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Integrability and global dynamics of the May–Leonard model

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ABSTRACT

We study when the celebrated May–Leonard model in $\mathbb{R}^3$, describing the competition between three species and depending on two positive parameters a and b, is completely integrable; i.e. when $a+b=2$ or $a=b$. For these values of the parameters we shall describe its global dynamics in the compactification of the positive octant, i.e. adding its infinity.

If $a+b=2$ and $a\neq 1$ (otherwise the dynamics is very easy) the global dynamics was partially known, and roughly speaking there are invariant topological half-cones by the flow of the system. These half-cones have a vertex at the origin of coordinates and surround the bisectrix $x=y=z$, and foliate the positive octant. The orbits of each half-cone are attracted to a unique periodic orbit of the half-cone, which lives on the plane $x+y+z=1$.

If $b=a\neq 1$ then we consider two cases. First, if $0<a<1$ then the unique positive equilibrium point attracts all the orbits of the interior of the positive octant. If $a>1$ then there are three equilibria in the boundary of the positive octant, which attract almost all the orbits of the interior of the octant, we describe completely their bassins of attractions.

1. Introduction

The Lotka–Volterra systems are classical differential systems introduced independently by Lotka and Volterra in the 1920s to model the interaction among species, see [1,2], see also [3]. They are equivalent to the replicator differential equations used in game theoretic applications to economics and evolution. See the good book of Hofbauer and Sigmund [4] for an introduction to the Lotka–Volterra and replicator systems and their applications. Nowadays the Lotka–Volterra systems continue being intensively studied; see for instance the recent works on them in [5–10] where they are studied from different points of view as their phase portraits, oscillations, traveling wave solutions, with delay, …

A particular class of the 3-dimensional Lotka–Volterra systems are the so called May–Leonard models. More precisely, in 1975 May and Leonard [11] studied the 3-dimensional Lotka–Volterra differential system

$$\begin{aligned}\dot{x} &= x(1-x-ay-bz),\\ \dot{y} &= y(1-bx-y-az),\\ \dot{z} &= z(1-ax-by-z),\end{aligned}\tag{1}$$

in $x\geq 0$, $y\geq 0$, and $z\geq 0$, describing the competition between three species and depending on two parameters $a>0$ and $b>0$. See [11] for the biological meaning of the variables x, y, z and the parameters a and b. Differential system (1) has been extensively studied by several authors, for instance see [12–17] and the references therein.

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