e (32 d_2 - 57 a_2 e)(d_2 - a_2 e)^7 \rho \right) \omega^2\\
&- 48 (d_2 - 3 a_2 e)(d_2 - a_2 e)^{13} \rho^6 \omega^4,
\end{aligned}
$$

$$
\begin{aligned}
l_{a7} =& a_2^3 b_2^4 e^3 k^8 \left(-30 d_2^4 \rho + 12 a_2^2 d_2^2 e^2 (24 d_2 + 7\rho) + 4 a_2^4 e^4 (20 d_2 + 21\rho)\right. \\
&\left. + a_2 d_2^3 e(-80 d_2 + 97\rho) - a_2^3 d_2 e^3 (240 d_2 + 131\rho)\right) \\
&- 4 a_2 b_2^2 e k^4 \rho^3 \left(8 d_2^{10} + 111 a_2 d_2^9 e - 1122 a_2^2 d_2^8 e^2\right. \\
&+ 4020 a_2^3 d_2^7 e^3 - 8568 a_2^4 d_2^6 e^4 + 8778 a_2^5 d_2^5 e^5 - 5796 a_2^6 d_2^4 e^6 \\
&+ 1812 a_2^7 d_2^3 e^7 + 112 a_2^8 d_2^2 e^8 - 257 a_2^9 d_2 e^9 + 54 a_2^{10} e^{10} \\
&\left. + 3 a_2 e(34 d_2 - 121 a_2 e)(d_2 - a_2 e)^7 \rho\right) \omega^2 \\
&+ 48 (d_2 - a_2 e)^{13} (5 d_2 - 11 a_2 e - 4\rho) \rho^6 \omega^4,
\end{aligned}
$$

$$
\begin{aligned}
l_{a8} =& a_2^2 b_2^4 e^2 k^8 \left(a_2 d_2^3 e(560 d_2 - 239\rho) + 68 d_2^4 \rho - 80 a_2^4 e^4 (7 d_2 + 6\rho)\right. \\
&\left. - 2 a_2^2 d_2^2 e^2 (996 d_2 + 239\rho) + a_2^3 d_2 e^3 (1680 d_2 + 857\rho)\right) \\
&+ 4 b_2^2 k^4 \rho^3 \left(2 \left(4 d_2^{10} + 53 a_2 d_2^9 e - 541 a_2^2 d_2^8 e^2 + 1940 a_2^3 d_2^7 e^3\right.\right. \\
&- 4284 a_2^4 d_2^6 e^4 + 4214 a_2^5 d_2^5 e^5 - 2758 a_2^6 d_2^4 e^6 + 836 a_2^7 d_2^3 e^7 \\
&\left. + 56 a_2^8 d_2^2 e^8 - 131 a_2^9 d_2 e^9 + 27 a_2^{10} e^{10}\right) \\
&\left. + 3 a_2 e(11 d_2 - 3 a_2 e)(d_2 - a_2 e)^7 \rho\right) \omega^2 \\
&+ 144 (d_2 - a_2 e)^{12} \rho^7 \omega^4, \\
l_{a9} =& a_2^3 b_2^4 e^3 k^8 \left(a_2 d_2^3 e(280 d_2 - 207\rho) + 62 d_2^4 \rho - 12 a_2^2 d_2^2 e^2 (84 d_2 + 17\rho)\right. \\
&\left. - 4 a_2^4 e^4 (70 d_2 + 51\rho) + 5 a_2^3 d_2 e^3 (168 d_2 + 65\rho)\right) \\
&+ 4 a_2 b_2^2 e k^4 \rho^3 \left(2 \left(6 d_2^{10} + 67 a_2 d_2^9 e - 728 a_2^2 d_2^8 e^2 + 2692 a_2^3 d_2^7 e^3\right.\right. \\
&- 6048 a_2^4 d_2^6 e^4 + 6370 a_2^5 d_2^5 e^5 - 4592 a_2^6 d_2^4 e^6 + 1828 a_2^7 d_2^3 e^7 \\
&\left. - 258 a_2^8 d_2^2 e^8 - 77 a_2^9 d_2 e^9 + 24 a_2^{10} e^{10}\right) \\
&\left. + 3 a_2 e(16 d_2 - 79 a_2 e)(d_2 - a_2 e)^7 \rho\right) \omega^2 \\
&- 48 (d_2 - a_2 e)^{12} \rho^6 \left(3 (d_2 - a_2 e)^2 - (3 d_2 + 5 a_2 e)\rho\right) \omega^4, \\
l_{a10} =& 6 a_1 b_2 d_2 (d_2 - a_2 e)^2 \rho \left(a_2^7 b_2^6 e^7 k^{12} - 2 a_2^4 b_2^4 e^4 (-d_2 + a_2 e)^5 k^8 \rho^2 \left(-4 d_2^2\right.\right. \\
&\left. + 2 d_2 (a_2 e - 5\rho) + a_2 e(2 a_2 e + 29\rho)\right) \omega^2 \\
&- 8 a_2^2 b_2^2 e^2 (d_2 - a_2 e)^{10} k^4 \rho^5 \left(d_2^2 + 4 d_2 (2 a_2 e + 3\rho)\right. \\
&\left.\left. - a_2 e(9 a_2 e + 25\rho)\right) \omega^4 + 32 (d_2 - a_2 e)^{16} (3 d_2 - a_2 e) \rho^8 \omega^6\right),
\end{aligned}
$$

$$
\begin{aligned}
l_{b1} =& a_1^{12} a_2^6 b_2^{16} d_2^{12} e^6 k^2 - 8 a_1^{11} a_2^6 b_2^{15} d_2^{11} e^6 k^3 + 28 a_1^{10} a_2^6 b_2^{14} d_2^{10} e^6 k^4 \\
&- 56 a_1^9 a_2^6 b_2^{13} d_2^9 e^6 k^5 + 70 a_1^8 a_2^6 b_2^{12} d_2^8 e^6 k^6 - 56 a_1^7 a_2^6 b_2^{11} d_2^7 e^6 k^7 \\
&+ 28 a_1^6 a_2^6 b_2^{10} d_2^6 e^6 k^8 - 8 a_1^5 a_2^6 b_2^9 d_2^5 e^6 k^9 + a_1^4 a_2^6 b_2^8 d_2^4 e^6 k^{10} \\
&+ 4 a_1^8 a_2^6 b_2^{14} d_2^{10} e^6 k^4 \rho^2 - 24 a_1^7 a_2^6 b_2^{13} d_2^9 e^6 k^5 \rho^2 + 60 a_1^6 a_2^6 b_2^{12} d_2^8 e^6 k^6 \rho^2 \\
&- 80 a_1^5 a_2^6 b_2^{11} d_2^7 e^6 k^7 \rho^2 + 60 a_1^4 a_2^6 b_2^{10} d_2^6 e^6 k^8 \rho^2 - 24 a_1^3 a_2^6 b_2^9 d_2^5 e^6 k^9 \rho^2 \\
&+ 4 a_1^2 a_2^6 b_2^8 d_2^4 e^6 k^{10} \rho^2,
\end{aligned}
$$

$$
\begin{aligned}
l_{b2} =& a_1^3 b_2 d_2^3 (-a_1 b_2 d_2 + k)^2 - 2 a_1^2 b_2^2 d_2^4 (d_2 - a_2 e) k\rho + 4 a_1 b_2 d_2^3 k^2 \rho^2 \\
&+ 2 e^2 (-d_2 + a_2 e) k^3 \rho^3, \\
l_{b3} =& a_1^2 d_2^2 (-a_1 b_2 d_2 + k)^2 - 8 a_1 b_2 d_2^3 (d_2 - a_2 e) k\rho + 4 \left(-2 a_2 d_2 e^3\right. \\
&\left. + a_2^2 e^4 + d_2^2 \left(1 + e^2\right)\right) k^2 \rho^2.
\end{aligned}
$$

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